Towards a Convenient Category of Topological Domains

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Abstract

We propose a category of topological spaces that promises to be convenient for the purposes of domain theory as a mathematical theory for modelling computation. Our notion of convenience presupposes the usual properties of domain theory, e.g. modelling the basic type constructors, fixed points, recursive types, etc. In addition, we seek to model parametric polymorphism, and also to provide a flexible toolkit for modelling computational effects as free algebras for algebraic theories. Our convenient category is obtained as an application of recent work on the remarkable closure conditions of the category of quotients of countably-based topological spaces. Its convenience is a consequence of a connection with realizability models.

1 Introduction

The title of this note deliberately echoes that of Steenrod's well-known paper: *A Convenient Category of Topological Spaces*, [46]. In his paper, Steenrod sets out to find a full subcategory of the category **Top** of all topological spaces that is "convenient" for the purposes of algebraic topology. One aspect of "convenience" that is particularly highlighted by Steenrod is cartesian closure, a property famously not enjoyed by **Top** itself. He argues at length that the category of compactly-generated Hausdorff spaces, which *is* cartesian closed, does provide all the "conveniences" needed for doing algebraic topology.

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Domain theory was originally developed by Dana Scott to meet the need for a mathematical theory capable of modelling a diversity of computational and programming language phenomena, see [15] for an overview. The goal of this note is to propose a category that is convenient for this purpose, where, by "convenience", we mean that the category should fulfill the general aim as amply as possible.

Such a notion of "convenience" is deliberately vague, and thus open to many interpretations. In Section 2, we formulate several specific demands on a "convenient" category, each elaborating an aspect of the general idea of "convenience". In the remainder of the note, we then develop a notion of domain that promises to meet all the requirements. The domains we settle on are certain topological spaces. Thus, once the programme of research outlined in this note has been completed, we expect to end up with a "convenient" category of "topological domains".

This note presents an early overview of ongoing research. No proofs are given. However, we distinguish clearly between results that have already been established and "conjectures" representing future work.

2 Requirements on a convenient category

In this section we expand upon the notion of "convenience" by placing five explicit requirements on a convenient category. We do not argue that the requirements we identify are the only possible ones, nor that they are essential for a category of domains to qualify as "convenient". Indeed, traditional domain theory has progressed a very long way without meeting all our requirements. The goal of this paper is rather to extend the achievements of traditional domain theory in new directions.

Our first requirement is based on the close connection between topology and computation, as developed, in particular, by Smyth, see [44] for a summary. A domain represents a "datatype" and must therefore have an underlying set of associated "data items". To each domain, it makes sense to associate the set of all "observable" (an abstraction of "semidecidable") properties over data items in the datatype. One can argue that the "logic of observable properties" should be closed under finite conjunctions and arbitrary disjunctions, i.e. that the extensions of the observable properties should form a topology on the underlying set, see [44, 48]. Thus a domain should, at least, be a topological space. Moreover, by allowing observations to make use of maps between domains, one can derive that every map between domains must be continuous. Furthermore, the notion of continuous function is itself an elegant mathematical abstraction of the notion of computable function, under which the finitary aspects of computability are modelled without recourse to any notion of algorithm. It is thus natural to impose the following requirement.

Requirement 1 (Topological category) The category of domains should be a full subcategory of the category **Top** of topological spaces and continuous functions.

The argument for Requirement 1 has been conceptual. However, an important benefit of the requirement is that domains lie in the realm of mainstream mathematical structures. For one thing, this means that we have a multitude of mathematical tools for manipulating domains. More importantly, the topological structure is indispensible for viewing familiar mathematical objects, such as the real numbers, metric spaces, etc., as embedded within domains. Such embeddings are essential for domains to be used to model computation over many forms of nondiscrete data, as in, e.g., [10, 9].

As a second requirement, we demand that the category of domains support all the standard constructions on domains, including the ability to model recursive definitions of both data and datatypes.

Requirement 2 (Basic structure) The category of domains should model (at least) the usual type constructors: cartesian product, $\times$; function space, $\to$; smash product, $\otimes$; strict function space, $\to_\perp$; coalesced sum, $\oplus$; and lifting, $(\cdot)_\perp$; see [1]. These should have the correct universal properties with respect to the categories of "strict" and "non-strict" maps between domains. The category should also model recursion and recursive types.

Requirement 2 is alone sufficient for modelling many forms of deterministic computation. Nevertheless, even in the realm of deterministic functional computation, recursive types are not sufficiently powerful for all applications. There are many additional forms of "type constructor" that one might also wish to model. In this note, we consider just one such feature, which is particularly important due to its power and its relationship with programming practice.

Requirement 3 (Parametric Polymorphism) The category of domains should model full second-order (parametric) polymorphism.

Here, it should, at least, be possible to incorporate parametricity using *relational parametricity* in the sense of Reynolds, [38, 29]. However, in the view of the author, it is not settled that relational parametricity is the final story in giving a mathematical account of parametricity. It is possible that alternative accounts of parametric polymorphism may yet emerge.

It is also vital that the category of domains should model a rich variety of programming language features and computational behaviours, going beyond deterministic functional computation. The non-functional aspects of real-world computation can often be encapsulated as *computational effects*. As Plotkin and Power have argued, many such effects are modelled by free algebras for algebraic theories, see [36]. These include the familiar powerdomains used for modelling forms of nondeterminism, as well as many other *computational monads* [31]. Accordingly, we make the following requirement, whose importance was first recognised by Plotkin.

Requirement 4 (Computational effects) The category of domains should provide free algebras for a wide class of algebraic theories.

