COMPREHENSION CHECK [1]

Taylor-Wiles system ([W], [TW], and [D1]).

1-1. Explain which part of the axioms of Taylor-Wiles system means "R is enough small", and which part means "T is enough large".
1-2. We need some numerical coincidence to make Taylor-Wiles system. Explain this.
1-3. How do we kill dual Selmer groups?
1-4. Explain how Auslander-Buchsbaum theorem was used in Diamond-Fujiwara's improvement of Taylor-Wiles system.
1-5. Explain $\mathcal{O}[\Delta_Q]$-structure of universal deformation rings.
1-6. Explain $\mathcal{O}[\Delta_Q]$-structure and its properties of Hecke modules.
1-7. Why is the Taylor-Wiles system applicable only for "minimal case"?
1-8. The theory of Taylor-Wiles system was improved, by Faltings ([TW, appendix]), and Diamond-Fujiwara ([D1]). Explain how the inputs and the outputs were changed about the following things.
   (a) $\mathbb{T}$ is locally complete intersection,
   (b) $R \xrightarrow{\sim} \mathbb{T}$, and
   (c) freeness of Hecke modules.
1-9. The Gorenstein-ness of Hecke algebras was used in three ways in the original arguments of [W] and [TW]. They are used in minimal case, non-minimal case, and a ring theoretic proposition [2]. Explain these.
1-10. What did we deduce the Gorenstein-ness of Hecke algebras from?
1-11. Now, we do not need to show the Gorenstein-ness of Hecke algebras to use Taylor-Wiles sytem. How was it improved?
1-12. How do we use the assumption that "$\overline{\rho}|_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$ is absolutely irreducible"? Explain at least two usage concretely.
1-13. Explain Ihara's lemma.
1-14. Explain $(3, 5)$-trick.
1-15. Which does not exist?
   (a) elliptic curve,
   (b) modular curve,
   (c) Shimura curve,
   (d) Frey curve, or
   (e) Fermat curve.

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[2] The assumption of Gorenstein-ness was removed soon by Lenstra about this ring theoretic proposition.

COMPREHENSION CHECK

Galois representations associated to Hilbert modular forms, and congruences ([T1]).

2-1. How do we use Shimura curves and the Jacquet-Langlands correspondence to construct Galois representations associated to Hilbert modular forms in the case where $[F : \mathbb{Q}]$ is odd or π is special or supercuspidal at some finite place?

2-2. How do we construct "congruences between old forms and new forms" to construct Galois representations associated to Hilbert modular forms in the case where $[F : \mathbb{Q}]$ is even?

2-3. How is the "error term" of congruences controllable?

2-4. Explain how we show there are enough congruences by using a Hilbert modular variety (not Shimura curve).

2-5. Explain in the point of view of applications to $R = T$ why it is useful to construct Galois representations associated to Hilbert modular form in the case of even degree.

COMPREHENSION CHECK

Global-local compatibility ([Ca1], [S3], and [S4]).

$\ell \neq p$ case:

3-1. Express supersingular loci adelically.

3-2. Explain how principal series representations appear in the cohomology of the normalization of the special fiber of Shimura curves.

3-3. Explain how (a part of) special representations appear in the cohomology of supersingular loci of the special fiber of Shimura curves.

3-4. Explain how supercuspidal representations appear in the cohomology of vanishing cycle sheaves of Shimura curves, and how we can show that $\sigma_{\mathfrak{p}}(\pi)$ depends only on $\pi_{\mathfrak{p}}$.

3-5. Explain the relation between the cohomology of vanishing cycle sheaves of Shimura curves, and (GL_2-case of) "vanishing cycle side" of Carayol's program.

3-6. Explain how we used the "congruence relation" in 3-2 and 3-3.

3-7. Explain how we can calculate the monodromy operator by using Picard-Lefschetz formula.

3-8. Explain how we can show the global-local compatibility for extraordinary representations.

$\ell = p$ case:

3-9 Explain how we use Lefschetz trace formulae and weight spectral sequences to compare ℓ-adic side and p-adic side of Weil-Deligne representations.

3-10 Explain how we use the weight-monodromy conjecture to compare ℓ-adic side and p-adic side of monodromy operators.

3-11 The only way to compare ℓ-adic side and p-adic side is to use geometry. However, there is not moduli interpretation of Shimura curves, so we cannot consider "Kuga-Sato variety" in a naive way. Explain how we overcome this difficulty.

COMPREHENSION CHECK

Kisin's modified Taylor-Wiles system ([K1]).

