Demonic orders and quasi-totality in Dedekind categories

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Abstract

This paper presents a proof of the associativity of demonic composition of relations in Dedekind categories and shows that the demonic composition is monotonic with respect to two demonic orderings on relations, which are defined by quasi-total relations, respectively.

1 Introduction

Relation algebras [8] are suitable for describing semantics of relational programming [4]. In particular demonic composition [2, 9, 1, 5, 10] and demonic orderings will be useful for designing nondeterministic programs [3, 10, 11]. For concrete relations R and S, the demonic composition $R \odot S$ relates elements x with elements y exactly if x is related with y by the usual relational composition RS and the image of x under R may not lie outside the domain of S (which should never be confused with the categorical concept of source of morphism):

$$(x, y) \in R \odot S \Leftrightarrow [\forall z : (x, z) \in R \Rightarrow z \in \mathrm{dom}(S)] \wedge (x, y) \in RS.$$

In this paper the demonic composition in Dedekind categories [6, 7] will be defined (without using complement operator). The proofs of associative law of demonic compositions are given earlier in [2, 9, 1, 5], here we give a proof using properties of Dedekind compositions. Moreover we study two demonic orderings of relations originally introduced by Desharnais et al. [5] and Xu et al. [10] and show several fundamental properties of them in Dedekind categories. In Section 2, we first review the definition of Dedekind categories. Then we introduce the demonic composition in a Dedekind category, and show some of its properties. In Section 3, we define quasi-totality of relations and give the definition of two refinement orderings, and provide existence conditions of the supremum and values of supremum and infimum of a set of relations with respect to both refinement orderings, respectively. Finally we prove the monotonicity of the demonic composition on these orderings.

2 Demonic Compositions

We will generalize demonic compositions into Dedekind categories and give a proof of associativity of the demonic compositions using properties of Dedekind compositions.

We first review the definition of a Dedekind category, a kind of relation category (following Olivier and Serrato, 1980) which is our general framework.

Throughout this paper, a morphism α from an object A into an object B in a Dedekind category (which will be defined below) will be called a relation, and denoted by a half arrow $\alpha : A \rightharpoondown B$. The composite of a relation $\alpha : A \rightharpoondown B$ followed by a relation $\beta : B \rightharpoondown C$ will be written as $\alpha\beta : A \rightharpoondown C$. We denote the identity relation on an object A by id_A. The composition operator will bind stronger than all other binary operators.

Definition 2.1 A *Dedekind category* $\mathcal{D}$ is a category satisfying the following:
D1. [Complete Heyting Algebra] For all pairs of objects X and Y the hom-set $\mathcal{D}(X, Y)$ consisting of all relations of X into Y is a complete Heyting algebra with the least relation 0_{XY} and the greatest relation ∇_{XY}. Its algebraic structure will be denoted by

$$\mathcal{D}(X, Y) = (\mathcal{D}(X, Y), \sqsubseteq, \sqcup, \sqcap, 0_{XY}, \nabla_{XY}).$$

That is, (a) $\sqsubseteq$ is a partial order on $\mathcal{D}(X, Y)$, (b) $\forall \alpha \in \mathcal{D}(X, Y) :: 0_{XY} \sqsubseteq \alpha \sqsubseteq \nabla_{XY}$, (c) $\sqcup_{\lambda \in \Lambda} \alpha_\lambda \sqsubseteq \alpha$ iff $\alpha_\lambda \sqsubseteq \alpha$ for all $\lambda \in \Lambda$, (d) $\alpha \sqsubseteq \sqcap_{\lambda \in \Lambda} \alpha_\lambda$ iff $\alpha \sqsubseteq \alpha_\lambda$ for all $\lambda \in \Lambda$, and (e) $\alpha \sqcap (\sqcup_{\lambda \in \Lambda} \alpha_\lambda) = \sqcup_{\lambda \in \Lambda}(\alpha \sqcap \alpha_\lambda)$.
D2. [Converse] There is given a converse operation $^\sharp : \mathcal{D}(X, Y) \to \mathcal{D}(Y, X)$. That is, for all relations $\alpha, \alpha' : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$, the following laws hold:
(a) $(\alpha\beta)^\sharp = \beta^\sharp\alpha^\sharp$, (b) $(\alpha^\sharp)^\sharp = \alpha$, (c) If $\alpha \sqsubseteq \alpha'$, then $\alpha^\sharp \sqsubseteq \alpha'^\sharp$.
D3. [Dedekind Formula] For all relations $\alpha : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the Dedekind formula $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$ holds.
D4. [Residue] For all relations $\beta : Y \rightharpoondown Z$ and $\gamma : X \rightharpoondown Z$ the residue (or division, weakest precondition) $\gamma \div \beta : X \rightharpoondown Y$ is a relation such that $\alpha\beta \sqsubseteq \gamma$ if and only if $\alpha \sqsubseteq \gamma \div \beta$ for all morphisms $\alpha : X \rightharpoondown Y$. $\square$

If all relations in a Dedekind category have complements, then the Dedekind category is called a Schröder category. It is well known that in a Schröder category the Dedekind formula is equivalent to an equivalence

$$\alpha\beta \sqsubseteq \gamma \;\Leftrightarrow\; \alpha^\sharp\gamma^- \sqsubseteq \beta^- \;\Leftrightarrow\; \gamma^-\beta^\sharp \sqsubseteq \alpha^-$$

which is called *Schröder rule*. A relation $f : X \rightharpoondown Y$ such that $f^\sharp f \sqsubseteq \mathrm{id}_Y$ (*univalent*) and $\mathrm{id}_X \sqsubseteq ff^\sharp$ (*total*) is called a *function* and may be introduced as $f : X \to Y$. A Dedekind category $\mathcal{D}$ is called *uniform* if $\nabla_{XY}\nabla_{YZ} = \nabla_{XZ}$ holds for all objects X, Y and Z in $\mathcal{D}$.

Before we define the demonic composition of relations in a Dedekind category, we consider the *Dedekind composition* $\alpha \ominus \beta$ defined by $\alpha^\sharp\gamma \sqsubseteq \beta$ iff $\gamma \sqsubseteq \alpha \ominus \beta$ for relations $\gamma : X \rightharpoondown Z$. It is easy to see that $\alpha \ominus \beta = (\beta^\sharp \div \alpha)^\sharp$.

The *demonic composition* in a Dedekind category $\mathcal{D}$ is defined by

$$\alpha \odot \beta = \alpha\beta \sqcap (\alpha \ominus \beta\nabla_{ZZ})$$

for relations $\alpha : X \rightharpoondown Y$ and $\beta : Y \rightharpoondown Z$. In Schröder categories it is clear that the demonic composition $\alpha \odot \beta$ can be rewritten to

$$\alpha \odot \beta = \alpha\beta \sqcap (\alpha(\beta\nabla_{ZZ})^-)^-.$$

The proofs of associativity of demonic composition using properties relate to complement were given in [2, 5]. Desharnais et al. [5] also give a proof of associativity by embedding a demonic semilattice in a relation algebra.

Proposition 2.2 *Let $\alpha : X \rightharpoondown Y$ and $\beta : Y \rightharpoondown Z$ be relations in a Dedekind category $\mathcal{D}$. If α is univalent or β is total, then $\alpha \odot \beta = \alpha\beta$. In particular, $\mathrm{id}_X \odot \alpha = \alpha \odot \mathrm{id}_X = \alpha$.*

Proof. First note that $\alpha \odot \beta = \alpha\beta$ iff $\alpha\beta \sqsubseteq \alpha \ominus \beta\nabla_{ZZ}$ iff $\alpha^\sharp\alpha\beta \sqsubseteq \beta\nabla_{ZZ}$. When α is univalent, $\alpha^\sharp\alpha\beta \sqsubseteq \beta \sqsubseteq \beta\nabla_{ZZ}$. Next assume β is total. Then $\nabla_{YZ} \sqsubseteq \beta\beta^\sharp\nabla_{YZ} \sqsubseteq \beta\nabla_{ZZ}$, and so $\alpha^\sharp\alpha\beta \sqsubseteq \beta\nabla_{ZZ}$. Consequently the last claim is clear from the fact that id_X is univalent and total. $\square$

The *domain* relation $\mathsf{dom}\ \alpha : X \rightharpoondown X$ and the *range* (codomain) relation $\mathsf{ran}\ \alpha : Y \rightharpoondown Y$ of $\alpha : X \rightharpoondown Y$ are defined by $\mathsf{dom}\ \alpha = \alpha\alpha^\sharp \sqcap \mathrm{id}_X$ and $\mathsf{ran}\ \alpha = \alpha^\sharp\alpha \sqcap \mathrm{id}_Y$, respectively.

