Effective Limit in Computable Analysis

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1 Introduction

Most of the studies of computable analysis have been capturing the computability of structures over real numbers. However, there are started some studies on other general mathematical structures recently.

Hertling studied computability over algebraic structures [H], however he did not discuss topology. Schröder studied week limit spaces [S], however he did not discuss modulus of convergence. The author believe that the notions of topology and modulus of convergence are most important in the study of computable analysis. This work aims to combine their works, and to apply to the discussion on modulus of convergence.

In discussing modulus of convergence, it seems natural to do with uniform topology, which Yasugi and others studied [Y]. They discuss the computability under uniform topology with computability structure, while Hertling and Schröder did with representation.

There are several methods in studying computable analysis. One of them is representation and another is computability structure.

The notion of representation has been studied by Weihrauch [W] and others. It is a kind of a mechanical method. A representation r of a mathematical structure X is a surjective partial function of Σ^ω into X. A function $f : X \to X$ is computable if the is a computable partial function $\bar{f}$ of Σ^* into Σ^* such that $f \circ r = r \circ \bar{f}$. We will use $\mathbf{N} \to \mathbf{N}$ as Σ^ω, the domain of representation. It seems that representations have too much information which comes from the details of implementation of computation models.

The notion of computability structure is studied by Pour-El et al. [P]. It is a kind of an axiomatic method. A computability structure over a mathematical structure X is a subset of X^ω which satisfies the axioms of computability structure. A function f over X is called computable if f is locally uniformly continuous and preserves the computability structure over X. In order to assert that a computability structure is natural, the mathematical structure X has been requested to have a distance. Yasugi et al. attempt to replace the distance with effective uniform topology in the literature [Y].

This work aims at comparing these two ways, that by computability structure and that by representation. The author would like to show that the essential notions are equivalently defined by both ways.

There are several ways to define effective uniform topology. Yasugi et al. defined it with effective uniform neighbourhood system in the literature [Y]. This work defines it with effective limit, which has not been used in any other works. An effective limit is a partial function of arity ω over the underlying set, while an effective neighbourhood system is a family of subsets. It seems that a partial function is more familiar in computational theory than a family of subset is.

There are three similar words 'recursive', 'effective' and 'computable', all of which appear in this paper. There are only little difference in the senses of these words. In this paper, the word 'recursive' is used only for the notions concerned to recursive functions over natural numbers. The word 'computable' is used only for the notion of computability structure which defined by Pour-El et al. [P]. The word 'effective' is used for other mathematical structures, Thus, a function represented by a recursive Type-2 function is called a recursive function in this paper, although Weihrauch and others call it a computable function [W].

2 Uniform Topology

Notation 2.1 We write $\mathbf{N}$ for $\{1, 2, 3, ...\}$.

Definition 2.2 (Partial function) A function f is a partial function of X into Y iff f is a function of $X' \subset X$ into $Y' \subset Y$. This X' is called the *domain* of f and written as $dom(f)$. We write $f : X \to_p Y$ when f is a partial function of X into Y. For $x \in X$, we say that $f(x)$ is *defined* when $x \in dom(f)$, and $f(x)$ is *undefined* if not. When we write $f(x) = y$, we implicitly assert $x \in dom(f)$.

The *range* of $f : X \to_p Y$ is the set $\{f(x) \in Y \mid x \in dom(f)\}$ and written as *range*(f). For $f : X \to_p Y$ and $g : Y \to_p Z$, the concatenation $g \circ f : X \to_p Z$ is defined as following: $dom(g \circ f) = \{x \in X \mid x \in dom(f), f(x) \in dom(g)\}$ and $g \circ f(x) = g(f(x))$ for $x \in dom(g \circ f)$.

For partial functions $f : X \to_p Y$ and $g : X' \to_p Y'$, the relation $f \subset_p g$ holds iff $dom(f) \subset dom(g)$ and $f(x) = g(x)$ for all $x \in dom(f)$. It may be the case that $X \neq X'$ or $Y \neq Y'$ although $f \subset_p g$. For partial functions f and g, it holds $f = g$ if $f \subset_p g$ and $g \subset_p f$.

Notation 2.3 We write $X \to Y$ for the function space $\{f \mid f : X \to Y\}$, and $X \to_p Y$ for the set of all the partial functions $\{f \mid f : X \to_p Y\}$. Thus, the notation $E \subset X \to Y$ does not mean $E : X \to_p Y$ but means $\forall f \in E.\ f : X \to Y$.

The operators $\to$ and $\to_p$ are right associative. Thus, the notation $X \to Y \to_p Z$ is an abbreviation of $X \to (Y \to_p Z)$.

