Embedding into Wreath Product and the Yoneda Lemma

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Abstract
The Kaloujnine-Krasner theorem is an immediate consequence of an easy theorem for the 2-category of groupoids and a 2-categorical variation of the Yoneda lemma.

1 Introduction

We give a new proof of the Kaloujnine-Krasner theorem in group theory [6]. This theorem asserts that every group extension embeds into a wreath product. Indeed we give a proof of the generalized theorem for twisted wreath product by Neumann [4].

The purpose of this note is to show that the pure group-theoretic Kaloujnine-Krasner theorem naturally arises as a consequence of a general fact for groupoids and a fundamental tool of general category theory, the Yoneda lemma. In particular, the use of the Yoneda lemma elucidates the reason wreath product naturally arises.

Given two groups N and G, an extension of N by G is a group F satisfying the short exact sequence

$$1 \longrightarrow N \longrightarrow F \longrightarrow G \longrightarrow 1.$$

In other words, an extension F is a group having N as a normal subgroup and satisfying $F/N \cong G$. The Kaloujnine-Krasner theorem asserts that every extension of N by G embeds into the standard wreath product $N \operatorname{Wr} G$.

The standard wreath product is defined as semidirect product $N^G \rtimes G$ where N^G is simply cartesian product of m copies of N where m is the order of G. An element of $N \operatorname{Wr} G$ is a pair of an element g and an m-tuple $\langle n_h \rangle_{h \in G}$ of elements of N. This is best illustrated in the form of matrices. Regarding G acting on the set G of m elements with regular action, each element $g \in G$ yields an $m \times m$ permutation matrix. Then the pair $(g, \langle n_h \rangle)$ may be regarded as a matrix where the non-zero entry in each column of the permutation matrix is replaced with the corresponding element n_h.

Example: (i) Let S_2 be the symmetric group of all permutations over two letters, and let C_n be the cyclic group of order n. Direct product $C_n \times S_2$ is an extension of C_n by S_2, generated by two element x and y satisfying three relations $x^2 = 1$, $y^n = 1$ and $xyx = y$. The group $C_n \times S_2$ embeds into $C_n \operatorname{Wr} S_2$ as

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad y \mapsto \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$$

where, in the last matrix, we abuse y also as the generator of C_n. It is easy to see that these matrices fulfill the three relations above.

(ii) The dihedral group D_{2n} is defined as the semidirect product $C_n \rtimes S_2$. Namely it is generated by two elements x and y subject to relations $x^2 = 1$, $y^n = 1$, and $xyx = y^{-1}$. The group D_{2n} embeds into C_n Wr S_2 as

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad y \mapsto \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$$

These matrices fulfill the defining three relations of the dihedral group.

We assume knowledge of category theory and bicategory theory. The standard literature is [3, 1]. With respect to the terminology for bicategories, we use pseudo-functors, quasi-natural transformation, and modifications. These are weak ones. For example, pseudo-functor preserves composition up to invertible 2-cells satisfying due coherence conditions.

2 Groupoids and the Yoneda lemma

A *groupoid* is a small category where all morphisms are invertible. Every group is regarded as a groupoid such that there is only one object: the elements of the group become the morphisms on the single object. Given a groupoid G, we write $G(x, y)$ the homset of all morphisms $x \xrightarrow{g} y$ in G.

A *groupoid homomorphism* is simply a functor between groupoids. A *natural isomorphism* (also called homotopy) between homomorphisms is defined as the usual natural transformation in category theory. Since all morphisms in groupoids are invertible, a natural transformation automatically turns out to be an isomorphism.

We let **Gpoid** denote the 2-category of groupoids, groupoid homomorphisms, and natural isomorphisms. Every 2-cell is invertible. Namely $\mathrm{Hom}(A, B)$ is a groupoid. In general, we call a 2-category where all 2-cells are invertible a *groupoid-enriched* category.

We will encounter several other groupoid-enriched categories in this paper:

Example: Let G be a groupoid.

(i) The groupoid enriched-category $\mathbf{Gpoid}^G$ is defined. Its objects are all pseudo-functors from G into $\mathbf{Gpoid}$. Its 1-cells are quasi-natural transformations, and its 2-cells are modifications.