We have the following properties relate to the domain and range relations.

Proposition 2.3 *Let $\alpha : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$ and $\gamma : X \rightharpoonup Z$ be relations in a Dedekind category $\mathcal{D}$. Then the following hold:*

(a) $\alpha(\mathsf{ran}\ \alpha) = \alpha$ *and* $(\mathsf{dom}\ \alpha)\alpha = \alpha$.

(b) $\mathsf{ran}\ \gamma \sqsubseteq \mathsf{ran}\ \beta \Leftrightarrow \gamma \sqsubseteq \nabla_{XY}\beta$ *and* $\mathsf{dom}\ \gamma \sqsubseteq \mathsf{dom}\ \alpha \Leftrightarrow \gamma \sqsubseteq \alpha\nabla_{YZ}$.

(c) $\alpha \odot \beta = (\mathsf{dom}\ \gamma)\alpha\beta$ *where* $\gamma = \alpha \ominus (\beta\nabla_{ZZ})$.

(d) *If* $u \sqsubseteq \mathrm{id}_X$ *and* $\nabla_{XZ}\nabla_{ZX} = \nabla_{XX}$, *then* $u \sqsubseteq \mathsf{dom}\ \gamma$ *iff* $\mathsf{ran}\ (u\alpha) \sqsubseteq \mathsf{dom}\ \beta$ *where* $\gamma = \alpha \ominus (\beta\nabla_{ZZ})$.

Proof. (a) It is clear from

$$\begin{array}{rcll} \alpha(\mathsf{ran}\ \alpha) & \sqsubseteq & \alpha & \{\ \mathsf{ran}\ \alpha \sqsubseteq \mathrm{id}_Y\ \} \\ & = & \alpha \sqcap \alpha\,\mathrm{id}_Y & \\ & \sqsubseteq & \alpha(\alpha^\sharp\alpha \sqcap \mathrm{id}_Y) & \{\ \text{Dedekind formula}\ \} \\ & = & \alpha(\mathsf{ran}\ \alpha). & \end{array}$$

(b) Assume that $\mathsf{ran}\ \gamma \sqsubseteq \mathsf{ran}\ \beta$. Then

$$\begin{array}{rcll} \gamma & = & \gamma(\mathsf{ran}\ \gamma) & \{\ \text{(a)}\ \} \\ & \sqsubseteq & \gamma(\mathsf{ran}\ \beta) & \{\ \text{assumption}\ \} \\ & \sqsubseteq & \gamma\beta^\sharp\beta & \{\ \mathsf{ran}\ \beta = \beta^\sharp\beta \sqcap \mathrm{id}_Z \sqsubseteq \beta^\sharp\beta\ \} \\ & \sqsubseteq & \nabla_{XY}\beta. & \{\ \gamma\beta^\sharp \sqsubseteq \nabla_{XY}\ \} \end{array}$$

Conversely assume that $\gamma \sqsubseteq \nabla_{XY}\beta$. Then

$$\begin{array}{rcll} \mathsf{ran}\ \gamma & = & \gamma^\sharp\gamma \sqcap \mathrm{id}_Z & \{\ \text{definition of range}\ \} \\ & \sqsubseteq & \gamma^\sharp\nabla_{XY}\beta \sqcap \mathrm{id}_Z & \{\ \text{assumption}\ \} \\ & = & (\gamma^\sharp\nabla_{XY} \sqcap \mathrm{id}_Z\beta^\sharp)\beta \sqcap \mathrm{id}_Z & \{\ \text{Dedekind Formula}\ \} \\ & \sqsubseteq & \beta^\sharp\beta \sqcap \mathrm{id}_Z & \\ & = & \mathsf{ran}\ \beta. & \{\ \text{definition of range}\ \} \end{array}$$

(c) Set $\gamma = \alpha \ominus \beta\nabla_{ZZ}$. First we show that $\gamma = (\mathsf{dom}\ \gamma)\nabla_{XZ}$. We have

$$\begin{array}{rcll} \gamma & = & (\mathsf{dom}\ \gamma)\gamma & \{\ \text{(a)}\ \} \\ & \sqsubseteq & (\mathsf{dom}\ \gamma)\nabla_{XZ} & \{\ \gamma \sqsubseteq \nabla_{XZ}\ \} \\ & = & (\gamma\gamma^\sharp \sqcap \mathrm{id}_X)\nabla_{XZ} & \{\ \text{definition of domain}\ \} \\ & \sqsubseteq & \gamma\nabla_{ZZ} & \{\ \gamma^\sharp\nabla_{XZ} \sqsubseteq \nabla_{ZZ}\ \} \\ & = & \gamma. & \{\ \text{Proposition A.3(e)}\ \} \end{array}$$

Hence $\gamma = (\mathsf{dom}\ \gamma)\nabla_{XZ}$. Thus $\alpha \odot \beta = \alpha\beta \sqcap \gamma = \alpha\beta \sqcap (\mathsf{dom}\ \gamma)\nabla_{XZ} = (\mathsf{dom}\ \gamma)\alpha\beta$ by Proposition A.1(a).
(d) Assume $u \sqsubseteq \mathrm{id}_X$ and $\nabla_{XZ}\nabla_{ZX} = \nabla_{XX}$, and set $\gamma = \alpha \ominus \beta\nabla_{ZZ}$. Then

$$\begin{array}{rcll} u \sqsubseteq \mathsf{dom}\ \gamma & \Leftrightarrow & \mathsf{dom}\ u \sqsubseteq \mathsf{dom}\ \gamma & \{\ u = \mathsf{dom}\ u\ \} \\ & \Leftrightarrow & u \sqsubseteq \gamma\nabla_{ZX} = \alpha \ominus \beta\nabla_{ZX} & \{\ \text{(b) and Proposition A.3(e):}\nabla_{XZ}\nabla_{ZX} = \nabla_{XX}\ \} \\ & \Leftrightarrow & \alpha^\sharp u \sqsubseteq \beta\nabla_{ZX} & \\ & \Leftrightarrow & u\alpha \sqsubseteq \nabla_{XZ}\beta^\sharp & \{\ \text{conversion}\ \} \\ & \Leftrightarrow & \mathsf{ran}\ (u\alpha) \sqsubseteq \mathsf{ran}\ \beta^\sharp = \mathsf{dom}\ \beta & \{\ \text{(b)}\ \} \end{array}$$

□

Backhouse and van der Woude [1] and Xu et al. [10] also gave the definition of demonic composition. The device used by them to restrict the domain of a relational composition is not intersection, but, instead, composition with a so-called 'monotype', that is, a relation below identity relation. The equivalence of their definition to our definition of demonic composition is clear from (c) and (d) of the last proposition. In [1] there is a proof of associative law for demonic composition using properties of monotype.

Before we see associativity of the demonic compositions we have to show the following lemma.

Lemma 2.4 *Let $\alpha : X \rightharpoondown Y$, $\beta : Y \rightharpoondown Z$ and $\gamma : Z \rightharpoondown W$ be relations in a uniform Dedekind category $\mathcal{D}$. Then the following hold:*

(a) $\alpha \ominus (\beta \odot \gamma)\nabla_{WW} = (\alpha \ominus \beta\gamma\nabla_{WW}) \sqcap (\alpha\beta \ominus \gamma\nabla_{WW})$.