Definition 2.4 (Modular Limit) Let X be a set. An partial function LM is a *modular limit* over X if there is a strictly increasing function $m : \mathbf{N} \to \mathbf{N}$ and the followings hold:

0. $LM : X^\omega \to_p X$

1. For each $x \in X$, $LM(x, x, x, ...) = x$
2. Let f be a function of $\mathbf{N} \to \mathbf{N}$ such that $f(n) \geq n$ for any $n \in \mathbf{N}$. If $(x_1, x_2, x_3, ...) \in dom(LM)$, then $(x_{f(1)}, x_{f(2)}, x_{f(3)}, ...) \in dom(LM)$ and $LM(x_{f(1)}, x_{f(2)}, x_{f(3)}, ...) = LM(x_1, x_2, x_3, ...)$
3. Let $\{x_{i,j}\}_{i,j}$ be a double sequence in X, that is, $x_{i,j} \in X$ for any $i, j \in \mathbf{N}$. Let $(y_1, y_2, y_3, ...)$ be a sequence in X. and z be a point in X. Suppose that $y_i = LM(x_{i,1}, x_{i,2}, x_{i,3}, ...)$ for each i.

   If $LM(y_1, y_2, y_3, ...) = z$, then $LM(x_{m(1),m(1)}, x_{m(2),m(2)}, x_{m(3),m(3)}, ...) = z$.
4. Let $\{x_{i,j}\}_{i,j}$ be a double sequence in X, that is, $x_{i,j} \in X$ for any $i, j \in \mathbf{N}$. Let $(y_1, y_2, y_3, ...)$ be a sequence in X. and z be a point in X. Suppose that $y_i = LM(x_{i,1}, x_{i,2}, x_{i,3}, ...)$ for each i.

   If $LM(x_{1,1}, x_{2,2}, x_{3,3}, ...) = z$, then $LM(y_{m(1)}, y_{m(2)}, y_{m(3)}, ...) = z$.

The function $m(\,)$, which appears in the conditions 3 and 4, is called a *modulus of diagonal convergence.*

Definition 2.5 (Effective Limit) A modular limit is called an *effective limit* when the modulus of diagonal convergence of it is a recursive function.

Remark 2.6 The conditions 3 and 4 in the definition of modular limit means that all the converging sequence under the modular limit are uniformly converging.

Definition 2.7 (Uniform Neighbourhood System) Let X be a set. The series of subsets $\{V_i(x)\}_{i\in\mathbf{N},x\in X}$ is a *uniform neighbourhood system* iff it satisfies the followings:

1. $x \in V_i(x) \subset X$ for each $i \in \mathbf{N}$ and each $x \in X$.
2. $V_i(x) \supset V_j(x)$ for $i < j$
3. There is a function $m : \mathbf{N} \to \mathbf{N}$ such that if $y \in V_{m(i)}(x)$ then $x \in V_i(y)$
4. There is a function $m' : \mathbf{N} \to \mathbf{N}$ such that if $y \in V_{m'(i)}(x)$ then $V_{m'(i)}(y) \subset V_i(x)$

The functions m and m' in the conditions 3 and 4 are called the *moduli.*

Definition 2.8 (Effective Uniform Neighbourhood System) The uniform neighbourhood system is *effective* if both of the moduli are recursive functions.

Remark 2.9 When we replace the condition 4 of the definition above with:
$\forall x \in X.\forall i \in \mathbf{N}.\exists j \in \mathbf{N}.\forall y \in V_j(x).\exists k \in \mathbf{N}.\ V_k(y) \subset V_i(x)$
then, the conditions defines the ordinary notion of countable neighbourhood systems. If we exchange the order of quantifiers $\forall y$ and $\exists k$, then, this condition is equivalent to the condition 4 of the definition above.

Proposition 2.10 *Let X be a set and $\{V_i(x)\}_{i,x}$ be a uniform neighbourhood system over X. Then, X is a Hausdorff space iff* $\bigcap_{i \in \mathbf{N}} V_i(x) = \{x\}$ *for any* $x \in X$.

Remark 2.11 There are several well-known definitions of uniform topology which are equivalent to each other. One of them is the definition by uniform neighbourhood systems. The following propositions show that modulus limit also gives the equivalent definition of uniform topology.

Definition 2.12 (Closeness relation) Let LM be the modular limit of X. The relation $x \xrightarrow{n} y$ holds iff there is a sequence $(x_1, x_2, x_3, ...)$ such that $x = x_i$ and $LM(x_1, x_2, x_3, ...) = y$. We call this relation *closeness relation*.