At this stage, we leave open the extent of the class of theories considered, but it should include, at the very least, all finitary equational algebraic theories.

Finally, we bring in an explicit connection with computability. It might be possible to satisfy the above requirements using topological spaces of large cardinality which have no possible computational significance. One would like a model in which all spaces have potential computational significance. One strong way of ensuring this is:

Requirement 5 (Effectivity) The notion of domain should have an effective counterpart, giving rise to a natural category of computable maps between effective domains, satisfying Requirements 2–4 above.

This requirement also has the direct benefit of establishing notions of computability that apply to all the constructions implicit in Requirements 2–4.

There are several conflicts between the above requirements and traditional domain theory, which we take to be the study of full subcategories of the category of directed-complete partial orders (dcpos) and continuous functions. By definition, all traditional categories of domains do satisfy Requirement 1. As we shall see, it is possible for them to satisfy many of the other requirements independently, but not all requirements in combination.

In order to satisfy Requirement 5, it seems necessary to restrict to full subcategories of ω-*continuous* dcpos, on which notions of computability can be defined using enumerations of the countable bases of such dcpos. The category of ω-continuous dcpos is not cartesian closed; so combining Requirements 5 and 2 requires, at least, finding cartesian-closed full subcategories. Such subcategories have been classified by Jung, see [1] for a survey. There are many, and it is indeed possible to satisfy Requirements 2 and 5 in combination.

Requirement 3 is more problematic. Indeed, to the best of my knowledge, no existing category of domains has been exhibited as a model of *parametric* polymorphism. There do exist models of non-parametric polymorphism, based on dependent product operations on domains, see e.g. [8]. However, Jung has shown that there is an essential difficulity in finding domain-theoretic models of polymorphism that are closed under the convex powerdomain [26]. Thus there is a problem in combining Requirement 4 with (even a non-parametric version of) Requirement 3.

Requirement 4 is also in conflict with Requirements 2 and 5, as the following example, due to Plotkin (private communication), shows. In the category of all ω-continuous dcpos, free algebras are available for any finitary equational theory. However, no nontrivial cartesian-closed full subcategory of ω-continuous dcpos is closed under the formation of free commutative monoids. Thus, in traditional domain theory, it is impossible to simultaneously satisfy Requirements 2, 4 and 5. This difficulty led Plotkin to first pose the problem of finding a category satisfying Requirements 2, 4 and 5 in combination. His recent work with Power on computational effects [36] has highlighted the computational importance of this problem.

The probabilistic powerdomain [40, 24] gives another example of a possible conflict between Requirements 2, 4 and 5. Although the category of all ω-continuous dcpos is closed under the probabilistic powerdomain [23], Jung and Tix have cast doubt on whether any cartesian-closed full subcategory remains closed under it [27]. In this case, there is no definitive negative result. Nonetheless, the formidable technical difficulties in the way of combining ω-continuous dcpos, probabilistic powerdomain and function space raise the question of whether traditional domain theory provides the right setting for combining probabilistic computation with Requirements 2 and 5.

In contrast, it does seem possible to satisfy all of Requirements 2–5, using notions of domain that arise in *realizability models* [28]. The type constructors and recursive types are worked out in [28, 37]. The interpretation of parametric polymorphism is being worked out in detail by Birkedal and Rosolini [7]. An indication of how to address free algebras appeared in [33], although the details there are for the special case of powerdomains only. Furthermore, many realizability models have an intrinsic notion of computability built into them, rendering Requirement 5 superfluous.

One major drawback of realizability models, however, is that the intrinsic mathematical structure of the objects modelling types is intangible. Objects are given as partial equivalence relations, possibly satisfying additional properties, over a partial combinatory algebra. In general, isomorphic objects may have very different partial equivalence relations underlying them. It is not at all straightforward to extract intrinsic properties of objects from their external presentations as partial equivalence relations.

What we would like instead is a notion of domain given by an explicit definition of the intrinsic mathematical structure involved. Requirement 1, whose many other benefits we have already discussed, is a strong constraint in this direction. It is this requirement that we take as our starting point for deriving our convenient category of domains.

3 Quotients of countably-based spaces

Requirement 1 demands that our category of domains be a full subcategory of **Top**. At the same time, Requirement 5 demands that an effective version of the category be available. Thus our domains must be topological spaces to which it is possible to associate a notion of effectivity. Such a notion of effectivity is known to be available for all countably-based (a.k.a. second-countable) topological spaces. For example, the topology of any such space A can be "presented" using a topological pre-embedding $\pi_A \colon A \longrightarrow \mathcal{P}\omega$,[1] where $\mathcal{P}\omega$ is the powerset of ω with the Scott topology. Given two such presentations $\pi_A \colon A \longrightarrow \mathcal{P}\omega$ and $\pi_B \colon B \longrightarrow \mathcal{P}\omega$, a continuous function $f \colon A \longrightarrow B$ is said to be *effective* if there is some computable $r \colon \mathcal{P}\omega \longrightarrow \mathcal{P}\omega$ (in the standard sense, see [43]) such that $r \circ \pi_A = \pi_B \circ f$.[2] For further discussion of related notions of effectivity, see e.g. [44, §5.1].

Restricting to countably-based spaces is unnecessarily constraining. It turns out to be possible to associate a notion of effectivity with a more general class of space. The spaces we consider are arbitrary quotient spaces of countably-based spaces, i.e. spaces X for which there exists a topological quotient $q \colon A \twoheadrightarrow X$, where A is countably based. We call such spaces *qcb spaces*, and we write **QCB** for the full subcategory of **Top** consisting of such spaces. Quite unexpectedly, **QCB** has very good categorical structure.