(ii) The slice groupoid-enriched category $\mathbf{Gpoid}/G$ is defined. The objects are all pairs (A, f) of a groupoid A and a groupoid homomorphism $A \xrightarrow{f} G$. The 1-cells $(A, f) \xrightarrow{(k,\mu)} (B, g)$ are such that $A \xrightarrow{k} B$ is a groupoid homomorphism and $gk \overset{\mu}{\Rightarrow} f$ is a natural isomorphism:

$$\begin{tikzcd} A \arrow[rr, "k"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\mu}{\Leftarrow}"] & B \arrow[dl, "g"] \\ & G & \end{tikzcd}.$$

The 2-cells $(k, \mu) \overset{\gamma}{\Rrightarrow} (k', \mu')$ are such that γ is a natural isomorphism between k and k' rendering the diagram

$$\begin{tikzcd} gk \arrow[rr, "g\gamma"] \arrow[dr, "\mu"'] & & gk' \arrow[dl, "\mu'"] \\ & f & \end{tikzcd}$$

of natural isomorphisms commutative.

The following definition plays an important role throughout the rest of the paper.

2.1 Definition

Let t be an object of $\mathbf{Gpoid}^G$ where G is a groupoid.

The *Grothendieck construction* $\int_{x\in G} t(x)$ is the groupoid defined as follows: Its objects are the pairs (x, a) of all objects $x \in G$ and all objects $a \in t(x)$; Its morphisms $(x, a) \xrightarrow{(g,\gamma)} (x', a')$ are such that $x \xrightarrow{g} x'$ is a morphism in G and $g \cdot a \xrightarrow{\gamma} a'$ a morphism in $t(x')$, where $g \cdot a$ denotes $t(g)(a)$.

Remark: The Grothendieck construction $\int_{x\in G} t(x)$ turns out to be a bicolimit of the pseudo-functor $G \xrightarrow{t} \mathbf{Gpoid}$ for groupoid G.

The following lemma gives a simple computational rule for the Grothendieck construction. The first one is an analogy of representing double integral by iteration of ordinary integral. The second is of Fubini's theorem.

2.2 Lemma

(i) *Let A be a groupoid. Moreover, let $A \xrightarrow{t} \mathbf{Gpoid}$ and $\int_{x\in A} t(x) \xrightarrow{u} \mathbf{Gpoid}$ be pseudo-functors. Then equivalence of groupoids*

$$\int_{(x,a)\in\int_{x\in A} t(x)} u(x,a) \;\cong\; \int_{x\in A}\int_{a\in t(x)} u(x,a)$$

holds.

(ii) *Let A and B be groupoids, and let $A\times B \xrightarrow{u}$ **Gpoid** be a pseudo-functor. Then equivalence of groupoids*

$$\int_{x\in A}\int_{y\in B} u(x,y) \;\cong\; \int_{x\in B}\int_{y\in A} u(x,y)$$

holds.

(*Proof*) (i) is straightforward (or follows from a general fact for bicolimits). (ii) is derived from (i) by taking as t the constant functor to B and remarking $A\times B \cong \int_{x\in A} B \cong B\times A$. $\square$

The Yoneda lemma is one of the most fundamental machinery in ordinary category theory [3]. The following lemma is an extension of the lemma to groupoid-enriched categories (the lemma for general bicategories is found in [5]). In the statement, x/G denotes the slice category (indeed groupoid) where the objects are all pairs (z,f) of an object z and a morphism $x \xrightarrow{f} z$ in G, and the morphisms $(z,f) \xrightarrow{g} (z',f')$ is a morphism $z \xrightarrow{g} z'$ in G satisfying $gf = f'$. In a dual way, G/x is defined.

2.3 Lemma
*Let G be a groupoid and let $G \xrightarrow{t}$ **Gpoid** be a pseudo-functor.*

The following equivalences of groupoids hold:

$$\varprojlim_{(z,f)\in x/G} t(z) \;\cong\; t(x) \;\cong\; \varinjlim_{(z,f)\in G/x} t(z).$$

Moreover these equivalences are quasi-natural in x.

The leftmost groupoid in this lemma denotes a bilimit of the pseudo-functor t preceded by obvious projection $x/G \longrightarrow G$. Likewise the rightmost is a bicolimit. The proof of the lemma is parallel to the one for ordinary categories.

Remark: The lemma remains to hold for general groupoid-enriched category $\mathbf{C}$ and a pseudo-functor $\mathbf{C} \xrightarrow{t}$ **Gpoid**. In this paper, we use only the special form above.