- $\boxed{4}$-1. What is the merit (for Taylor-Wiles system) of using automorphic forms on a quaternion, which ramifies at all archemedian places?
- $\boxed{4}$-2. When we consider deformations, which are Barsotti-Tate at p only after wildly ramified extensions, then the universal deformation ring has bad singularity, and we cannot expect that it is locally complete intersection. How did Kisin's modified Taylor-Wiles system overcome this difficulty?
- $\boxed{4}$-3. Explain how Kisin's modified Talyor-Wiles system can treat "non-minimal case".
- $\boxed{4}$-4. The properties of local deformation rings are important for Kisin's modified Taylor-Wiles system. Explain how the following things are used in the "$R^{\mathrm{red}} = \mathbb{T}$" theorem.
  - (a) local deformation rings are domain,
  - (b) local deformation rings are formally smooth after inverting p, and
  - (c) calculations of the dimensions of local deformation rings.
- $\boxed{4}$-5. Calculate the dimension of
  - (a) local deformation rings for $v \in \Sigma$,
  - (b) local deformation rings for $v \mid p$, and
  - (c) global deformation rings.
- $\boxed{4}$-6. Explain the technique of investigating stalks of points in generic fiber of local deformation rings.
- $\boxed{4}$-7. How did we get the needed information about local deformation rings from the moduli of finite flat group schemes?
- $\boxed{4}$-8. Explain the following things, which we did to get the needed information about the moduli of finite flat group schemes:
  - (a) To relate it with complete local rings of a Hilbert modular variety.
  - (b) To calculate linear algebraic data.
- $\boxed{4}$-9.
  - (a) Where is a "geometric incarnation" of Tate's theorem $\{p\text{-divisible groups}/O_K\} \hookrightarrow \underline{\mathrm{Rep}}_{\mathbb{Z}_p}(G_K)$ in [K1]?
  - (b) Where did we use Breuil's theorem $\underline{\mathrm{Rep}}^{\mathrm{fl}}_{\mathrm{tor}}(G_K) \hookrightarrow \underline{\mathrm{Rep}}_{\mathrm{tor}}(G_{K_\infty})$ in [K1]?
  - (c) Where did we use Breuil's theorem "crystalline representations of Hodge-Tate weights $\subset \{0, 1\}$ come from p-divisible groups" in [K1]?
- $\boxed{4}$-10. Explain Kisin's generalization or another proof of the above $\boxed{4}$-9 (b) and (c), in terms of modules with connection on open unit disk.
- $\boxed{4}$-11. Explain Skinner-Wiles' base change arguments [SW3].

COMPREHENSION CHECK

Modularity lifting theorem for crystalline representasions of intermediate weights ([K3], [BLZ], [BB1], [C1], and [C2]).

5-1. What do we use in the case of crystalline deformations of intermediate weights instead of the moduli of finit flat group schemes?

5-2. How did we use the information of mod p reduction of crystalline representations of intermediate weights to show the modularity lifting?

5-3. Explain the relation between the category of Wach modules (lattices) and the category of crystalline representations (and their lattices).

5-4. Explain the following two methods of determing mod p reduction of crystalline representations of intermediate weights:
 (a) the method of Berger-Li-Zhu ([BLZ]), and
 (b) the method of Berger-Breuil ([BB1]).

5-5. Explain how the p-adic local Langlands correspondence was used in [BB1].

5-6. Explain the compatibility of p-adic local Langlands correspondence and mod p local Langlands correspondence.

5-7. Explain how the technique of "flatening" was used in Gabber's appendix in [K3].

COMPREHENSION CHECK

Modularity lifting theorem for residually reducible representations ([SW1]).

6-1. Explain pseudo representaions.

6-2. We do not have deformations to Hecke algebras in the residually reducible case, and only have pseudo deformations to Hecke algebras. This makes arguments complicated. Explain this, and how we solved this technical problem.

6-3. In Skinner-Wiles' technique, we have to make base changes to make codimension of loci of reducible representaion larger in the spectrum of a universal deformation ring. Explain this.

6-4. We use Hida theoretic Hecke algebras in the Skinner-Wiles case. So, the quotient algebras modulo prime ideals do not have finite cardinality in general. Thus, we have to modify Taylor-Wiles' patching arguments. Explain this.

6-5. We do not have $\overline{\rho}|_{F(\zeta_p)}$ is absolutely irreducible in the residually reducible case [SW1], and we do not need to assume it even in the residually irreducible and nearly ordinary deformation case [SW2]. How do we modify the Taylor-Wiles arguments, especially killing dual Selmer groups?

6-6. How do we show the existence of Eisenstein ideals by using p-adic L-function in [SW1]?

6-7. How do we construct "nice" primes in [SW1]?