[1] Being a *pre-embedding* means that, for every open $U \subseteq A$, there exists an open $V \subseteq B$ such that $U = \pi_A^{-1}(V)$. A pre-embedding is a topological embedding if and only if it is an injective function.

[2] In general, r only determines f when π_B is a topological embedding, i.e. when B is T_0. A reader who prefers effective maps to be determined by their computational component may adapt the discussion throughout the paper by assuming all spaces to be T_0.

Theorem 3.1 *The category* **QCB** *has all countable limits and colimits and is cartesian closed.*

For proofs of the theorem see [30, 41]. N.b., the countable colimits are inherited from **Top**, but limits are *not* inherited; both finite products and equalizers differ in **QCB** and **Top**.

Theorem 3.1 is a major reason for considering the category of all quotients of countably-based spaces rather than simply restricting to countably-based spaces themselves. The category of countably-based spaces does not have coequalizers, and, more importantly, it is not cartesian closed.

In the remainder of this section we give two other characterizations of qcb spaces. The first shows explicitly how qcb spaces can be provided with an associated notion of effectivity. The second gives a more intrinsic characterization of qcb spaces.

To give an account of effectivity, we introduce the following definitions, which are motivated by the straightforward Proposition 3.4 below.

Definition 3.2 An *ω-representation* of a topological space X is given by a topological quotient $q\colon A \twoheadrightarrow X$, where A is a countably-based space.

Definition 3.3 An ω-representation $q\colon A \twoheadrightarrow X$ is said to be *ω-projecting* if, for every countably-based B and continuous $f\colon B \longrightarrow X$, there exists a continuous $g\colon B \longrightarrow A$ such that $q \circ g = f$.

Proposition 3.4 *Suppose that $q\colon A \twoheadrightarrow X$ and $r\colon B \twoheadrightarrow Y$ are ω-projecting ω-representations. Then, for any continuous $f\colon X \longrightarrow Y$, there exists a continuous $g\colon A \longrightarrow B$ making the square below commute.*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, two heads, "q"'] & B \arrow[d, two heads, "r"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Also, given arbitrary functions f, g making the square commute, if g is continuous then so is f.

Thus ω-projecting ω-representations determine the topological structure of the represented spaces to the extent that continuous maps between X and

Y are completely determined by continuous maps between the representing countably-based spaces A and B.

The above definitions relate to the desire of having an associated notion of effectivity as follows. The notion of an effective map between X and Y can be derived from the notion of effective map between A and B as countably-based spaces. Specifically, we stipulate that a continuous function $f: X \longrightarrow Y$ is *effective* if there exists an effective continuous $g: A \longrightarrow B$ making the diagram commute. The notion of effectivity thus depends upon presentations of A and B, and also upon q and r, but such dependency is unavoidable. Effectivity is always associated with the presentation of mathematical structure, rather than directly with the structure itself.

We have shown that a notion of effective map is available between those spaces for which there exists an ω-projecting ω-representation. Such spaces were first introduced in [30], where, using a result due to Schröder, it is proved that they coincide with qcb spaces.

Another result of Schröder's gives a more intrinsic characterization of qcb spaces. Recall that the relation of sequence convergence on a topological space X is defined by $(x_i) \to x$ if, for every open $U \ni x$, the sequence (x_i) is eventually in U (i.e. there exists n such that, for all $i \geq n$, $x_i \in U$). A subset $W \subseteq X$ is said to be *sequentially open* if $(x_i) \to x \in W$ implies that (x_i) is eventually in W. Trivially, every open subset is sequentially open. We say that X is *sequential* if every sequentially open subset is open. A *sequential pseudobase* of a topological space X is a family $\mathcal{B}$ of subsets of X satisfying: for every open $U \subseteq X$ and convergent $(x_i) \to x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$ and (x_i) is eventually in B. This notion is due to Schröder [42]. Note that a family of open sets is a sequential pseudobase if and only if it is a base for the topology.

We can now summarise the main characterizations of qcb spaces.

Theorem 3.5 *The following are equivalent for a topological space X.*

1. *X is a qcb space.*

2. *X has an ω-projecting ω-representation.*

3. *X is sequential and has a countable sequential pseudobase.*

The implication 2 implies 1 is trivial. Lawson has given a direct proof that 1 implies 3 [11]. Bauer gives the construction of an ω-projecting ω-representation from a countable sequential pseudobase for a sequential space [3]. Nevertheless, all the main ingredients of the theorem are, in a slightly different context, due to Schröder, see [42, Theorem 13] and and [41, Theorem 3.2.4]. The latter result provides another interesting characterization of qcb spaces using a generalization of Weihrauch's notion of Baire space representation [49]. This leads to an alternative (but presumably equivalent) account of effectivity in terms of Type-2 Turing Machines.

There are many ways of understanding the cartesian-closed structure of **QCB**. As in [30], the structure of **QCB** is preserved by an embedding $\mathbf{QCB} \hookrightarrow \mathbf{Equ}$, where **Equ** is Scott's category of *equilogical spaces* [4].[3] This embedding allows one to understand limits in **QCB** in terms of limits in **Equ**, which are defined using limits in **Top**, but with the caveat that spaces are considered modulo an equivalence relation. Alternatively, the cartesian-closed structure of **QCB** can be understood via structure-preserving embeddings into many cartesian-closed coreflective hulls of **Top**. For example, there are such embeddings into the categories of sequential spaces [30], of (not necessarily Hausdorff) compactly-generated spaces, and of core-compactly-generated spaces [11].