Proposition 2.13 *The followings hold. The function $m(\ )$ is the modulus of diagonal convergence of the modular limit in the followings.*

1. *If $i \leq j$ and $x \xrightarrow{j} y$, then $x \xrightarrow{i} y$.*
2. *If $m(i) \leq j$ and $x \xrightarrow{j} y$, then $y \xrightarrow{i} x$.*
3. *If $m(i) \leq j$, $x \xrightarrow{j} y$ and $y \xrightarrow{j} z$, then $x \xrightarrow{i} z$.*
4. *If $m(m(i)) \leq j$ and $x \xrightarrow{j} z \xleftarrow{j} y$, then $x \xrightarrow{i} y$.*
5. *If it holds that $x_i \xrightarrow{m(i)} y$ for each i, then $LM(x_1, x_2, x_3, ...) = y$.*

Lemma 2.14 *Let X be a set and LM be a modular limit of modulus m over X. For each $i \in \mathbf{N}$ and each $x \in X$, a subet $V_i(x) \subset X$ is defined as $V_i(x) = \{y \mid y \xrightarrow{i} x\}$ Then, $\{V_i(x)\}_{i,x}$ is a uniform neighbourhood system of modulus m over X, and it makes X a Hausdorff space.*

Moreover, if LM is an effective limit, then $\{V_i(x)\}_{i,x}$ is an effective uniform neighbourhood system.

Lemma 2.15 *Let X be a set and $\{V_i(x)\}_{i,x}$ be a uniform neighbourhood system over X which makes X a Hausdorff space. A partial function $LM : X^\omega \rightarrow_p X$ is defined as follows:*

$$dom(LM) = \{(x_1, x_2, ...) \mid \exists y \in X.\ \forall i \in \mathbf{N}.\ x_i \in V_i(y)\}$$
$$LM(x_1, x_2, ...) = y \text{ iff } \forall i \in \mathbf{N}.\ x_i \in V_i(y)$$

Note that such x is uniquely determined, because X is a Hausdorff space by the neighbourhood system $\{V_i(x)\}_{i,x}$.

Then, LM is a modular limit of modulus m over X.

Moreover, if $\{V_i(x)\}_{i,x}$ is an effective uniform neighbourhood system, then LM is an effective limit.

Remark 2.16 The previous lemmata 2.14 and 2.15 mean that modular limit is corresponds to uniform neighbourhood system, and effective limit is corresponds to effective uniform neighbourhood system.

Example 2.17 Let $\mathbf{R}$ be the set of all the real numbers. Define a partial function $LimE : \mathbf{N}^\omega \rightarrow_p \mathbf{N}$ as:

$$dom(LimE) = \left\{ (x_1, x_2, x_3, ...) \;\middle|\; |x_i - x_j| < \frac{1}{i} + \frac{1}{j} \right\}$$
$$LimE(x_1, x_2, ...) = x \text{ iff } \lim x_i = x$$

Then, this partial function *LimE* is an effective limit with the modulus of diagonal convergence $n \mapsto 2n$.

Example 2.18 For $i \in \mathbf{N}$ and $x \in \mathbf{R}$, put $V_i^{\mathbf{R}}(x) \subset \mathbf{R}$ as $y \in V_i^{\mathbf{R}}(x)$ iff $|y - x| < 1/i\}$. Then, $V_i^{\mathbf{R}}(x) = \{y \mid y \overset{n}{\to} x\}$ and $\{V_i^{\mathbf{R}}(x)\}_{i,x}$ is an effective uniform neighbourhood system.

Example 2.19 The functional space $\mathbf{N} \to \mathbf{N}$ is regarded as a topological space. Define a partial function $LM_{\mathbf{N},\mathbf{N}} : (\mathbf{N} \to \mathbf{N})^{\omega} \to_p \mathbf{N} \to \mathbf{N}$ as:

1. $(x_1, x_2, ...) \in dom(LM_{\mathbf{N},\mathbf{N}})$ iff for any $i \leq j < k$, $x_j(i) = x_k(i)$
2. For $(x_1, x_2, ...) \in dom(LM_{\mathbf{N},\mathbf{N}})$, $LM_{\mathbf{N},\mathbf{N}}(x_1, x_2, ...) = (i \mapsto x_i(i))$

The condition 1 means that x_j and x_k have the common initial segment of length j where $j < k$. Then, this $LM_{\mathbf{N},\mathbf{N}}$ is a modular limit with the identity function $i \mapsto i$ as the modulus of diagonal convergence, and it induces the ordinary topology of $\mathbf{N} \to \mathbf{N}$.