(b) $\alpha(\beta \odot \gamma) \sqcap (\alpha\beta \ominus \gamma\nabla_{WW}) = \alpha\beta\gamma \sqcap (\alpha\beta \ominus \gamma\nabla_{WW})$.

(c) $\alpha \odot (\beta \odot \gamma) = \alpha\beta\gamma \sqcap (\alpha \ominus (\beta\gamma\nabla_{WW} \sqcap (\beta \ominus \gamma\nabla_{WW})))$.

(d) $(\alpha \odot \beta)\gamma = \alpha\beta\gamma \sqcap (\alpha \ominus \beta\nabla_{ZW})$.

(e) $(\alpha \odot \beta) \ominus \gamma\nabla_{WW} = (\alpha \ominus \beta\nabla_{ZW}) \Rightarrow (\alpha\beta \ominus \gamma\nabla_{WW})$.

(f) $(\alpha \odot \beta) \odot \gamma = \alpha\beta\gamma \sqcap (\alpha \ominus (\beta\nabla_{ZW} \sqcap (\beta \ominus \gamma\nabla_{WW})))$.

Proof. (a) It follows from

$$\begin{array}{cll} & \alpha \ominus (\beta \odot \gamma)\nabla_{WW} & \\ = & \alpha \ominus (\beta\gamma \sqcap (\beta \ominus \gamma\nabla_{WW}))\nabla_{WW} & \\ = & \alpha \ominus (\beta\gamma\nabla_{WW} \sqcap (\beta \ominus \gamma\nabla_{WW})) & \{\text{ Propositions A.3(e) and A.1(b) }\} \\ = & (\alpha \ominus \beta\gamma\nabla_{WW}) \sqcap (\alpha \ominus (\beta \ominus \gamma\nabla_{WW})) & \{\text{ Proposition A.3(c) }\} \\ = & (\alpha \ominus \beta\gamma\nabla_{WW}) \sqcap (\alpha\beta \ominus \gamma\nabla_{WW}). & \{\text{ Proposition A.3(d) }\} \end{array}$$

(b) It follows from

$$\begin{array}{cll} & \alpha(\beta \odot \gamma) \sqcap (\alpha\beta \ominus \gamma\nabla_{WW}) & \\ = & \alpha(\beta\gamma \sqcap (\beta \ominus \gamma\nabla_{WW})) \sqcap (\alpha \ominus (\beta \ominus \gamma\nabla_{WW})) & \{\text{ Proposition A.3(d) }\} \\ = & \alpha\beta\gamma \sqcap (\alpha \ominus (\beta \ominus \gamma\nabla_{WW})) & \{\text{ Proposition A.3(f) }\} \\ = & \alpha\beta\gamma \sqcap (\alpha\beta \ominus \gamma\nabla_{WW}). & \{\text{ Proposition A.3(d) }\} \end{array}$$

(c) It is a direct corollary of (a) and (b):

$$\begin{array}{rcll} \alpha \odot (\beta \odot \gamma) & = & \alpha(\beta \odot \gamma) \sqcap (\alpha \ominus (\beta \odot \gamma)\nabla_{WW}) & \\ & = & \alpha(\beta \odot \gamma) \sqcap (\alpha \ominus \beta\gamma\nabla_{WW}) \sqcap (\alpha\beta \ominus \gamma\nabla_{WW}) & \{\text{ (a) }\} \\ & = & \alpha\beta\gamma \sqcap (\alpha \ominus \beta\gamma\nabla_{WW}) \sqcap (\alpha\beta \ominus \gamma\nabla_{WW}) & \{\text{ (b) }\} \\ & = & \alpha\beta\gamma \sqcap (\alpha \ominus \beta\gamma\nabla_{WW}) \sqcap (\alpha \ominus (\beta \ominus \gamma\nabla_{WW})) & \{\text{ Proposition A.3(d) }\} \\ & = & \alpha\beta\gamma \sqcap (\alpha \ominus (\beta\gamma\nabla_{WW} \sqcap (\beta \ominus \gamma\nabla_{WW}))). & \{\text{ Proposition A.3(c) }\} \end{array}$$

(d) It follows from

$$\begin{array}{rcll} (\alpha \odot \beta)\gamma & = & (\alpha\beta \sqcap (\alpha \ominus \beta\nabla_{ZZ}))\gamma & \\ & = & (\alpha\beta \sqcap (\alpha \ominus \beta\nabla_{ZZ})\nabla_{ZZ})\gamma & \{\text{ Proposition A.3(e) }\} \\ & = & \alpha\beta\gamma \sqcap (\alpha \ominus \beta\nabla_{ZZ})\nabla_{ZW} & \{\text{ Proposition A.1(b) }\} \\ & = & \alpha\beta\gamma \sqcap (\alpha \ominus \beta\nabla_{ZW}). & \{\text{ Proposition A.3(e) }\} \end{array}$$

(e) It is immediate from

$$\begin{array}{cll} & (\alpha \odot \beta) \ominus \gamma\nabla_{WW} & \\ = & (\alpha\beta \sqcap (\alpha \ominus \beta\nabla_{ZZ})) \ominus \gamma\nabla_{WW} & \\ = & (\alpha\beta \sqcap (\alpha \ominus \beta\nabla_{ZZ})\nabla_{ZZ}) \ominus \gamma\nabla_{WW} & \{\text{ Proposition A.3(e) }\} \\ = & (\alpha \ominus \beta\nabla_{ZZ})\nabla_{ZW} \Rightarrow (\alpha\beta \ominus \gamma\nabla_{WW}) & \{\text{ Proposition A.3(h) }\} \\ = & (\alpha \ominus \beta\nabla_{ZW}) \Rightarrow (\alpha\beta \ominus \gamma\nabla_{WW}). & \{\text{ Proposition A.3(e) }\} \end{array}$$

(f) It is a corollary of (d) and (e):

$$\begin{array}{cll} & (\alpha \odot \beta) \odot \gamma & \\ = & (\alpha \odot \beta)\gamma \sqcap ((\alpha \odot \beta) \ominus \gamma\nabla_{WW}) & \\ = & \alpha\beta\gamma \sqcap (\alpha \ominus \beta\nabla_{ZW}) \sqcap ((\alpha \ominus \beta\nabla_{ZW}) \Rightarrow (\alpha\beta \ominus \gamma\nabla_{WW})) & \{\text{ (d), (e) }\} \\ = & \alpha\beta\gamma \sqcap (\alpha \ominus \beta\nabla_{ZW}) \sqcap (\alpha\beta \ominus \gamma\nabla_{WW}) & \{\text{ Proposition A.2(c) }\} \\ = & \alpha\beta\gamma \sqcap (\alpha \ominus (\beta\nabla_{ZW} \sqcap (\beta \ominus \gamma\nabla_{WW}))). & \{\text{ Propositions A.3(d) and A.3(c) }\} \end{array}$$

□

Now we show the associative law of the demonic compositions.