The existence of so many structure-preserving embeddings persuades the author that **QCB** is an "inevitable" category, arising as a subcategory of any of the main approaches to reconciling topological continuity and cartesian closure.[4] People differ in whether they prefer to consider cartesian-closed subcategories of **Top**, but lose topological limits; or to consider cartesian-closed supercategories of **Top**, retaining topological limits, but going outside the familiar world of topology. In the author's view, these two alternatives are not in conflict. In either case, **QCB** lives as a full subcategory via a structure-preserving embedding. Moreover, all reasonable spaces lie inside **QCB**. Furthermore, the existence of many categories embedding **QCB** provides a range of alternative tools for understanding constructions in **QCB** (e.g. one can use the sequential function space, the compactly-generated function space, the function space in **Equ**, etc.).

On the other hand, in spite of so many available tools, aspects of the cartesian closure of **QCB** remain hard to understand. For example, consider the "type hierarchies" over $\mathbb{N}$ (with the discrete topology) and $\mathbb{R}$ (with the Euclidean topology) given by $\mathbb{N}, \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}, \ldots$ and $\mathbb{R}, \mathbb{R}^{\mathbb{R}}, \mathbb{R}^{\mathbb{R}^{\mathbb{R}}}, \ldots$. Here, $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ and $\mathbb{R}^{\mathbb{R}^{\mathbb{R}}}$ are examples of non-countably-based spaces that are nonetheless qcb spaces. It is easily shown that these spaces are Hausdorff. Also, $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is totally disconnected. As the following open questions illustrate, other basic

[3]Because we are not requiring qcb spaces to be T_0, the T_0 condition in the definition of equilogical space must be omitted.

[4]It remains to be checked that there is a structure-preserving embedding of **QCB** into Hyland's category of "filter spaces" [18]. The author strongly expects this to be the case.

properties of their topologies remain, however, tantalisingly elusive.

Question 3.6 Are the spaces $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ and $\mathbb{R}^{\mathbb{R}^{\mathbb{R}}}$ regular?

Question 3.7 Is $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ zero dimensional (i.e. does it have a base of clopen subsets)?

Question 3.7 was posed in [5], where an application of a positive answer to the question is given.

4 Topological (pre)domains

Quotients of countably-based spaces form, apparently, the largest class of topological spaces to which it is possible to associate a notion of effectivity. The category **QCB** thus addresses Requirements 1 and 5. Moreover, Theorem 3.1 shows that the category has surprisingly rich categorical structure. Nonetheless, it does not satisfy Requirements 2–4. In this section, we address Requirement 2, by cutting down to a full subcategory of **QCB**.

It is easiest to address Requirement 2 by identifying, in the first place, a category of *predomains* within **QCB**. In traditional domain theory, predomains are distinguished from domains by not being required to have least element in the partial order. (Thus predomains are dcpos, and domains are *pointed* dcpos.) This relaxation allows, for example, the category of predomains to have countable coproducts. Although endomorphisms on predomains need not have fixed points, the familiar cartesian-closed category of domains, which does have a fixed-point operator, is recovered simply as the full sucategory of those predomains that do have least element. The fixed-point operator exists because partial orders are required to be directed complete and because continuous functions (with respect to the Scott topology) preserve directed suprema. In fact, as is well known, the weaker properties of ω-completeness and ω-continuity suffice.

In traditional domain theory, the topology (the Scott topology) is derived from the partial order. To define our notion of predomain, we also work with order-theoretic properties, but we take the topology as primary and the order as derived. Recall that the *specialization order* $\sqsubseteq$ on a topological space X is defined by $x \sqsubseteq y$ if, for all open $U \subseteq X$, $x \in U$ implies $y \in U$. In general $\sqsubseteq$ is a preorder on X. The space X is said to be T_0 if $\sqsubseteq$ is a partial order. We can now give a definition of topological predomain.

Definition 4.1 (Topological predomain) A *topological predomain* is a topological space X satisfying the following properties:

1. X is a qcb space;
2. $\sqsubseteq$ is an ω-complete partial order (in particular, X is T_0); and
3. every open $U \subseteq X$ is inaccessible by ω-lubs (i.e., for any ascending sequence $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$, if $\bigsqcup_i x_i \in U$ then $x_i \in U$ for some i).

Apart from being phrased with respect to ω-lubs rather than directed lubs, this definition recalls the notion of *monotone convergence space* [14].

Definition 4.2 A topological space X is a *monotone convergence space* if: the specialization order on X is a dcpo (in particular, X is T_0), and every open subset of X is Scott-open with respect to the order.

Monotone convergence spaces include: all T_1 spaces, all sober spaces, and all dcpos with the Scott topology.

Proposition 4.3 *A qcb space is a topological predomain if and only if it is a monotone convergence space.*

Thus, because of the restriction to qcb spaces, it makes no difference whether topological predomains are defined using ω-lubs or using directed lubs.

Next, we give a useful category-theoretic characterization of topological predomains. We write ω for the set of natural numbers under the Alexandroff topology on the their usual linear ordering. We write $\overline{\omega}$ for $\omega \cup \{\infty\}$, where $\infty = \bigsqcup_i i$, with the Scott topology.

Proposition 4.4 *In the category* **QCB** *the following are equivalent.*

1. *X is a topological predomain.*
2. *For all qcb spaces Z and maps $f\colon Z \times \omega \longrightarrow X$, there exists a unique $g\colon Z \times \overline{\omega} \longrightarrow X$ such that the diagram below commutes:*

$$\begin{tikzcd}
Z \times \overline{\omega} \arrow[r, "g"] & X \\
Z \times \omega \arrow[u, hook] \arrow[ur, "f"'] &
\end{tikzcd}$$

We write **TP** for the category of topological predomains and continuous functions.