3 Algebraic Structure

Definition 3.1 (Algebraic structure) A sequence $X = (|X|, f_1, f_2, f_3, ..., R_1, R_2, R_3, ...)$ is an *algebraic structure* iff it consists of the underlying set $|X|$, partial functions $f_i : |X|^{arity(f_i)} \to_p |X|$, and subsets $R_i \subset |X|^{arity(R_i)}$. An algebraic structure X has finite partial functions or countably many infinite partial functions, and also X has finite relations or countably many infinite relations. For any f_i and any R_i, the arities *arity*(f_i) and *arity*(R_i) belong to $0 \cup \mathbf{N} \cup \{\omega\}$. The set of the functions and relations $\{f_1, f_2, f_3, ..., R_1, R_2, R_3, ...\}$ are called the signature of X.

We sometimes identify X to $|X|$, and simply write X for $|X|$.

We may define that an algebraic structure has many sorted, or it refers some other algebraic structures.

Definition 3.2 (Algebraic Structure With Uniform Topology) Let $X = (|X|, f_1, f_2, f_3, ..., R_1, R_2, R_3, ...)$ be an algebraic structure. Suppose that a partial function $f_i = LM$ belongs to the signature, and $f_i = LM$ satisfies the definition of modular limit over the domain $|X|$. Then, this X is an *algebraic structure with uniform topology* by the modular limit LM.

If the modular limit is effective, then it is called an *algebraic structure with effective uniform topology*

Example 3.3 we define an algebraic structure $\mathbf{R} = (|\mathbf{R}|, 0, 1, +, -, \times, /, LimE, <)$ as follows:

The set $|\mathbf{R}|$ is the set of all the real numbers. The constants 0 and 1 are the numbers 0 and 1 themselves. The functions $+$, $-$ and $\times$ are the ordinary summation, subtraction and multiplication. The partial function / is the ordinary division. The partial function $LimE : \mathbf{R}^\omega \to_p \mathbf{R}$ is defined as in the example 2.17, which is a limit operation with modulus $1/n$. The relation $<$ is the ordinary inequality without equality.

Then, This $\mathbf{R}$ is an algebraic structure with effective uniform topology.

We sometimes identify $\mathbf{R}$ and $|\mathbf{R}|$.

4 Representation

Notation 4.1 We identify a function $f : \mathbf{N} \to \mathbf{N}$ to a infinite sequence $(f(0), f(1), f(2), ...) \in \mathbf{N}^\omega$. For a finite sequence $x \in \mathbf{N}^*$ and a finite or infinite sequence $y \in \mathbf{N}^* \cup \mathbf{N}^\omega$, we write $x \sqsubseteq y$ when x is an initial segment of y. Put $\phi_{\mathbf{N}^*} : \mathbf{N} \to \mathbf{N}^*$ as the standard enumeration of $\mathbf{N}^*$ and $\psi_{\mathbf{N}^*} : \mathbf{N}^* \to \mathbf{N}$ as its inverse.

Definition 4.2 (Recursive Type-2 Function) A partial function of $\mathbf{N}^\omega \to_p \mathbf{N}^\omega$ is called a *partial Type-2 function* or *partial functional*. A partial functional $F : \mathbf{N}^\omega \to_p \mathbf{N}^\omega$ is *recursive* iff there is a partial recursive function $f : \mathbf{N} \to_p \mathbf{N}$ which satisfies the following three conditions. We write $\hat{f}$ for $\phi_{\mathbf{N}^*} \circ f \circ \psi_{\mathbf{N}^*} : \mathbf{N}^* \to \mathbf{N}^*$. Then, the conditions are the following:
1. The function $\hat{f}$ is monotone with respect to $\sqsubseteq$, that is, for any $y \sqsubseteq z \in \mathbf{N}^*$, $\hat{f}(y) \sqsubseteq \hat{f}(z)$.
2. For each $x : \mathbf{N}^\omega$, $x \in dom(F)$ iff there are arbitrary long $\hat{f}(y)$ such that $y \sqsubseteq x$, that is, for every $n \in \mathbf{N}$, there exists $y \sqsubseteq x$ such that $\hat{f}(y)$ is longer than the length n.
3. For $x \in dom(F)$, $F(x)$ is the infinite sequence such that $\hat{f}(y) \sqsubseteq F(x)$ for any $y \sqsubseteq x$.

Remark 4.3 This definition is equivalent to the notion of recursive functions relative to another functions (which is given by Odifretti [O]). And also this is equivalent to the notion defined by Type-2 machines (by Weihrauch [W]).

Definition 4.4 (Representation) A partial function $r : \mathbf{N}^\omega \to_p X$ is a *representation* of X iff it is a surjection, that is, $range(r) = X$.

Definition 4.5 (Pairing) A *pairing function* $\langle -, - \rangle : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ is defined as

$$\langle m, n \rangle = \frac{(m+n-1)(m+n-2)}{2} + m,$$

which is a standard bijection of $\mathbf{N} \times \mathbf{N} \to \mathbf{N}$.