Theorem 2.5 *Let $\alpha : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$ and $\gamma : Z \rightharpoonup W$ be relations in a uniform Dedekind category $\mathcal{D}$. Then the associative law $\alpha \odot (\beta \odot \gamma) = (\alpha \odot \beta) \odot \gamma$ of the demonic compositions holds.*

Proof. By Lemmas 2.4(c) and (f) it suffices to see an equality $\beta\nabla_{ZW} \sqcap (\beta \ominus \gamma\nabla_{WW}) = \beta\gamma\nabla_{WW} \sqcap (\beta \ominus \gamma\nabla_{WW})$. Applying Proposition A.3(f) one can see that

$$\begin{array}{rll} & \beta\nabla_{ZW} \sqcap (\beta \ominus \gamma\nabla_{WW}) & \\ = & \beta(\nabla_{ZW} \sqcap \gamma\nabla_{WW}) \sqcap (\beta \ominus \gamma\nabla_{WW}) & \{ \text{ Proposition A.3(f) } \} \\ = & \beta\gamma\nabla_{WW} \sqcap (\beta \ominus \gamma\nabla_{WW}). & \end{array}$$

$\square$

Example 2.6 Take the following homogeneous relations α, α' and β on a set $X = \{1, 2\}$ represented by Boolean matrices:

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \alpha' = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\alpha \sqsubseteq \alpha'$, but $\alpha \odot \beta \not\sqsubseteq \alpha' \odot \beta$ since

$$\alpha \odot \beta = \alpha\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and}$$

$$\alpha' \odot \beta = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

3 Demonic Orderings

As we see in Example 2.6 the demonic composition is not monotonic with respect to the ordering $\sqsubseteq$ on relations. For ensuring the existence of the fixed points of a recursively defined program, we need other orderings among relations on which the demonic composition is monotonic. There are two refinement orderings which are introduced by Xu et al. [10] and Desharnais et al. [5], respectively. In this section we define these refinement orderings in Dedekind categories, and show some of their properties, and finally prove the monotonicity of the demonic composition on these two refinement orderings.

We first recall that each hom-set $\mathcal{D}(X, Y)$ has relative pseudo-complement, that is, for any two relations α and β in $\mathcal{D}$ there is a relation $\alpha \Rightarrow \beta$ such that $\alpha \sqcap \gamma \sqsubseteq \beta$ iff $\gamma \sqsubseteq \alpha \Rightarrow \beta$ for all relations γ.

Define $\alpha^+ = \alpha\nabla_{YY} \Rightarrow \alpha$ for every relation $\alpha : X \rightharpoonup Y$ in a Dedekind category $\mathcal{D}$. A relation α is called *quasi-total* if $\alpha^+ = \alpha$. We can easily see that all total relations are quasi-total as follows: If α is total, then $\nabla_{XY} = \mathrm{id}_X\nabla_{XY} \sqsubseteq \alpha\alpha^\sharp\nabla_{XY} \sqsubseteq \alpha\nabla_{YY}$. Hence $\alpha^+ = \alpha\nabla_{YY} \Rightarrow \alpha = \nabla_{XY} \Rightarrow \alpha = \alpha$. All quasi-total relations are total in uniform Schröder categories. To prove this claim it is enough to show that $\alpha\nabla_{YX} = \nabla_{XX}$ for each quasi-total relation α, because of the fact that $\mathrm{id}_X \sqsubseteq \alpha\alpha^\sharp$ iff $\alpha\nabla_{YX} = \nabla_{XX}$. If α is quasi-total then $\alpha\nabla_{YY} \Rightarrow 0_{XY} \sqsubseteq \alpha\nabla_{YY} \Rightarrow \alpha = \alpha \sqsubseteq \alpha\nabla_{YY}$ and so $\alpha\nabla_{YY} \Rightarrow 0_{XY} = (\alpha\nabla_{YY} \Rightarrow 0_{XY}) \sqcap \alpha\nabla_{YY} = 0_{XY}$. In boolean lattices (or equivalently, in Schröder categories) $\delta \Rightarrow 0_{XY} = \delta^-$ for each relation $\delta : X \rightharpoonup Y$, and so $\alpha\nabla_{YY} = (\alpha\nabla_{YY})^{--} = (\alpha\nabla_{YY} \Rightarrow 0_{XY}) \Rightarrow 0_{XY} = 0_{XY} \Rightarrow 0_{XY} = \nabla_{XY}$. Therefore $\alpha\nabla_{YX} = \alpha\nabla_{YY}\nabla_{YX} = \nabla_{XY}\nabla_{YX} = \nabla_{XX}$ by the uniformity.

Proposition 3.1 *Let $\alpha : X \rightharpoonup Y$ be a relation in a Dedekind category $\mathcal{D}$.*

(a) $\alpha \sqsubseteq \alpha^+$ *and* $\alpha^{++} = \alpha^+$. *(Every α^+ is quasi-total.)*

(b) $\alpha\nabla_{YY} = \alpha$ *iff* $\alpha^+ = \nabla_{XY}$. *In particular* $0_{XY}^+ = \nabla_{XY}$ *and* $(\alpha\nabla_{YY})^+ = \nabla_{XY}$.

Proof. (a) It is trivial that $\alpha \sqsubseteq \alpha^+$. Also $\alpha^{++} = \alpha^+\nabla_{YY} \Rightarrow (\alpha\nabla_{YY} \Rightarrow \alpha) = (\alpha^+\nabla_{YY} \sqcap \alpha\nabla_{YY}) \Rightarrow \alpha = \alpha\nabla_{YY} \Rightarrow \alpha = \alpha^+$ by Proposition A.2(d).
(b) Assume $\alpha\nabla_{YY} = \alpha$. Then $\alpha^+ = \alpha \Rightarrow \alpha = \nabla_{XY}$. Conversely assume $\alpha\nabla_{YY} \Rightarrow \alpha = \nabla_{XY}$. Then $\alpha\nabla_{YY} = \alpha\nabla_{YY} \sqcap \nabla_{XY} = \alpha\nabla_{YY} \sqcap \alpha^+ = \alpha$ by Proposition A.2(c). Hence $\alpha = \alpha\nabla_{YY}$. $\square$

In a Dedekind category $\mathcal{D}$ two demonic refinement orderings $\leq$ and $\preceq$ of relations $\alpha, \alpha' : X \rightharpoonup Y$ are respectively defined in [10] and [5] as follows:

$$\begin{array}{llll} \alpha \leq \alpha' & \overset{def}{\Leftrightarrow} & \alpha \sqsubseteq \alpha' \sqsubseteq \alpha^+ & \{ \text{ Xu et al. [10] } \} \\ & \Leftrightarrow & \alpha' \sqcap \alpha\nabla_{YY} = \alpha & \\ & \Leftrightarrow & \alpha^+ = \alpha\nabla_{YY} \Rightarrow \alpha' & \end{array}$$

$$\alpha \preceq \alpha' \quad \overset{def}{\Leftrightarrow} \quad \alpha\nabla_{YY} \sqsubseteq \alpha'\nabla_{YY} \ \wedge\ \alpha' \sqsubseteq \alpha^+ \quad \{ \text{ Desharnais et al. [5] } \}$$

We can obtain straightforwardly from the above definitions that $\alpha \leq \alpha'$ implies $\alpha \preceq \alpha'$.

Proposition 3.2 *Let $\alpha : X \rightharpoonup Y$ and $\alpha' : X \rightharpoonup Y$ be relations in a Dedekind category $\mathcal{D}$. Then the following hold:*

(a) *If $\alpha\nabla_{YY} = \alpha$, then $\alpha \leq \alpha'$ iff $\alpha \sqsubseteq \alpha'$. In particular $0_{XY} \leq \alpha$ and $0_{XY} \preceq \alpha$.*

(b) *If $\alpha\nabla_{YY} = \alpha$, then $\alpha \preceq \alpha'$ iff $\alpha \sqsubseteq \alpha'\nabla_{YY}$. In particular $\alpha\nabla_{YY} \preceq \alpha$.*

(c) *$\alpha \leq \alpha^+$ and $\alpha \preceq \alpha^+$.*

(d) *$\alpha\alpha^\sharp\alpha \preceq \alpha$.*

(e) *If $\alpha\nabla_{YY} \sqsupseteq \alpha'\nabla_{YY}$ and $\alpha \preceq \alpha'$, then $\alpha' \sqsubseteq \alpha$.*

Proof. (a) Assume $\alpha\nabla_{YY} = \alpha$. Then the assertion is trivial since $\alpha^+ = \nabla_{XY}$ by Proposition 3.1(b).
(b) It is trivial from the definition.
(c) By Proposition 3.1(a) we have $\alpha \sqsubseteq \alpha^+ \sqsubseteq \alpha^+$ which means $\alpha \leq \alpha^+$, and so $\alpha \preceq \alpha^+$.
(d) It follows from $\alpha\alpha^\sharp\alpha\nabla_{YY} \sqsubseteq \alpha\nabla_{YY}$ and $\alpha \sqsubseteq \alpha\alpha^\sharp\alpha \sqsubseteq (\alpha\alpha^\sharp\alpha)^+$ by Proposition 3.1(a).
(e) Assume that $\alpha\nabla_{YY} \sqsupseteq \alpha'\nabla_{YY}$ and $\alpha \preceq \alpha'$. Then we have $\alpha' = \alpha'\nabla_{YY} \sqcap \alpha' \sqsubseteq \alpha\nabla_{YY} \sqcap \alpha^+ = \alpha$ by Proposition A.2(c). $\square$

Next we see the demonic refinement orderings are orderings on the hom-set $\mathcal{D}(X, Y)$.