6-8. How do we use Washington's theorem about p-rank of ideal class groups of cyclotomic $\mathbb{Z}_\ell$-extensions in [SW1]?

6-9. How do we use Mme Raynaud's theorem in [SW1]?

6-10. Explain how we show pro-modularity of other irreducible components from pro-modularity of an irreducible component.

6-11. How do we use the (technical) concept of "nice" prime?

COMPREHENSION CHECK

Taylor's potential modularity theorem ([T2], and [T3]).

7-1. How do we use a Hilbert-Blumenthal modular variety to do "(ℓ, ℓ')-trick"?.

7-2. How do we use Moret-Bailly's theorem to do "(ℓ, ℓ')-trick"?.

7-3. Explain the following things to construct "local points":
  (a) construction of Hilbert-Blumenthal abelian varieties over a finite field, and Honda-Tate theory, and
  (b) construction of a lifting of it to a local field, and Serre-Tate theory.

7-4. Explain the arguments of raising level, and congruences between different weights.

7-5. How do we use principal ideal theorem in [T3]?

7-6. Explain the following applications:
  (a) Fontaine-Mazur conjecture of degree 2, and
  (b) meromorphic continuation and functional equation of L-functions for (strongly compatible system of) ℓ-adic representations of degree 2.

7-7. Explain how we use these potential modularity theorems for Khare-Wintenberger's proof of Serre's conjecture.

COMPREHENSION CHECK

Modularity lifting theorem for unitary groups, and proof of Sato-Tate conjecture ([CHT], [T4], and [HSBT]).

8-1. Explain why we use unitary groups to consider GL_n.
8-2. We need some numerical coincidence to use Taylor-Wiles system. Explain this coincidence for unitary groups and essentially self-dual representations.
8-3. Explain Taylor-Wiles type deformations and $\mathcal{O}[\Delta_Q]$-structure of universal deformation rings for unitary groups.
8-4. Explain $\mathcal{O}[\Delta_Q]$-structure and its properties of Hecke modules for unitary groups.
8-5. Explain how we used "Ramakrishna deformations" and "one more deformations" in [CHT].
8-6. How do we avoid using Ihara's lemma, and make the modularity lifting theorem for non-minimal case unconditional?
8-7. We have to know some information about the action of local deformation rings on Hecke modules to use Kisin's modified Taylor-Wiles arguments. How do we get the information about it?
8-8. How do we use the fact that the Calabi-Yau family we are considering has big monodromy to show the potential modularity?
8-9. Explain how the potential modularity of $\mathrm{Sym}^{\mathrm{n}} H^1(E)$ deduce from the modularity lifting thoerem for unitary groups, and a Calabi-Yau family.
8-10. Explain how we deduce Sato-Tate conjecture from the potential modularity of $\mathrm{Sym}^{\mathrm{n}} H^1(E)$ for odd n's.

COMPREHENSION CHECK

Serre's conjecture for $p = 2, 3$, and 5 ([Ta2], [Se2], and [Sc]).

9-1. How do we use Minkowski's (root) discriminant bound and global class field theory in Tate's proof of Serre's conjecture for $p = 2$?

9-2. How do we use Odlyzko's (root) discriminant bound and global class field theory in Serre's proof of Serre's conjecture for $p = 3$?

9-3. How do we use Odlyzko's (root) discriminant bound, Fontaine's (root) discriminant bound, and global class field theory in Schoof's proof of Serre's conjecture for $p = 5$ [1]?

9-4. How do we show Odlyzko's (root) discriminant bound by using Dedekind zeta function?

9-5. How was the PD-structure used for Fontaine's (root) discriminant bound?

9-6. Explain the following things in Schoof's proof [Sc]:

(a) How do we use Herbrand's theorem and p-adic L-function when we consider extensions of μ_p by $\mathbb{Z}/p\mathbb{Z}$ over Spec $\mathbb{Z}[1/\ell]$?

(b) How do we use Burnside's basis theorem?

(c) How do we use Taussky's theorem about 2-groups?

9-7. Show that any group of order less than 60 is solvable as follows [2]:

(a) Show that any p-group is nilpotent.

(b) Show that any group G of order $p^n q$ (p, q: primes) is solvable as follows:

(i) Assume that G does not have non-trivial normal subgroup. Show that the number of p-Sylow subgroups is q.

(ii) Let D be one of the maximal subgroups, which can be expressed as the intersection of two p-Sylow subgroups. Put H be the normalizer of D in G. Then, show that H has at least two p-Sylow subgroups. (Hint: any p-Sylow subgroup of H can be expressed as the intersection of a p-Sylow subgroup with H.)

(iii) Show that H has exactly q p-Sylow subgroups, and all of them contain D.