Remark 4.6 With this paring function $\langle -, - \rangle$, we can regard a unary function as a binary function, such as a unary unction $(n \mapsto f(n)) : \mathbf{N} \to \mathbf{N}$ as a binary function $((m, n) \mapsto f(\langle m, n \rangle)) : \mathbf{N}^2 \to \mathbf{N}$. As inverse, we can regard a binary function as a unary function, such as a binary function $((m, n) \mapsto f(m, n)) : \mathbf{N}^2 \to \mathbf{N}$ as a unary function $(\langle m, n \rangle \mapsto f(m, n)) : \mathbf{N} \to \mathbf{N}$.

Definition 4.7 (Recursive function with respect to representations) Let X and Y be sets. Let r and r' be representations of X and of Y respectively. A partial function $f : X \to_p Y$ is *recursive* with respect to r and r' iff there is a Type-2 partial recursive functional $F : \mathbf{N}^\omega \to_p \mathbf{N}^\omega$ such that $f \circ r \subset_p r' \circ \bar{F}$. If $X = Y$ and $r = r'$, then we say simply that the f is recursive with respect to r.

Remark 4.8 We say that $f : X^2 \to_p X$ is recursive with respect to r iff it is recursive with respect to (r', r) where r' is defined as:

$$r'(n \mapsto \langle w_1(n), w_2(n)\rangle) = (r(w_1), r(w_2)) \text{ for any } w_1, w_2 \in \mathbf{N}^\omega.$$

We say similarly for $f : X^3 \to_p X$, $f : X^4 \to_p X$ and so forth.

We say that $f : X^\omega \to_p X$ is recursive with respect to r iff it is recursive with respect to (r'', r) where r'' is defined as:

$$r'(n \mapsto (i \mapsto w(\langle n, i\rangle)) = r(w) \text{ for any } w \in \mathbf{N}^\omega.$$

Definition 4.9 (Reducibility) Let X be a set, and r be a representation of X. A partial function $f : \mathbf{N}^\omega \to_p X$ is *continuously reducible* to r if there is a partial continuous functional $F : \mathbf{N}^\omega \to \mathbf{N}^\omega$ such that $dom(f) = dom(F)$ and $f = r \circ F$. The partial function f is *reducible* to r if such F is recursive.

Definition 4.10 (Admissibility) Let X be a topological space. A representation $r : \mathbf{N}^\omega \to_p X$ is *admissible* if the following conditions hold:
1. (continuity) r is continuous.
2. (finality) Every partial continuous function $q : \mathbf{N}^\omega \to_p X$ is continuously reducible to r.

Remark 4.11 The notion of admissibility is just a topological notion, and does not include none of effectivity.

Lemma 4.12 *Let X be an algebraic structure with uniform topology, and LM be the modular limit of X, Let r be a representation of X. Suppose that LM is computable with respect to r. Then any continuous partial function $f : \mathbf{N}^\omega \to X$ r is continuously reducible to r.*

Theorem 4.13 *If a representation r of X is continuous, and the modular limit of X is recursive with respect to r, then r is admissible.*

5 Computability Structure

Remark 5.1 A binary partial recursive function $f : \mathbf{N} \times \mathbf{N} \to_p \mathbf{N}$ induces a total function of natural numbers into partial recursive functions $(m \mapsto (n \mapsto f(m, n))) : \mathbf{N} \to (\mathbf{N} \to_p \mathbf{N})$. When we say a recursive function $g : \mathbf{N} \to (\mathbf{N} \to_p \mathbf{N})$, we will denote that there is a binary partial recursive function $f : \mathbf{N} \times \mathbf{N} \to_p \mathbf{N}$ such that $g : m \mapsto g(m) = (n \mapsto f(m, n))$.

Definition 5.2 (Term) Let $\{f_1, f_2, ..., R_1, R_2, ...\}$ be a signature and $V = \{v_1, v_2, v_3, ...\}$ be a set of countably infinitely many variables. Then we define the set of formal expressions $Term_\alpha$ which is defined for each ordinal α as follows:

$Term_0 = V$

$Term_{\alpha+1} = Term_\alpha$

$\quad \cup \{f_i\langle t_1, t_2, ..., t_{arity(f_i)}\rangle \mid arity(f_i) \in \{0\} \cup \mathbf{N}, t_j \in Term_\alpha\}$

$\quad \cup \{f_i\langle t_1, t_2, t_3, ...\rangle \mid arity(f_i) = \omega, t_j \in Term_\alpha\}$

$Term_\alpha = \bigcup_{\beta < \alpha} Term_\beta$ for a limit ordinal α

It is easy to seen that $Term_\alpha$ is saturated for uncountable ordinals α, because the arity of every function is countable. Thus $Term_\alpha$ is unique for any uncountable ordinal α. We define $Term$ as:

$Term = Term_\alpha$ for the least uncountable ordinal α.