Proposition 3.3 *Relations $\leq$ and $\preceq$ on the hom-set $\mathcal{D}(X, Y)$ are orderings.*

Proof. (Reflexive law) $\alpha \leq \alpha$ and $\alpha \preceq \alpha$ follows from a fact $\alpha \sqsubseteq \alpha \sqsubseteq \alpha^+$ by Proposition 3.1(a).
(Transitive law) Assume that $\alpha \leq \alpha'$ and $\alpha' \leq \alpha''$, that is, $\alpha \sqsubseteq \alpha' \sqsubseteq \alpha^+$ and $\alpha' \sqsubseteq \alpha'' \sqsubseteq \alpha'^+$. Hence $\alpha \sqsubseteq \alpha' \sqsubseteq \alpha''$ and

$$\begin{array}{llll} \alpha'' & \sqsubseteq & \alpha'\nabla_{YY} \Rightarrow \alpha' & \{ \alpha'' \sqsubseteq \alpha'^+ \} \\ & \sqsubseteq & \alpha'\nabla_{YY} \Rightarrow (\alpha\nabla_{YY} \Rightarrow \alpha) & \{ \alpha' \sqsubseteq \alpha^+ \} \\ & = & (\alpha'\nabla_{YY} \sqcap \alpha\nabla_{YY}) \Rightarrow \alpha & \{ \text{Proposition A.2(d)} \} \\ & = & \alpha\nabla_{YY} \Rightarrow \alpha. & \{ \alpha \sqsubseteq \alpha' \} \\ & = & \alpha^+ & \end{array}$$

Similarly $\alpha \preceq \alpha'$ and $\alpha' \preceq \alpha''$ imply $\alpha \preceq \alpha''$.
(Anti-symmetric law) Assume that $\alpha \preceq \alpha'$ and $\alpha' \preceq \alpha$. First note that $\alpha\nabla_{YY} = \alpha'\nabla_{YY}$. Then using Proposition 3.2(e) we have $\alpha \sqsubseteq \alpha'$ and $\alpha' \sqsubseteq \alpha$. Hence $\alpha = \alpha'$. Anti-symmetry of $\leq$ is trivial. $\square$

Example 3.4 Consider the following relations on a set $X = \{1, 2\}$ represented by matrices:

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \alpha\nabla_{XX} \text{ and } \alpha' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{id}_X.$$

Then $\alpha \preceq \alpha'$ ($\alpha\nabla_{XX} \sqsubseteq \nabla_{XX} = \alpha'\nabla_{XX}$ and $\alpha' \sqsubseteq \nabla_{XX} = \alpha^+$), but $\alpha \not\leq \alpha'$ because $\alpha \not\sqsubseteq \alpha'$.

Lemma 3.5 *Let $\alpha, \alpha' : X \rightharpoonup Y$ be relations in a Dedekind category $\mathcal{D}$. If α' is univalent and $\alpha \sqsubseteq \alpha'$, then $\alpha' \sqsubseteq \alpha^+$ and consequently $\alpha \leq \alpha'$ and $\alpha \preceq \alpha'$.*

Proof. Assume $\alpha'^\sharp \alpha' \sqsubseteq \mathrm{id}_Y$ and $\alpha \sqsubseteq \alpha'$. Then

$$\begin{array}{rcll} \alpha' \sqcap \alpha \nabla_{YY} & \sqsubseteq & \alpha\alpha^\sharp\alpha' & \{ \text{Dedekind Formula} \} \\ & \sqsubseteq & \alpha\alpha'^\sharp\alpha' & \{ \alpha \sqsubseteq \alpha' \} \\ & \sqsubseteq & \alpha. & \{ \alpha'^\sharp\alpha' \sqsubseteq \mathrm{id}_Y \} \end{array}$$

Hence $\alpha' \sqsubseteq \alpha^+$. □

By the above lemma if α is quasi-total and α' is univalent, then $\alpha \sqsubseteq \alpha'$ implies $\alpha = \alpha'$.

The following proposition characterizes maximal elements in the demonic orderings:

Proposition 3.6 (a) *A relation $\alpha : X \rightharpoonup Y$ is maximal in $(\mathcal{D}(X,Y), \leq)$ iff it is quasi-total ($\alpha = \alpha^+$).*

(b) *Suppose a relational axiom*[1] *of choice. Then a relation $\alpha : X \rightharpoonup Y$ is maximal in $(\mathcal{D}(X,Y), \preceq)$ iff $\alpha = \alpha^+$ and $\alpha^\sharp\alpha \sqsubseteq \mathrm{id}_Y$.*

Proof. (a) Assume that $\alpha = \alpha^+$ and $\alpha \leq \alpha'$. Then $\alpha \sqsubseteq \alpha'$ and $\alpha' \sqsubseteq \alpha^+ = \alpha$. Hence $\alpha = \alpha'$ and so α is maximal. Conversely assume that α is maximal in $(\mathcal{D}(X,Y), \leq)$. Then $\alpha = \alpha^+$ follows from the maximality of α since $\alpha \leq \alpha^+$ by Proposition 3.2(c).
(b) Let $\alpha = \alpha^+$ and $\alpha^\sharp\alpha \sqsubseteq \mathrm{id}_Y$. Assume $\alpha \preceq \alpha'$. Then $\alpha' \sqsubseteq \alpha^+ = \alpha$ and so $\alpha' \preceq \alpha$ by Lemma 3.5. Hence $\alpha = \alpha'$ by the anti-symmetric law of $\preceq$, which proves the maximality of α. Conversely assume that α is maximal in $(\mathcal{D}(X,Y), \preceq)$. Since $\alpha \preceq \alpha^+$ by Proposition 3.2(c) the maximality of α leads $\alpha = \alpha^+$. Now by the relational axiom* of choice there exists a univalent relation $f : X \to Y$ such that $f \sqsubseteq \alpha$ and $f\nabla_{YY} = \alpha\nabla_{YY}$. Then $\alpha \preceq f$ since $\alpha\nabla_{YY} = f\nabla_{YY}$ and $f \sqsubseteq \alpha = \alpha^+$. Again by the maximality of α we have $\alpha = f$, which proves that α is univalent. □

Theorem 3.7 *Let A be a nonempty subset of $\mathcal{D}(X,Y)$.*

(a) *The supremum of the set A in $(\mathcal{D}(X,Y), \leq)$ exists if and inly if*

$$\sqcup_{\alpha \in A}\alpha \sqsubseteq \sqcap_{\alpha \in A}\alpha^+.$$

When this condition is satisfied, the supremum is

$$\sup_{\leq} A = \sqcup_{\alpha \in A}\alpha.$$

(b) *The infimum of A in $(\mathcal{D}(X,Y), \leq)$ always exists, that is,*

$$\inf_{\leq} A = \sqcup\{\alpha_0 \mid \alpha_0 \sqsubseteq \sqcap_{\alpha \in A}\alpha \text{ and } \sqcup_{\alpha \in A}\, \alpha \sqsubseteq \alpha_0^+\}.$$

In particular, $\inf_{\leq} A = \sqcap_{\alpha \in A}\alpha$ when $\sqcup_{\alpha \in A}\alpha \sqsubseteq (\sqcap_{\alpha \in A}\alpha)^+$.