We can write the equivalences of the lemma in terms of hom-groupoid and the Grothendieck construction:

$$\mathrm{Hom}_{\mathbf{Gpoid}^G}(G(x,\text{-}),\, t) \;\cong\; t(x) \;\cong\; \int_{z\in G} t(z)G(z,x).$$

In the rightmost, concatenation is cartesian product of groupoid $t(z)$ with set (i.e., discrete groupoid) $G(z,x)$.

The following simple fact is the first vehicle for verification of the Kaloujnine-Krasner theorem.

2.4 Theorem

Let G be a groupoid.

Then biequivalence $\mathbf{Gpoid}^G \cong \mathbf{Gpoid}/G$ between groupoid-enriched categories holds.

(*Proof*) A 2-functor $\mathbf{Gpoid}^G \xrightarrow{F} \mathbf{Gpoid}/G$ is simply the Grothendieck construction. Namely F carries each object $t \in \mathbf{Gpoid}^G$ to $\int_{z\in G} t(z)$ with an obvious projection to G. Also a 2-functor in the reverse direction $\mathbf{Gpoid}/G \xrightarrow{U} \mathbf{Gpoid}^G$ involves the construction. Given an object (A,p) of $\mathbf{Gpoid}/G$, the object $U(A,p)$ is defined as the strict (pseudo-)functor $\int_{x\in A} G(p(x),\text{-})$.

Let us prove $UF \cong id$ (an identity functor). We have

$$UF(t) \;\cong\; \int_{(z,a)\,\int_{z\in G} t(z)} G(z,\text{-}).$$

By Lem. 2.2, it is equivalent to $\int_{z\in G}\int_{a\in t(z)} G(z,\text{-})$, that is, $\int_{z\in G} t(z)G(z,\text{-})$. By the Yoneda lemma, the last is equivalent to t. Next we prove $FU \cong id$. We have

$$FU(A,p) = \int_{z\in G}\int_{x\in A} G(p(x),z),$$

which is equivalent to $\int_{x\in A}\int_{z\in G} G(p(x),z)$ by Fubini. The internal integral is equivalent to a trivial groupoid 1 by the Yoneda lemma, noticing $G(p(x),z) \cong G(z,p(x))$ (or by a simple direct argument). Hence $FU(A,p) \cong A$. □

Remark: The construction $FU(A,p)$ in the proof above gives an fibration [2] (also called opfibration in the literature) equivalent to the original $A \xrightarrow{p} G$.

This theorem looks straightforward, but it conceals a deep consequence. Let us consider the special case where G is a group. Given another group N, we want to characterize all groups F satisfying the short exact sequence

$$1 \longrightarrow N \longrightarrow F \longrightarrow G \longrightarrow 1.$$

In other words, F is a group having (a copy of) N as a normal subgroup and satisfying $F/N \cong G$. By the theorem, we can associate an object of $\mathbf{Gpoid}^G$ given as $t = \int_{x\in F} G(px, \text{-})$ where $F \xrightarrow{p} G$ is the canonical surjection. For a unique object $z \in G$, the groupoid $t(z)$ is equivalent to the group N. Conversely, if $t(z)$ is a group, the associated Grothendieck construction $\int_{z\in G} t(z)$ is obviously a group endowed with a surjection onto G. Therefore the problem to characterize all group extensions F amounts to the problem to give all pseudo-functors $t \in \mathbf{Gpoid}^G$ such that $t(z) = N$.

This observation yields the traditional theory of Schreier's factor sets in group theory [6]. This theory tells us that all group extensions are obtained from the following data: a family of automorphisms $g \cdot (\text{-})$ on N for $g \in G$, and a family of elements $\tilde{\varphi}_{g,h}$ of N for $g, h \in G$, all these satisfying the equalities:

$$\begin{aligned} gh \cdot x &= \tilde{\varphi}_{g,h}(g \cdot h \cdot x)\tilde{\varphi}_{g,h}^{-1} \\ \tilde{\varphi}_{gh,k}\tilde{\varphi}_{g,h} &= \tilde{\varphi}_{g,hk}(g \cdot \tilde{\varphi}_{h,k}) \end{aligned}$$

where $g, h, k \in G$ and $x \in N$. The second equality is exactly the coherence condition for pseudo-functors. The first simply says that $\tilde{\varphi}_{g,h}$ is a natural isomorphism. (For a general pseudo-functor, we have also structural 2-cells involving identities. In this case, however, we do not need them, for they are determined from the condition of $\tilde{\varphi}$.) For the reader's convenience, we record the construction of group F from a factor set. The underlying set is $N \times G$. Multiplication is given as $(x, g) \cdot (y, h) = (x(g \cdot y)\tilde{\varphi}_{g,h}^{-1}, gh)$. The unit is $(\tilde{\varphi}_{e,e}, e)$ where e is the unit of G.