Corollary 4.5 **TP** *is an exponential ideal of* **QCB**.

Explicitly, this means that: (i) for topological predomains X and Y, the product $X \times Y$ in **QCB** is a topological predomain; and (ii) for any qcb space X and topological predomain Y, the function space Y^X in **QCB** is a topological predomain. Thus, in particular, **TP** is cartesian closed and the embedding $\mathbf{TP} \hookrightarrow \mathbf{QCB}$ preserves the cartesian-closed structure. It is also easy to see, directly from the definition of topological predomain, that **TP** is closed under countable coproducts in **QCB** (and hence in **Top**). The existence of more intricate (co)limits is a consequence of the result below.

Proposition 4.6 (Schröder) *The category of monotone convergence spaces is a full reflective subcategory of* **Top**, *and the reflection functor cuts down to a refection* $R\colon \mathbf{QCB} \to \mathbf{TP}$.

Corollary 4.7 **TP** *has all countable limits and colimits.*

Limits in **TP** are inherited from **QCB**, but colimits are constructed using the reflection. So neither limits nor colimits in **TP** are inherited, in general, from **Top**. (This is a fact of life, not a problem.)

We now briefly discuss how Requirement 2 has been met. First, another definition is needed.

Definition 4.8 (Topological domain) A *topological domain* is a topological predomain for which $\sqsubseteq$ has a least element.

The various type constructors listed in Requirement 2 can all be defined on topological domains entirely in the expected way. Their universal properties are also as expected. The important categories here are: **TD**, the category of continuous functions between topological domains; and $\mathbf{TD}_\perp$, its subcategory of strict maps (those preserving the least element). The category **TD** is cartesian closed (it is an exponential ideal of **TP**), with a least-fixed-point operator; the category $\mathbf{TD}_\perp$ is symmetric monoidal closed, with respect to $\otimes$ and $\to_\perp$, and has $\oplus$ as coproduct. Moreover, the inclusion of $\mathbf{TD}_\perp$ in **TD** has a left adjoint, given by $(\cdot)_\perp$, with the adjunction giving rise to a model of intuitionistic linear type theory. Furthermore, $\mathbf{TD}_\perp$ is also characterized as the Eilenberg-Moore category of the monad given by $(\cdot)_\perp$ on **TP**.[5] Finally, the standard technology for recursive domain equations, [45],

[5] As in ordinary domain theory, $\mathbf{TD}_\perp$ is also equivalent to the Kleisli category of the lifting monad.

applies directly to $\mathbf{TD}_\perp$, which can be shown to be algebraically compact in the sense of Freyd [12, 13].

We end this section by relating **TP** and **TD** to traditional categories of domains. Firstly, note that **TP** contains very many topological spaces that are not normally included in categories of predomains; for example: n-dimensional Euclidean space $\mathbb{R}^n$, and also $\mathbb{R}^\omega$; Baire space, $\mathbb{N}^\omega$; Cantor space 2^ω; etc. Also, by Corollary 4.5, the type hierarchies $\mathbb{N}, \mathbb{N}^\mathbb{N}, \mathbb{N}^{\mathbb{N}^\mathbb{N}}, \ldots$ and $\mathbb{R}, \mathbb{R}^\mathbb{R}, \mathbb{R}^{\mathbb{R}^\mathbb{R}}, \ldots$ are in **TP**.

Obviously, **TP** contains every dcpo whose Scott topology has a countable base. In particular, every ω-continuous dcpo is contained in **TP**. Actually, it can be shown that a continuous dcpo is contained in **TP** if and only if it is ω-continuous. Thus the extra generality of allowing non-countably-based spaces does not manifest itself with continuous dcpos.[6] As **TP** is cartesian closed, it should be interesting to examine function spaces over the notorious examples of ω-algebraic dcpos that are not contained within any of the cartesian-closed categories of ω-continuous cpos, see e.g. [1]. We have not yet performed the required calculations, but we strongly expect that such function spaces in **TP** go outside the category of dcpos, i.e. that they result in spaces whose topology is not the Scott topology.

On the other hand, well-behaved categories of dcpos do turn out to live faithfully inside **TD** (and so also inside **TP**). Let $\omega\mathbf{S}$ be the category of continuous functions between ω-continuous Scott domains (i.e. bounded-complete pointed ω-continuous dcpos).[7]

Proposition 4.9 *$\omega\mathbf{S}$ is an exponential ideal of* **QCB** *(hence also of* **TP**, *and of* **TD***).*

Conjecture 4.10 *$\omega\mathbf{S}$ is the largest full subcategory of pointed ω-continuous dcpos that forms an exponential ideal of* **TD**.

Being an exponential ideal is a very strong requirement. A weaker requirement is merely to ask for the existing cartesian-closed structure of a category of domains to be preserved. Jung has identified the largest cartesian-closed full subcategory, $\omega\mathbf{FS}$, of the category of pointed ω-continuous dcpos [25].

Conjecture 4.11 *The embedding $\omega\mathbf{FS} \hookrightarrow \mathbf{TD}$ preserves the cartesian-closed structure.*

[6] This is a special case of a more general result: a core-compact space is qcb if and only if it is countably based, see [11].

[7] One reason for requiring Scott domains to be pointed is that the category of all bounded-complete ω-continuous dcpos is not cartesian closed.

5 Topological predomains and realizability

Proposition 4.9 and, if true, Conjecture 4.11 show that, to a large extent, traditional domain theory lives faithfully inside **TD**. The benefit of the much richer world provided by **TD** (and also **TP**) is that Requirements 3 and 4 can also, apparently, be addressed. To appreciate this, we now give an alternative characterization of the category **TP** using the techniques of realizability semantics.