Elements of $Term$ are called *terms*.

Definition 5.3 (Evaluation) Let X be a structure. We define a partial function $[\![-]\!] : Term \times (\mathbf{N} \to_p X) \to_p X$

– For a variable $v_i \in V = Term_0$ and a partial function $\rho : \mathbf{N} \to_p X$,

$[\![v_i]\!]_\rho$ is defined iff so is $\rho(i)$, and $[\![v_i]\!]_\rho = \rho(i)$.

– For a term $f_i\langle t_1, t_2, ...\rangle \in Term_{\alpha+1} - Term_\alpha$ and a partial function $\rho : \mathbf{N} \to_p X$,

$[\![f_i\langle t_1, t_2, ...\rangle]\!]_\rho$ is defined iff $f_i([\![t_1]\!]_\rho, [\![t_2]\!]_\rho, ...)$ is defined

and $[\![f_i\langle t_1, t_2, ...\rangle]\!]_\rho = f_i([\![t_1]\!]_\rho, [\![t_2]\!]_\rho, ...)$

Definition 5.4 (Standard representation of terms) We define a representation r_T of $Term$ with representations r_α of $Term_\alpha$ for any ordinals α.

Let $j_i : \mathbf{N} \to \mathbf{N}$ be a function such that $j_i(n) = 1 + \langle i, n\rangle$. Note that $\bigcup_{i \in \mathbf{N}} j_i(\mathbf{N}) = \{2, 3, 4, ...\}$ and all the summands are disjoint to each other.

The function r_0 of $V = Term_0$ is defined as follows.

For $x : \mathbf{N}^\omega$, $x \in dom(r_0)$ iff $x(1) = 1$

For $x \in dom(r_0)$, $r_0(x) = v_{x(2)}$

The representation $r_{\alpha+1}$ of $Term_{\alpha+1}$ is defined as follows.

For $x : \mathbf{N}^\omega$, $x \in dom(r_0)$ iff either

$x \in dom(r_\alpha)$

or,

$x(1) \neq 1$,

the signature has the function $f_{x(1)-1}$

and $x \circ j_i \in dom(r_\alpha)$ for each $i < arity(f_{x(1)-1})$

For $x \in dom(r_{\alpha+1})$,

if $x \in dom(r_\alpha)$ then $r_{\alpha+1}(x) = r_\alpha(x)$,

otherwise $r_{\alpha+1}(x) = f_{x(1)-1}\langle r_\alpha(x \circ j_1), r_\alpha(x \circ j_2), r_\alpha(x \circ j_3), ...\rangle$

The representation r_α of $Term_\alpha$ for a limit ordinal α is defined as follows.

$dom(r_\alpha) = \bigcup_{\beta < \alpha} dom(r_\beta)$

For $a \in dom(r_\alpha)$,

if $x \in dom(r_0)$ then $r_\alpha(x) = r_0(x) = v_{x(1)}$,

and if $x \in dom(r_{\beta+1}) - dom(r_\beta)$ then $r_\alpha(x) = r_{\beta+1}(x)$

This r_α is saturate as α is uncountable. We define r_T as

$r_T(x) = r_\alpha(x)$ for the least uncountable ordinal α.

Remark 5.5 It is obvious that this $r_\alpha : \mathbf{N}^\omega \to_p Term_\alpha$ is surjective. Thus, r_T is a surjection into $Term$.

Definition 5.6 (Computability Structure) Let X be an algebraic structure. A subset $S \subset \mathbf{N} \to X$ is a *computability structure* over X if S satisfies the followings:
1. (Permutation) For each $s \in S$ and each total recursive function $f : \mathbf{N} \to \mathbf{N}$, it holds that $s \circ f \in S$.
2. (Merging) For any $s, s' \in S$, there is $s'' \in S$ such that $s''(2n) = s(n)$ and $s''(2n+1) = s'(n)$.
3. (Effective sequence of terms) For each $s \in S$ and each recursive function $f : \mathbf{N} \to (\mathbf{N} \to_p \mathbf{N})$, if $f(m) \in dom(r_T)$ and $[\![r_T(f(m))]\!]_{(i \mapsto s(\langle n,i \rangle))}$ is defined for each $m, n \in \mathbf{N}$, then there is $s' \in S$ such that $s'(\langle m, n\rangle) = [\![r_T(f(m))]\!]_{(i \mapsto s(\langle n,i \rangle))}$

Example 5.7 Regard $\mathbf{N}^\omega$ as an algebraic structure with uniform topology $(\mathbf{N}^\omega, LM_{\mathbf{N},\mathbf{N}})$, as in Example 2.19. Put $S_{\mathbf{N}^\omega}$ as

$$\{f : \mathbf{N} \to \mathbf{N}^\omega \mid f(i)(j) = g(i,j),\ g \text{ is a binary total recursive function}\}.$$

Then, this $S_{\mathbf{N}^\omega}$ is a computability structure over $\mathbf{N}^\omega$. This $S_{\mathbf{N}^\omega}$ is the standard computability structure of it.