Proof. (a) Set $\alpha_0 = \sqcup_{\alpha \in A}\alpha$. We prove the existence condition and the value of the supremum. Let α' be any relation. Then

$$\begin{array}{ll} & \forall \alpha \in A : \alpha \leq \alpha' \\ \Leftrightarrow & \quad \{ \text{definition} \} \\ & \forall \alpha \in A : \alpha' \sqcap \alpha\nabla_{YY} = \alpha \wedge \alpha_0 \sqsubseteq \sqcap_{\alpha \in A}\alpha^+ \\ \Leftrightarrow & \quad \{ \Rightarrow:\ \alpha' \sqcap \alpha_0\nabla_{YY} = \sqcup_{\alpha \in A}(\alpha' \sqcap \alpha\nabla_{YY}) = \alpha_0 \\ & \qquad \Leftarrow:\ \text{Because } \alpha \sqsubseteq \alpha_0 \sqsubseteq \alpha' \text{ and } \alpha' \sqcap \alpha\nabla_{YY} = \alpha' \sqcap \alpha_0\nabla_{YY} \sqcap \\ & \quad \alpha\nabla_{YY} = \alpha_0 \sqcap \alpha\nabla_{YY} \sqsubseteq \alpha^+ \sqcap \alpha\nabla_{YY} = \alpha. \} \\ & \alpha' \sqcap \alpha_0\nabla_{YY} = \alpha_0 \wedge \alpha_0 \sqsubseteq \sqcap_{\alpha \in A}\alpha^+ \\ \Leftrightarrow & \quad \{ \text{definition} \} \\ & \alpha_0 \leq \alpha' \wedge \alpha_0 \sqsubseteq \sqcap_{\alpha \in A}\alpha^+. \end{array}$$

[1]A relational axiom of choice: for every relation $\alpha : X \rightharpoonup Y$ there exists a univalent relation $f : X \to Y$ such that $f \sqsubseteq \alpha$ and $f\nabla_{YY} = \alpha\nabla_{YY}$.

(b) Denote by A_0 the set of all lower bounds α_0 of A, that is $A_0 = \{\alpha_0 | \alpha_0 \sqsubseteq \sqcap_{\alpha\in A}\alpha \wedge \sqcup_{\alpha\in A}\alpha \sqsubseteq \alpha_0^+\}$, and set $\alpha_* = \sqcup_{\alpha_0\in A_0}\alpha_0$. Obviously A_0 is a nonempty set, since a zero relation 0_{XY} is a lower bound of A. Let α' be any relation, then we obtain

$$
\begin{array}{ll}
 & \forall\alpha \in A : \alpha' \leq \alpha \\
\Leftrightarrow & \quad \{ \text{ definition } \} \\
 & \forall\alpha \in A : \alpha \sqcap \alpha'\nabla_{YY} = \alpha' \\
\Leftrightarrow & \quad \{ \Rightarrow\!: \ \alpha_* \sqcap \alpha'\nabla_{YY} \sqsubseteq \alpha \sqcap \alpha'\nabla_{YY} = \alpha' \sqcap \alpha'\nabla_{YY} \sqsubseteq \alpha_* \sqcap \alpha'\nabla_{YY} \\
 & \quad \text{since } \alpha' \in A_0. \ \ \} \\
 & \alpha_* \sqcap \alpha'\nabla_{YY} = \alpha' \\
\Leftrightarrow & \quad \{ \text{ definition } \} \\
 & \alpha' \leq \alpha_*,
\end{array}
$$

where the second $\Leftarrow$ follows from $\alpha' \sqsubseteq \alpha_* \sqsubseteq \alpha$ and the next computation

$$
\begin{array}{lll}
\alpha & \sqsubseteq & \sqcap_{\alpha_0\in A_0}\alpha_0^+ \\
 & = & \sqcap_{\alpha_0\in A_0}(\alpha_0\nabla_{YY} \Rightarrow \alpha_0) \\
 & \sqsubseteq & \sqcap_{\alpha_0\in A_0}(\alpha_0\nabla_{YY} \Rightarrow \alpha_*) \quad \{ \text{ Proposition A.2(g): } \alpha_0 \sqsubseteq \alpha_* \ \} \\
 & = & (\sqcup_{\alpha_0\in A_0}\alpha_0\nabla_{YY}) \Rightarrow \alpha_* \quad \{ \text{ Proposition A.2(e) } \} \\
 & = & \alpha_*\nabla_{YY} \Rightarrow \alpha_* \\
 & \sqsubseteq & \alpha'\nabla_{YY} \Rightarrow \alpha'^+ \quad \{ \text{ Proposition A.2(g): } \alpha' \sqsubseteq \alpha_* \sqsubseteq \alpha'^+ \ \} \\
 & = & \alpha'^+. \quad \{ \text{ Proposition A.2(d) } \}
\end{array}
$$

□

We next see the supremum and the infimum of a chain with respect to $\leq$ found in [10].

Proposition 3.8 *Every chain A in $(\mathcal{D}(X,Y), \leq)$ has the supremum $\sup_{\leq} A = \sqcup_{\alpha\in A}\alpha$ and the infimum $\inf_{\leq} A = \sqcap_{\alpha\in A}\alpha$.*

Proof. (i) By the virtue of the last theorem it suffices to see that every chain A in $(\mathcal{D}(X,Y), \leq)$ satisfies $\sqcup_{\alpha\in A}\alpha \sqsubseteq \sqcap_{\alpha\in A}\alpha^+$. The inequality is equivalent to a fact that $\alpha' \sqsubseteq \alpha^+$ for all $\alpha', \alpha \in A$. But A is a chain, so $\alpha \leq \alpha'$ or $\alpha' \leq \alpha$. In the case of $\alpha \leq \alpha'$ it is trivial that $\alpha' \sqsubseteq \alpha^+$. Also in the case of $\alpha' \leq \alpha$ we have $\alpha' \sqsubseteq \alpha \sqsubseteq \alpha^+$.
(ii) It suffices to show that $\sqcap_{\alpha\in A}\alpha$ is a lower bound of A, that is, $\sqcup_{\alpha\in A}\alpha \sqsubseteq (\sqcap_{\alpha\in A}\alpha)^+$, which is equivalent to $\alpha' \sqcap (\sqcap_{\alpha\in A}\alpha)\nabla_{YY} \sqsubseteq \alpha$ for all $\alpha', \alpha \in A$. But A is a chain in $(\mathcal{D}(X,Y), \leq)$, so $\alpha \leq \alpha'$ or $\alpha' \leq \alpha$. In the case of $\alpha \leq \alpha'$ we have $\alpha' \sqcap (\sqcap_{\alpha\in A}\alpha)\nabla_{YY} \sqsubseteq \alpha' \sqcap \alpha\nabla_{YY} = \alpha$. Also in the case of $\alpha' \leq \alpha$ it is trivial that $\alpha' \sqcap (\sqcap_{\alpha\in A}\alpha)\nabla_{YY} \sqsubseteq \alpha' \sqsubseteq \alpha$. □

We now see the supremum and the infimum with respect to $\preceq$ found in [5].