(iv) Show that $D = \{1\}$.

(v) Show that any intersection of different two p-Sylow groups is $\{1\}$.

(vi) Show that by counting elements, G has unique q-Sylow subgroup, so it is normal.

(c) Show that any group of order $p^n q^2$ ($p > q$ primes) is solvable by similar arguments as above.

(d) Show that groups of order $30 = 2.3.5$ and of order $42 = 2.3.7$ are solvable.

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[1] Precisely speaking, he proved non-existence of certain abelian varieties having restricted reduction conditions.

[2] If we use Feit-Thompson's theorem (any group of odd order is solvable) and Burnside's theorem (any group of order $p^n q^m$ (p, q: primes) is solvable), then it is easy. However, these theorems are very difficult.

COMPREHENSION CHECK

Existence of strictly compatible system, and proof of Serre's conjecture ([KW1], [Kh1], [KW2], and [KW3]).

- 10-1. Prove Fermat's Last Theorem without "Langlands-Tunnell's theorem" and "Ribet's level lowering" by the method of strictly compatible systems.
- 10-2. How do we use Taylor's potential modularity (twice) for the proof of the existence of strictly compatible systems?
- 10-3. For the existence of strictly compatible systems, we show the following things about global deformation rings. Explain the proof of them.
  - (a) Lower bound of the dimension, and
  - (b) Flatness over $\mathcal{O}$.
- 10-4. Explain kinds of the strictly compatible systems (3 kinds in the level one case, 4 kinds in general).
- 10-5. Explain Dieulefait's another proof of Serre's conjecture of level one case, which do not use a distribution of Fermat primes [Di2].
- 10-6. Write down how the inductions are arranged in [Kh1], and [KW2].
- 10-7. Explain the following induction steps:
  - (a) "add a good dihedral prime",
  - (b) "killing ramification", and
  - (c) "weight reduction".
- 10-8. Explain applications of Serre's conjecture about:
  - (a) some Fermat-like diophantine equations,
  - (b) non-existence of some finite flat group schemes,
  - (c) the finiteness of isomorphism classes of mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of bounded ramifications,
  - (d) modularity of abelian varieties of GL_2-type,
  - (e) Artin's conjecture of degree 2, and
  - (f) Fontaine-Mazur conjecture of degree 2.

COMPREHENSION CHECK

Modularity lifting theorem and potential modularity for $p = 2$ ([Kh1], [KW3], and [K5]).

11-1. We have $(k \cdot \mathrm{id}) \subset \mathrm{ad}^0\overline{\rho}$ in the case of $p = 2$. So, we do not have $(\mathrm{ad}^0\overline{\rho})^* \cong \mathrm{ad}^0\overline{\rho}$ in this case. How do we modify the usual Galois cohomology calculations in this case ([KW3])?

11-2. How do we modify the arguments of killing dual Selmer groups in the case of $p = 2, 3$ in [Kh1] ($p = 3$) and [KW3] ($p = 2, 3$) [1]?

11-3. How do we treat the "neatness problem" in the case of $p = 2, 3$ in [Kh1] ($p = 3$) and [KW3] ($p = 2, 3$)?

11-4. We cannot take a finite place v above an odd prime such that

$$(1 - Nv)((1 + Nv)^2 \det \overline{\rho}(\mathrm{Fr}_v) - Nv(\mathrm{tr}\overline{\rho}(\mathrm{Fr}_v))^2) \in \mathbb{F}^\times$$

in the case of $p = 2$. How do we overcome this difficulty in [KW3] and [K5]?

11-5. We cannot use Breuil's theory in the case of $p = 2$. So, we use Zink's theory of displays and windows in [K5]. How do we use this to overcome the difficulty?

[1] In [T3], the case $p = 2, 3$ are excluded.

COMPREHENSION CHECK

Modularity lifting theorem and Breuil-Mézard conjecture ([K6], [BM], [C1], and [C2]).

12-1. Explain Breuil-Mézard conjecture.

12-2. How do we use Breuil-Mézard conjecture to show the modularity lifting theorem for potentially semistable deformations? In particular, how do we overcome the difficulty that we do not know that the local deformation rings are domains?

12-3. How do we use the p-adic local Langlands correspondence in Kisin's proof of many cases of Breuil-Mézard conjecture.

12-4. Explain Colmez' functor in the p-adic local Langlands correspondence.

12-5. Explain the compatibility of the p-adic local Langlands correspondence and classical local Langlands correspondence.

12-6. Explain how we use the compatibility of p-adic local Langlands correspondence and classical local Langlands correspondence in Kisin's proof of many cases of Breuil-Mézard conjecture.