Pointwise and Sequential Continuity in Constructive Analysis

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We discuss various continuity properties, especially pointwise and sequential continuity, in Bishop's constructive mathematics; see [1, 2, 11] for Bishop's constructive mathematics and [3, 4, 5, 9] for various continuity properties. We say that a mapping f between metric spaces X and Y is *sequentially continuous* if $x_n \to x$ implies that $f(x_n) \to f(x)$; *pointwise continuous* if for each $x \in X$ and $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$ for all $y \in X$. We first show the following theorem.

Theorem 1 *The following are equivalent.*

1. *Every sequentially continuous mapping of a separable metric space into a metric space is pointwise continuous.*

2. *Every sequentially continuous mapping of a complete separable metric space into a metric space is pointwise continuous.*

3. BD-N. *Every countable pseudo-bounded subset of* $\mathbf{N}$ *is bounded.*

Here a subset A of $\mathbf{N}$ is said to be *pseudo-bounded* if for each sequence $\{a_n\}$ in A, $a_n < n$ for all sufficiently large n. Note that although BD-N holds in classical mathematics, intuitionistic mathematics and constructive recursive mathematics of Markov's school, a natural recursivisation of BD-N is independent of Heyting arithmetic [3, 5, 8, 10].

We also show that very important theorems in functional analysis – Banach's inverse mapping theorem, the open mapping theorem, the closed graph theorem, the Banach-Steinhaus theorem and the Hellinger-Toeplitz theorem – can be proved in Bishop's constructive mathematics for *sequentially continuous* linear mappings [6, 7]. However it has emerged that the theorems for *pointwise continuous* linear mappings are equivalent to BD-N [8].

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