In this section, we assume the reader has some acquaintance with realizability models, as presented in [28]. We only consider models built over Scott's combinatory algebra $\mathcal{P}\omega$ [43]. We write $\mathbf{Asm}(\mathcal{P}\omega)$ for the associated category of *assemblies* and $\mathbf{Mod}(\mathcal{P}\omega)$ for its full subcategory of *modest sets*. Using the terminology of [28], consider the *divergence* $D = \{\emptyset\}$. As in [28, §4], the divergence determines a *lifting functor* L and *dominance* Σ. Using the lifting functor L, we identify the notion of a *complete* object of $\mathbf{Asm}(\mathcal{P}\omega)$, see [28, §5]. Using the dominance Σ, we define the notion of an *extensional* object of $\mathbf{Asm}(\mathcal{P}\omega)$, see [28, §10]. Let $\mathbf{CE}(\mathcal{P}\omega)$ be the full subcategory of complete extensional objects of $\mathbf{Asm}(\mathcal{P}\omega)$. As in [28], $\mathbf{CE}(\mathcal{P}\omega)$ is in fact a full subcategory of $\mathbf{Mod}(\mathcal{P}\omega)$. Moreover, for very general reasons, $\mathbf{CE}(\mathcal{P}\omega)$ is a well-behaved category of predomains in $\mathbf{Mod}(\mathcal{P}\omega)$; see, for example, [32] (where extensional objects are called *regular Σ-posets*).

In fact, $\mathbf{CE}(\mathcal{P}\omega)$ is much more that a well-behaved category of predomains. It is a complete internal full subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$ in the sense explained in [19, 21]. This implies many properties: $\mathbf{CE}(\mathcal{P}\omega)$ is a full reflective subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$; as an internal category, $\mathbf{CE}(\mathcal{P}\omega)$ is cocomplete; $\mathbf{CE}(\mathcal{P}\omega)$ is a model of the polymorphic λ-calculus; and, by [39, 7], this model can be refined to yield a model of parametric polymorphism. Finally, following the approach of Phoa and Taylor [33], it should be possible to use the internal completeness of $\mathbf{CE}(\mathcal{P}\omega)$ to construct free algebras for (at least finitary) algebraic theories. Although the details of this construction need further work, there seems to be no fundamental obstacle to it succeeding. In summary, the category $\mathbf{CE}(\mathcal{P}\omega)$ promises to satisfy Requirements 3 and 4.

At this point, the reader may be wondering what all this has to do with topological (pre)domains. The connection is provided by the result below.

Theorem 5.1 *The categories* **TP** *and* $\mathbf{CE}(\mathcal{P}\omega)$ *are equivalent.*

This theorem is a consequence of properties of the embedding of **QCB** into the category of countably-based equilogical spaces given in [30], using the fact that the latter category is equivalent to $\mathbf{Asm}(\mathcal{P}\omega)$ [5]. Several further ingredients are needed in the proof, including the characterization of **TP** given by Proposition 4.4.

6 Summary and further work

We have introduced a category **TP** of *topological predomains*, with a subcategory **TD** of *topological domains*, that together promise to meet our 5 requirements for "convenience". In the definition, Requirements 1 and 5 were explicitly taken account of by the use of qcb spaces, as in Section 3. Requirement 2 was established directly in terms of the the topological definition of the categories **TP** and **TD**. On the other hand, our arguments for Requirements 3 and 4 were indirect, making use of a connection with realizability models. (It should be repeated that the argument for Requirement 4 is not yet rigorous.)

As was emphasised in Section 2, one of the major benefits of working with a category of topological spaces is that one is working with (at least to some extent) familiar mathematical structures. We thus view it as very desirable to establish Requirements 3 and 4 in direct topological terms. It should not be necessary to take a detour through a realizability model in order to understand the interpretation of polymorphic types and computational effects.

In the case of Requirement 3, our results indicate the existence of a purely topological model of parametric polymorphism. In fact, by constructing the model over $\mathbf{TD}_\perp$, it should be possible to obtain a concrete model combining parametric polymorphism, intuitionistic linear type theory and fixed points. As has been argued by Plotkin [34, 35], such a combination of features is immensely powerful. The existence of realizability models of this setting was first outlined by Plotkin, and is being worked out in detail (and in much greater generality) by Birkedal and Rosolini [7]. A syntactic model, based on a term calculus quotiented by operational equivalence, has been presented by Bierman, Pitts and Russo [6]. However, to the best of my knowledge, the only existing model defined in terms of concrete mathematical structures is due to R. Hasegawa, using his bicategorical theory of *twiners* [16]. Our work demonstrates that it is also possible to obtain a purely topological model of linear parametricity and fixed points. It is to be hoped that a direct topological interpretation of parametric polymorphism in the model will prove forthcoming.

For Requirement 4, the presence of arbitrary countable colimits in **TD** suggests that free algebras should be available for algebraic theories that are, in a suitable sense, countably presented. For defining free algebras for computational effects, countable presentations should be of similar usefulness to the *countable enriched Lawvere theories* of Plotkin and Power [36], although the details will necessarily be different as **TD** is not locally countably presentable.[8] It is plausible that, in **TD**, it will also be possible to allow operations of uncountable arity, as long as the "arity" is given by a countably-based space, possibly even an arbitrary qcb space. Precise conditions on the algebraic theories should emerge when Requirement 4 is worked out in detail.