Definition 5.8 (Computable function) Let X and Y be algebraic structure with effective uniform topology, and LM^X and LM^Y be the effective limits of them, respectively. Let S_X and S_Y be computability structures of X and Y, respectively.

Then, a partial function $f : X \to Y$ is *computable* if there exists a total recursive function $m : \mathbf{N}^2 \to \mathbf{N}$ and the following five conditions hold:

1. There is $a \in S_X$ such that $s(\mathbf{N})$ is dense in $dom(f)$.
2. For each $s \in S_X$, there is a partial recursive function $i : \mathbf{N}^2 \to_p \mathbf{N}$ such that for each $n \in \mathbf{N}$, if $s(n) \in dom(f)$, then $LM^X(a(i(n,1)), a(i(n,2)), a(i(n,3)), ...) = s(n)$.
3. $f \circ a \in S_Y$
4. For every $(i_1, i_2, i_3, ...) \in \mathbf{N}^\omega$, if $(a(i_1), a(i_2), a(i_3), ...) = dom(LM^X)$, then there is a sequence $(j_1, j_2, j_3, ...) \in \mathbf{N}^\omega$ such that $m(i_{j_n}, n) \leq j_n$ for any $n \in \mathbf{N}$.
5. if $LM^X(a(i_1), a(i_2), a(i_3), ...) = x$, and $m(i_{j_n}, n) \leq j_n$ for any $n \in \mathbf{N}$, then $LM^Y(f(a(i_{j_1})), f(a(i_{j_2})), f(a(i_{j_3})), ...) = f(x)$.

Proposition 5.9 *Let $f : X \to_p Y$ and $g : Y \to_p Z$ be computable functions with respect to (S_X, S_Y) and (S_Y, S_Z), respectively. If $range(f) \subset dom(g)$, then $g \circ f$ is computable with respect to (S_X, S_Z).*

Notation 5.10 We abbreviate the following condition over a quadruple (X, LM, r, S) as the condition $(*)$:
– The first component X is an algebraic structures with effective uniform topology.
– The second LM is the effective limits of X.
– The third r is a representation of X such that all the partial functions of the signature of X are recursive with respect to r. Thus, S_r is a computable structures over X.
– As for the forth component, $S = S_r$.

Lemma 5.11 *Let (X, LM, r, S_r) be a quadruple which satisfies the condition $(*)$. Then, for each function $f : \mathbf{N}^\omega \to X$, if f is computable with respect to $(S_{\mathbf{N}^\omega}, S_r)$, then f is reducible to r.*

Remark 5.12 This proof follows the steps similar to those in the proof of Lemma 4.12.

Corollary 5.13 *A partial function $f : \mathbf{N}^\omega \to_p \mathbf{N}^\omega$ is recursive if it is computable with respect to $S_{\mathbf{N}^\omega}$.*

Lemma 5.14 *Let f be a partial function of $\mathbf{N}^\omega \to_p \mathbf{N}^\omega$. Suppose that f is recursive and there is $a \in S_{\mathbf{N}^\omega}$ such that $a(\mathbf{N})$ is dense in dom(f). Then, f is computable with respect to $S_{\mathbf{N}^\omega}$.*

Corollary 5.15 *Let f be a partial function of $\mathbf{N}^\omega \to_p \mathbf{N}^\omega$. Suppose that there is $a \in S_{\mathbf{N}^\omega}$ such that $a(\mathbf{N})$ is dense in dom(f). Then, f is recursive iff it is computable with respect to $S_{\mathbf{N}^\omega}$.*

Lemma 5.16 *Let (X, LM^X, r, S_X) be a quadruple which satisfies the condition $(*)$. Let Y be an algebraic structure with effective uniform topology, and LM^Y be the effective limit of it. Let S_Y be a computability structure over Y. Suppose that r is computable with respect to $(S_{\mathbf{N}^\omega}, S_X)$.*

Then, for any $f : X \to_p Y$, if $f \circ r$ is computable with respect to $(S_{\mathbf{N}^\omega}, S_Y)$, then f is computable with respect to (S_X, S_Y).