Proposition 3.9 *Let A be a nonempty subset of $\mathcal{D}(X,Y)$.*

(a) *The supremum of the set A in $(\mathcal{D}(X,Y), \preceq)$ exists if and only if*

$$\sqcup_{\alpha\in A}\alpha\nabla_{YY} \sqsubseteq (\sqcap_{\alpha\in A}\alpha^+)\nabla_{YY}.$$

When this condition is satisfied, the supremum is

$$\sup_{\preceq} A = (\sqcup_{\alpha\in A}\alpha\nabla_{YY}) \sqcap (\sqcap_{\alpha\in A}\alpha^+).$$

(b) *The infimum of A in $(\mathcal{D}(X,Y), \preceq)$ always exists, that is,*

$$\inf_{\preceq} A = (\sqcup_{\alpha\in A}\alpha) \sqcap (\sqcap_{\alpha\in A}\alpha\nabla_{YY}),$$

Proof. (a) Set $\alpha_0 = (\sqcup_{\alpha\in A}\alpha\nabla_{YY}) \sqcap (\sqcap_{\alpha\in A}\alpha^+)$. Noting that when the condition $\sqcup_{\alpha\in A}\alpha\nabla_{YY} \sqsubseteq (\sqcap_{\alpha\in A}\alpha^+)\nabla_{YY}$ holds

$$\alpha_0\nabla_{YY} = (\sqcup_{\alpha\in A}\alpha\nabla_{YY}) \sqcap (\sqcap_{\alpha\in A}\alpha^+)\nabla_{YY} = \sqcup_{\alpha\in A}\alpha\nabla_{YY}$$

and so α_0 can be rewritten to

$$\alpha_0 = \alpha_0\nabla_{YY} \sqcap (\sqcap_{\alpha\in A}\alpha^+),$$

we prove the existence condition and the value of supremum of A. Let α' be any relation. We have

$$\begin{array}{ll} & \forall\alpha \in A : \alpha \preceq \alpha' \\ \Leftrightarrow & \quad \{ \text{ definition } \} \\ & \sqcup_{\alpha\in A}\alpha\nabla_{YY} \sqsubseteq \alpha'\nabla_{YY} \wedge \alpha' \sqsubseteq \sqcap_{\alpha\in A}\alpha^+ \wedge \sqcup_{\alpha\in A}\alpha\nabla_{YY} \sqsubseteq (\sqcap_{\alpha\in A}\alpha^+)\nabla_{YY} \\ \Leftrightarrow & \quad \{ \Rightarrow\!: \alpha' \sqcap \alpha_0\nabla_{YY} \sqsubseteq (\sqcap_{\alpha\in A}\alpha^+) \sqcap \alpha_0\nabla_{YY} = \alpha_0 \ \} \\ & \alpha_0\nabla_{YY} \sqsubseteq \alpha'\nabla_{YY} \wedge \alpha' \sqsubseteq \alpha_0^+ \wedge \sqcup_{\alpha\in A}\alpha\nabla_{YY} \sqsubseteq (\sqcap_{\alpha\in A}\alpha^+)\nabla_{YY} \\ \Leftrightarrow & \quad \{ \text{ definition } \} \\ & \alpha_0 \preceq \alpha' \wedge \sqcup_{\alpha\in A}\alpha\nabla_{YY} \sqsubseteq (\sqcap_{\alpha\in A}\alpha^+)\nabla_{YY}, \end{array}$$

where the second $\Leftarrow$ follows from

$$\begin{array}{lll} & \alpha' \sqcap \alpha\nabla_{YY} & \\ \sqsubseteq & \alpha_0^+ \sqcap \alpha\nabla_{YY} & \{ \ \alpha' \sqsubseteq \alpha_0^+ \ \} \\ = & (\alpha_0\nabla_{YY} \Rightarrow \alpha_0) \sqcap \alpha\nabla_{YY} & \\ = & \alpha_0 \sqcap \alpha\nabla_{YY} & \{ \text{ Proposition A.2(f): } \alpha\nabla_{YY} \sqsubseteq \alpha_0\nabla_{YY} \ \} \\ \sqsubseteq & \alpha_0 & \\ \sqsubseteq & \alpha^+, & \end{array}$$

which implies $\alpha' \sqsubseteq \alpha\nabla_{YY} \Rightarrow \alpha^+ = \alpha^+$ for each $\alpha \in A$ by Proposition A.2(d).
(b) Set $\alpha_0 = (\sqcup_{\alpha\in A}\alpha) \sqcap (\sqcap_{\alpha\in A}\alpha\nabla_{YY})$. Then $\alpha_0\nabla_{YY} = \sqcap_{\alpha\in A}\alpha\nabla_{YY}$ and so $\alpha_0 = (\sqcup_{\alpha\in A}\alpha) \sqcap \alpha_0\nabla_{YY}$. So we have the following equivalences for any given relation α'

$$\begin{array}{lll} & \forall\alpha \in A : \alpha' \preceq \alpha & \\ \Leftrightarrow & \forall\alpha \in A : \alpha'\nabla_{YY} \sqsubseteq \alpha\nabla_{YY} \wedge \alpha \sqsubseteq \alpha'^+ & \{ \text{ definition } \} \\ \Leftrightarrow & \alpha'\nabla_{YY} \sqsubseteq \alpha_0\nabla_{YY} \wedge \sqcup_{\alpha\in A}\alpha \sqsubseteq \alpha'^+ & \{ \text{ definition } \} \\ \Leftrightarrow & \alpha'\nabla_{YY} \sqsubseteq \alpha_0\nabla_{YY} \wedge \alpha_0 \sqsubseteq \alpha'^+ & \{ \Rightarrow\!: \alpha_0 \sqsubseteq \sqcup_{\alpha\in A}\alpha \sqsubseteq \alpha'^+ \ \} \\ \Leftrightarrow & \alpha' \preceq \alpha_0, & \{ \text{ definition } \} \end{array}$$

where the third $\Leftarrow$ is shown as follows. Consider the following computation

$$(\sqcup_{\alpha\in A}\alpha) \sqcap \alpha'\nabla_{YY} \sqsubseteq (\sqcup_{\alpha\in A}\alpha) \sqcap \alpha_0\nabla_{YY} = \alpha_0 \sqsubseteq \alpha'^+,$$

which implies $\sqcup_{\alpha\in A}\alpha \sqsubseteq \alpha'^+$ by Proposition A.2(d). □

Lemma 3.10 *Let $\alpha : X \rightharpoondown Y$ and $\beta : X \rightharpoondown Z$ be relations. Then $\alpha \odot \beta = \alpha\beta \sqcap (\alpha^+ \ominus \beta\nabla_{ZZ})$.*

Proof.

$$\begin{array}{rcll} \alpha \odot \beta & = & \alpha\beta \sqcap (\alpha \ominus \beta\nabla_{ZZ}) & \\ & = & \alpha\beta \sqcap (\alpha\nabla_{YY} \sqcap \alpha^+) \ominus \beta\nabla_{ZZ} & \{ \ \alpha = \alpha\nabla_{YY} \sqcap \alpha^+ \ \} \\ & = & \alpha\beta \sqcap (\alpha\nabla_{YZ} \Rightarrow (\alpha^+ \ominus \beta\nabla_{ZZ})) & \{ \text{ Proposition A.3(h) } \} \\ & = & \alpha\beta \sqcap (\alpha^+ \ominus \beta\nabla_{ZZ}) & \{ \text{ Proposition A.2(f) } \} \end{array}$$

□

In the following discussion, a map which is monotonic with respect to $\le$ or $\preceq$ is called $\le$-monotonic or $\preceq$-monotonic, respectively. The next proposition shows that the demonic composition $\odot$ is $\le$-monotonic and $\preceq$-monotonic.

Proposition 3.11 *Let $\alpha, \xi : X \rightharpoondown Y$ and $\beta : Y \rightharpoondown Z$ be relations. Then the following hold:*

(a) *If* $\alpha \le \alpha'$ *and* $\beta \le \beta'$, *then* $\alpha \odot \beta \le \alpha' \odot \beta'$.

(b) *If* $\alpha \preceq \alpha'$ *and* $\beta \le \beta'$, *then* $\alpha \odot \beta \preceq \alpha' \odot \beta'$.