In addition to providing the general construction needed to verify Requirement 4, it will also be of interest to look at specific examples of computationally interesting effects; for example, probabilistic choice, which is traditionally modelled by the probabilistic powerdomain [40, 24]. As has already been reported, in the context of traditional domain theory, there are (possibly insurmountable) difficulties in combining the probabilistic powerdomain with any cartesian-closed category of continuous dcpos [27]. However, it is well established that the probabilistic powerdomain lives naturally in wider categories of topological spaces than dcpos, see e.g. [17, 2]. Our categories **TP** and **TD** will allow such a wider topological notion of probabilistic powerdomain to be combined with the usual domain-theoretic constructions (as summarized in Requirement 2) and also with polymorphism. At the time of writing, we have an explicit definition of a probabilistic powerdomain in **TP**. However, much remains to be done to relate it to the established definitions for ω-continuous dcpos (and wider classes of spaces). Also, we would like to characterize the probabilistic powerdomain as a free algebra, as in [23].

Another very interesting topic for future research is to combine Requirements 3 and 4 in the stronger sense of extending the notion of parametric polymorphism to incorporate parametricity for the operators associated with computational effects.

In this note, we have used effectivity as a motivating factor in the identification of qcb spaces and in the subsequent definition of topological (pre)domains. However, Requirement 5 has only been dealt with in a cursory manner. In particular, we have glossed over one important issue. The simple account of effectivity for **QCB** given in Section 3 is, in itself, insufficient as an account of effectivity in **TP**. The problem is that an effective version of Proposition 4.4 is required in order for domain-theoretic constructions, such as fixed points, to exist in the category of effective maps. In order to

[8] It is almost certainly impossible for any locally presentable category to satisfy either of Requirements 3 and 5.

implement this concretely, additional effective information is needed in the "presentation" of a topological predomain. Obtaining a palatable explicit description of this effective structure is one possible goal for future research. Moreover, it should also be established that all the constructions needed in fulfilling Requirements 2–4 behave well with respect to effective maps.

There is, however, an alternative more conceptual approach to effectivity in **TP**. This is, simply, to extract the notion of effectivity and its properties as a consequence of all constructions being formalizable in the internal logic of the realizability category $\mathbf{Asm}(\mathcal{P}\omega)$, or rather in its associated subcategory of effective maps. This approach should, of course, be equivalent to the external approach discussed above.

Ultimately, it will thus be beneficial to have both the topological account of **TD** and the alternative realizability account worked out in detail. Moreover, the realizability account suggests other perspectives. Theorem 5.1 gives one example of a situation in which the category of complete extensional objects in a realizability category turns out to have a very elegant concrete description (as the category **TP**). This is consistent with the established observation that, in many different realizability models, the complete extensional objects form the category of predomains with the most natural external decription, see the introduction to [32] for other examples. Nevertheless, several other categories of predomains are available in realizability models, see [28, 32] for overviews. It would be particularly interesting to obtain a concrete description of the category $\mathbf{Rep}(\mathcal{P}\omega)$ of *replete objects* in $\mathbf{Asm}(\mathcal{P}\omega)$, as defined by Hyland and Taylor [20, 47], which is nicely characterized as the smallest full reflective exponential ideal of $\mathbf{Asm}(\mathcal{P}\omega)$ containing the object Σ. Considered as a category of topological spaces, $\mathbf{Rep}(\mathcal{P}\omega)$ forms a category of predomains contained in **TP**. This containment is strict. For example, the space of natural numbers with the topology of cofinite subsets is a topological predomain that is not replete. This space is a well-known example of a T_1 space that is not sober. Indeed, the very definition of repleteness suggests a connection between repleteness and sobriety. In an attempt to make this connection rigorous, recall that the category of sober topological spaces is a full reflective subcategory of **Top**, see e.g. [22]. The following question, which should be compared with Proposition 4.6, seems nontrivial.

Question 6.1 Is the sober reflection of a qcb space also a qcb space?

If (and only if) the answer to this question is positive, then it holds that the category $\mathbf{Rep}(\mathcal{P}\omega)$ is equivalent to the category of sober qcb spaces.

Such an equivalence would provide a very elegant concrete description of the category $\mathbf{Rep}(\mathcal{P}\omega)$. This would be interesting as, at the time of writing, no concrete description of the category of replete objects in a realizability model has ever been established.

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References

[1] S. Abramsky and A. Jung. Domain theory. In *Handbook of Logic in Computer Science, Vol. 3*, pages 1–168. OUP, 1994.

[2] M. Alvarez-Manilla. Extension of valuations on locally compact sober spaces. *Topology Appl.*, 124:397–433, 2002.

[3] A. Bauer. A relationship between equilogical spaces and type two effectivity. *Electronic Notes in Theoretical Computer Science*, 45, 2001.

[4] A. Bauer, L. Birkedal, and D.S. Scott. Equilogical spaces. *Theoretical Computer Science*, to appear.

[5] A. Bauer, M.H. Escardó, and A.K. Simpson. Comparing functional paradigms for exact real-number computation. In *Proceedings ICALP 2002*, pages 488–500. Springer LNCS 2380, 2002.

[6] G.M. Bierman, A.M. Pitts, and C.V. Russo. Operational properties of Lily, a polymorphic linear lambda calculus with recursion. *Electronic Notes in Theoretical Computer Science*, 41, 2000.

[7] L. Birkedal and G. Rosolini. A parametric domain-theoretic model for the polymorphic lambda calculus. In preparation, 2003.

[8] Th. Coquand, C.A. Gunter, and G. Winskel. Domain theoretic models of polymorphism. *Information and Computation*, 81:123–167, 1989.