Theorem 5.17 (Main Theorem) *Let (X, LM^X, r_X, S_X) and (Y, LM^Y, r_Y, S_Y) be quadruples which satisfy the condition $(*)$. Suppose that r_X and r_Y are computable with respect to $(S_{\mathbf{N}^\omega}, S_X)$ and $(S_{\mathbf{N}^\omega}, S_Y)$, respectively.*

Let $f : X \to_p Y$ and $F : \mathbf{N}^\omega \to_p \mathbf{N}^\omega$ be partial functions which satisfy the following:

1. *$f \circ r_X = r_Y \circ F$*
2. *range$(F) \subset$ dom(R_Y)*
3. *There is $a \in S_{\mathbf{N}^\omega}$ such that $a(\mathbf{N})$ is dense in dom(F).*

Then, F is recursive iff f is computable with respect to (S_X, S_Y).

6 Computability in Yasugi's sense

Remark 6.1 There is another definition of computable functions, which is given by Yasugi et al. [Y]. Their definition is defined by using uniform neighbourhood system. We call the computability defined by their definition the computability in Yasugi's sense, or Y-computability.

Definition 6.2 (Computable function in Yasugi's sense) Let $X = (|X|, \{V_i^X(x)\}_{i,x}, S_X)$ be a triples which consists of an underlying space X, an effective uniform neighbourhood system $\{V_i^X(x)\}$, and a computability structure S_X. As is usual, we identify $|X|$ to X.

A partial function $f : X \to_p \mathbf{R}$ is *computable in Yasugi's sense*, or *Y-computable*, iff the following hold:
1. For $s \in S_X$, if $s(\mathbf{N}) \in dom(f)$, then $f \circ s \in S_{\mathbf{R}}$.
2. For any $s \in S_X$, there is a total recursive function $j_s : \mathbf{N} \to \mathbf{N}^\omega$ such that $f(V^X_{j_s(i,n)}(x_i)) \subset V_n^{\mathbf{R}}(f(x_i))$ for any i and n.
3. There are $e \in S_X$ and a total recursive function $j_e : \mathbf{N} \to \mathbf{N}^\omega$ such that
   (3.1) The range $e(\mathbf{N})$ is dense in X.
   (3.2) $f(V^X_{j_e(i,n)}(x_i)) \subset V_n^{\mathbf{R}}(f(x_i))$ for any i and n.
   (3.3) $\bigcup_{i \in \mathbf{N}} V^X_{j_e(i,n)}(x_i) = X$.

Lemma 6.3 *Let X be an algebraic structure with effective uniform topology, and LM be the effective limit of it. Put $\{V_i(x)\}$ be the effective uniform neighbourhood system defined as Lemma 2.14. Let S_X be a computable structure over X.*

If a function $f : X \to \mathbf{R}$ is computable, then it is Y-computable.

Lemma 6.4 *Let X be an algebraic structure with effective uniform topology, and LM be the effective limit of it. Put $\{V_i(x)\}$ be the effective uniform neighbourhood system defined as Lemma 2.14. Let S_X be a computable structure over X. Suppose the following set is recursively enumerable for each $s \in S_X$:*

$$\{(i, j, n, n') \in \mathbf{N}^4 \mid V_j(s(n')) \subset V_i(s(n))\}$$

If a function $f : X \to \mathbf{R}$ is Y-computable, then it is computable.

Corollary 6.5 *Under the same assumption as the previous lemma (6.4), a function $f : X \to \mathbf{R}$ is computable iff it is Y-computable.*

Remark 6.6 The algebraic structure $\mathbf{R}$ satisfies the assumption of the previous lemma (6.4). Therefore, for each function of $\mathbf{R} \to \mathbf{R}$, it is computable iff it is Y-computable.

Remark 6.7 This condition appears in the definition of computability by Yasugi et al.:

$$\bigcup_{i \in \mathbf{N}} V^X_{j_e(i,n)}(x_i) = X.$$

The function j_e corresponds to the modulus m in our definition.

In their definition, the condition asserts only the existence of i such that $x \in V^X j_e(i,n)(x_i)$ for x, but not the effectivity of such i. Our definition asserts the effectivity of such i, because our definition asserts the numerical inequality, which is recursively justified.

7 Conclusional remark

The essential definition of this work is Definition 5.8, which defines computable functions over algebraic structures with effective uniform topology. The main theorem (Theorem 5.17) says that the computability in our definition is equivalent to that in the definition by Weihrauch and others, with some suitable assumption. And also, Corollary 6.5 says that our computability is equivalent to that by Yasugi and others, with some suitable assumption.

The author think that this definition 5.8 has some conditions on the domain of the function, which seems inessential. The author would like to make this condition more natural as a future work.

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