Proof. (a) Assume that $\alpha \sqsubseteq \alpha' \sqsubseteq \alpha^+$ and $\beta \sqsubseteq \beta' \sqsubseteq \beta^+$. Then

$$\begin{array}{rcll} \alpha \odot \beta & = & \alpha\beta \sqcap (\alpha^+ \ominus \beta\nabla_{ZZ}) & \{ \text{ Lemma 3.10 } \} \\ & \sqsubseteq & \alpha'\beta' \sqcap (\alpha' \ominus \beta'\nabla_{ZZ}) & \{ \alpha \sqsubseteq \alpha' \sqsubseteq \alpha^+,\ \beta \sqsubseteq \beta' \ \} \\ & = & \alpha' \odot \beta'. & \end{array}$$

and

$$\begin{array}{cll} & (\alpha' \odot \beta') \sqcap (\alpha \odot \beta)\nabla_{ZZ} & \\ \sqsubseteq & \alpha'\beta' \sqcap \alpha\nabla_{YZ} \sqcap (\alpha^+ \ominus \beta\nabla_{ZZ}) & \\ & \{ (\alpha \odot \beta)\nabla_{ZZ} = \alpha\beta\nabla_{ZZ} \sqcap (\alpha^+ \ominus \beta\nabla_{ZZ}) \sqsubseteq \alpha\nabla_{YZ} \sqcap (\alpha^+ \ominus \beta\nabla_{ZZ}) \ \} & \\ \sqsubseteq & \alpha^+\beta' \sqcap (\alpha^+ \ominus \beta\nabla_{ZZ}) \sqcap \alpha\nabla_{YZ} & \{ \alpha' \sqsubseteq \alpha^+ \ \} \\ \sqsubseteq & \alpha^+(\beta' \sqcap \beta\nabla_{ZZ}) \sqcap \alpha\nabla_{YZ} & \{ \text{ Dedekind formula and Proposition A.3(a) } \} \\ \sqsubseteq & \alpha^+\beta \sqcap \alpha\nabla_{YZ} & \{ \beta' \sqcap \beta\nabla_{ZZ} \sqsubseteq \beta \text{ by } \beta' \sqsubseteq \beta^+ \ \} \\ = & (\alpha^+ \sqcap \alpha\nabla_{YY})\beta & \{ \text{ Proposition A.1(b) } \} \\ = & \alpha\beta. & \end{array}$$

Hence $(\alpha' \odot \beta') \sqcap (\alpha \odot \beta)\nabla_{ZZ} \sqsubseteq \alpha \odot \beta$ and so $\alpha' \odot \beta' \sqsubseteq (\alpha \odot \beta)^+$.
(b) Assume that $\alpha\nabla_{YY} \sqsubseteq \alpha'\nabla_{YY}$, $\beta\nabla_{ZZ} \sqsubseteq \beta'\nabla_{ZZ}$, $\alpha' \sqsubseteq \alpha^+$ and $\beta' \sqsubseteq \beta^+$. First note that $\alpha^+ \ominus \beta\nabla_{ZZ} \sqsubseteq \alpha' \ominus \beta'\nabla_{ZZ}$ by the assumptions $\alpha' \sqsubseteq \alpha^+$, $\beta\nabla_{ZZ} \sqsubseteq \beta'\nabla_{ZZ}$ and Proposition A.3(i). Then

$$\begin{array}{cll} & (\alpha \odot \beta)\nabla_{ZZ} & \\ = & \alpha\beta\nabla_{ZZ} \sqcap (\alpha^+ \ominus \beta\nabla_{ZZ}) & \{ \text{ Lemma 3.10, Proposition A.1(b) and A.3(e) } \} \\ \sqsubseteq & \alpha\beta\nabla_{ZZ} \sqcap (\alpha' \ominus \beta'\nabla_{ZZ}) & \\ \sqsubseteq & \alpha'\nabla_{YZ} \sqcap (\alpha' \ominus \beta'\nabla_{ZZ}) & \{ \text{ assumption } \} \\ \sqsubseteq & \alpha'\beta'\nabla_{ZZ} \sqcap (\alpha' \ominus \beta'\nabla_{ZZ}) & \{ \text{ Dedekind formula and Proposition A.3(a) } \} \\ \sqsubseteq & (\alpha' \odot \beta')\nabla_{ZZ}. & \end{array}$$

We have to see $\alpha' \odot \beta' \sqsubseteq (\alpha \odot \beta)^+$, but this claim can be shown by the same argument of the second part in the proof for (a). □

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A Basic Properties of Relations

In this section we list a few basic properties of relations.

Proposition A.1 *Let $\alpha : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$, $\eta : X \rightharpoonup W$ and $u : X \rightharpoonup X$ be relations in a Dedekind category $\mathcal{D}$. Then the following hold:*

(a) *If $u \sqsubseteq \mathrm{id}_X$ then $u\nabla_{XY} \sqcap \alpha = u\alpha$.*

(b) $(\alpha \sqcap \eta\nabla_{WY})\beta = \alpha\beta \sqcap \eta\nabla_{WZ}$.

□

Proposition A.2 *Let $\alpha, \beta, \gamma : X \rightharpoonup Y$ be relations in a Dedekind category $\mathcal{D}$. Then the following hold:*

(a) $\beta \sqsubseteq \alpha \Rightarrow \beta$.

(b) $\alpha \Rightarrow \alpha = \nabla_{XY}$ *and* $\nabla_{XY} \Rightarrow \alpha = \alpha$.

(c) $\alpha \sqcap (\alpha \Rightarrow \beta) = \alpha \sqcap \beta$. *In particular* $\alpha\nabla_{YY} \sqcap \alpha^+ = \alpha$.

(d) $\alpha \Rightarrow (\beta \Rightarrow \gamma) = (\alpha \sqcap \beta) \Rightarrow \gamma$. *In particular* $\alpha\nabla_{YY} \Rightarrow \alpha^+ = \alpha^+$.

(e) $(\alpha \sqcup \alpha') \Rightarrow \beta = (\alpha \Rightarrow \beta) \sqcap (\alpha' \Rightarrow \beta)$.

(f) *If $\alpha \sqsubseteq \beta$, then $\alpha \sqcap (\beta \Rightarrow \gamma) = \alpha \sqcap \gamma$.*

(g) *If $\alpha \sqsupseteq \alpha'$ and $\beta \sqsubseteq \beta'$, then $\alpha \Rightarrow \beta \sqsubseteq \alpha' \Rightarrow \beta'$.*

□

Proposition A.3 *Let $\alpha, \alpha' : X \rightharpoonup Y$, $\beta, \beta' : Y \rightharpoonup Z$, $\delta : Z \rightharpoonup W$ and $\xi : X \rightharpoonup W$ be relations in a Dedekind category $\mathcal{D}$. Then the following hold:*

(a) $\alpha^\sharp(\alpha \ominus \beta) \sqsubseteq \beta$.

(b) $(\alpha \ominus \beta)\delta \sqsubseteq \alpha \ominus (\beta\delta)$.

(c) $\alpha \ominus (\beta \sqcap \beta') = (\alpha \ominus \beta) \sqcap (\alpha \ominus \beta')$.

(d) $(\alpha\beta) \ominus \delta = \alpha \ominus (\beta \ominus \delta)$.

(e) *If $\nabla_{WZ}\nabla_{ZW} = \nabla_{WW}$, then $(\alpha \ominus \beta\nabla_{ZZ})\nabla_{ZW} = \alpha \ominus \beta\nabla_{ZW}$.*

(f) $\alpha(\beta \sqcap \beta') \sqcap \alpha \ominus \beta' = \alpha\beta \sqcap \alpha \ominus \beta'$.

(g) $(\xi\nabla_{WX} \sqcap \mathrm{id}_X) \ominus \alpha = \xi\nabla_{WY} \Rightarrow \alpha$.

(h) $(\xi\nabla_{WY} \sqcap \alpha) \ominus \beta = \xi\nabla_{WZ} \Rightarrow (\alpha \ominus \beta)$.

(i) *If $\alpha \sqsupseteq \alpha'$ and $\beta \sqsubseteq \beta'$, then $\alpha \ominus \beta \sqsubseteq \alpha' \ominus \beta'$.*

□