[9] A. Edalat. Domains for computation in mathematics, physics and exact real arithmetic. *Bulletin of Symbolic Logic*, 3:401–452, 1997.

[10] M.H. Escardó. PCF extended with real numbers. *Theoretical Computer Science*, 162:79–115, 1996.

[11] M.H. Escardó, J.D. Lawson, and A.K. Simpson. Core compactly generated spaces. In preparation, 2003.

[12] P.J. Freyd. Algebraically complete categories. In *Category Theory, Proceedings Como 1990*, pages 95–104. Springer LNM 1488, 1991.

[13] P.J. Freyd. Remarks on algebraically compact categories. In *Applications of Categories in Computer Science*, pages 95–106. LMS Lecture Notes 177, CUP, 1992.

[14] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices.* Springer-Verlag, 1980.

[15] C.A. Gunter and D.S. Scott. Semantic domains. In *Handbook of Theoretical Computer Science, Vol. B*, pages 633–674. Elsevier, 1990.

[16] R. Hasegawa. The theory of twiners and linear parametricity. In preparation, 2003.

[17] R. Heckmann. Spaces of valuations. *Annals of the New York Academy of Sciences*, 806:174–200, 1996.

[18] J.M.E. Hyland. Filter spaces and continuous functionals. *Annals of Mathematical Logic*, 16:101–143, 1979.

[19] J.M.E. Hyland. A small complete category. *Annals of Pure and Applied Logic*, 40:135–165, 1988.

[20] J.M.E. Hyland. First steps in synthetic domain theory. In *Category Theory, Proceedings Como 1990*, pages 131–156. Springer LNM 1488, 1991.

[21] J.M.E. Hyland, E.P. Robinson, and G. Rosolini. The discrete objects in the effective topos. *Proc. Lond. Math. Sci.*, 60:1–36, 1990.

[22] P.T. Johnstone. *Stone Spaces.* CUP, 1982.

[23] C. Jones. *Probabilistic Non-determinism.* PhD thesis, University of Edinburgh, 1990.

[24] C. Jones and G.D. Plotkin. A probabilistic powerdomain of evaluations. In *Proceedings 4th IEEE Symposium on Logic in Computer Science*, pages 186–195, 1989.

[25] A. Jung. The classification of continuous domains. In *Proceedings 5th IEEE Symposium on Logic in Computer Science*, pages 35–40, 1990.

[26] A. Jung. The dependent product construction in various categories of domains. *Theoretical Computer Science*, 79:359–363, 1991.

[27] A. Jung and R. Tix. The troublesome probabilistic powerdomain. *Electronic Notes in Theoretical Computer Science*, 13, 1998.

[28] J.R. Longley and A.K. Simpson. A uniform account of domain theory in realizability models. *Math. Struct. in Comp. Sci.*, 7:469–505, 1997.

[29] Q. Ma and J.C. Reynolds. Types, abstraction and parametric polymorphism, part 2. In *Mathematical Foundations of Programming Semantics*, pages 1–40. Springer, 1992.

[30] M. Menni and A.K. Simpson. Topological and limit space subcategories of countably-based equilogical spaces. *Math. Struct. in Comp. Sci.*, 12, 2002.

[31] E. Moggi. Notions of computation and monads. *Information and Computation*, 93:55–92, 1991.

[32] J. van Oosten and A.K. Simpson. Axioms and (counter)examples in synthetic domain theory. *Annals of Pure and Applied Logic*, 104:233–278, 2000.

[33] W.K.-S. Phoa and P. Taylor. The synthetic Plotkin powerdomain. Unpublished manuscript, 1990.

[34] G.D. Plotkin. Second-order type theory and recursion. Talk at the Scott Fest, 1993.

[35] G.D. Plotkin. Type theory and recursion. Invited talk at *8th IEEE Symposium on Logic in Computer Science*, 1993.

[36] G.D. Plotkin and A.J. Power. Computational effects and operations: an overview. *Electronic Notes in Theoretical Computer Science*, to appear. Special Issue for Prodeedings Domains VI.

[37] B. Reus and Th. Streicher. General synthetic domain theory — a logical approach. *Math. Struct. in Comp. Sci.*, 9:177–223, 1999.

[38] J.C. Reynolds. Types, abstraction and parametric polymorphism. In *Information Processing '83*, pages 513–523. North Holland, 1983.

[39] E.P. Robinson and G. Rosolini. Reflexive graphs and parametric polymorphism. In *Proceedings 9th IEEE Symposium on Logic in Computer Science*, pages 364–371, 1994.

[40] N. Saheb Djahromi. CPO's of measures for nondeterminism. *Theoretical Computer Science*, 12:19–37, 1980.

[41] M. Schröder. *Admissible Representations for Continuous Computations*. PhD thesis, FernUniversität, Hagen, 2002.

[42] M. Schröder. Extended admissibility. *Theoretical Computer Science*, 284:519–538, 2002.

[43] D.S. Scott. Data types as lattices. *SIAM J. Comput.*, 5:522–587, 1976.

[44] M.B. Smyth. Topology. In *Handbook of Logic in Computer Science, Vol. 1*, pages 641–761. OUP, 1992.

[45] M.B. Smyth and G.D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM J. Comput.*, 11:761–783, 1982.

[46] N.E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.

[47] P. Taylor. The fixed point property in synthetic domain theory. In *Proceedings 6th IEEE Symposium on Logic in Computer Science*, pages 152–160, 1991.

[48] S. Vickers. *Topology via Logic*. CUP, 1989.

[49] K. Weihrauch. *Computable Analysis*. Springer, 2000.