Two different factor sets may yield isomorphic groups. Quasi-natural equivalence between pseudo-functors yields the condition for that. Two factors sets $(\cdot, \tilde{\varphi})$ and $(\cdot', \tilde{\psi})$ give isomorphic groups if there is a family of elements $z_g \in N$ for $g \in G$, subject to the equalities:

$$\begin{aligned} g \cdot' x &= z_g(g \cdot x)z_g^{-1} \\ \tilde{\psi}_{g,h} &= z_{gh}\tilde{\varphi}_{g,h}z_g^{-1}(g \cdot' z_h)^{-1} \end{aligned}$$

where $x \in N$ and $g, h \in G$. Again the second equality is exactly the coherence condition for quasi-natural transformations.

3 Twisted wreath product and the Kaloujnine-Krasner theorem

Let G be a group acting on a set Y from right. For a group N, the *general wreath product* is defined as semidirect product $N^Y \rtimes G$ [6]. Here G acts on cartesian product N^Y from left as permutation of components. Namely, for a function $Y \xrightarrow{\theta} N$ and $g \in G$, we define the function $g \cdot \theta$ by equality $(g \cdot \theta)(y) = \theta(y \cdot g)$ for $y \in Y$.

The group N^Y is equivalent to the hom-groupoid $\mathrm{Hom}_{\mathbf{Gpoid}}(Y, N)$, where Y is regarded as a discrete groupoid and N as a groupoid with a single object. Therefore the general wreath product can be written $\int_{z\in G} \mathrm{Hom}_{\mathbf{Gpoid}}(\varphi(z), N)$ where $G^{\mathrm{op}} \xrightarrow{\varphi} \mathbf{Gpoid}$ is the functor carrying the unique object $z \in G$ to the discrete groupoid Y and morphisms of G to the corresponding right actions on Y.

Our proof of the Kaloujnine-Krasner theorem involves standard wreath product as well as twisted wreath product [6].

3.1 Definition
Let N and G be a group. Let us assume for (ii), in addition, that H is a subgroup of G endowed with a left action on N (note that H acts also on G by left multiplication).

(i) The *standard wreath product* $N \,\mathrm{Wr}\, G$ is defined as the semidirect product $N^G \rtimes G$ where N^G is the group of all functions on G into N with pointwise multiplication and the left action of G on N^G is induced by the regular right action of G on G itself.

(ii) The *twisted wreath product* $N \,\mathrm{Wr}_H\, G$ is defined similarly to the standard wreath product, except that the first component is restricted to those functions $G \to N$ commuting with left actions of H.

The standard wreath product $N \,\mathrm{Wr}\, G$ is a special case of general wreath product $N^Y \rtimes G$ where Y is equal to G. Furthermore, the standard wreath product is a special case of the twisted wreath product $N \,\mathrm{Wr}_H\, G$ where H is the unit group.

From the observation above for general wreath product, the standard wreath product $N \,\mathrm{Wr}\, G$ can be written $\int_{z\in G} \mathrm{Hom}_{\mathbf{Gpoid}}(\varphi(z), N)$ where the functor $G^{\mathrm{op}} \xrightarrow{\varphi} \mathbf{Gpoid}$ carries the unique object $z \in G$ to the set (i.e., discrete groupoid) of elements of G, and the morphisms of G to the right multiplication by the corresponding elements.

We want to give a similar characterization for twisted wreath product. Let us consider the groupoid-enriched category $\mathbf{Gpoid}^H$. Then φ in the previous paragraph, in fact, yields a functor into $\mathbf{Gpoid}^H$, where the left action of H on $\varphi(z) = G$ is given by left multiplication. Moreover, we can consider the given action $H \to \mathrm{Aut}(N)$ as the (strict) functor $t \in \mathbf{Gpoid}^H$ carrying the unique object of H to N. We show that $\int_{z\in G} \mathrm{Hom}_{\mathbf{Gpoid}^H}(\varphi(z), t)$ is equivalent to the twisted wreath product $G \,\mathrm{Wr}_H\, N$.

3.2 Lemma
Let H be a subgroup of a group G, and let N be a group endowed with a left action $H \to \mathrm{Aut}\, N$, regarded as a functor $H \xrightarrow{t} \mathbf{Gpoid}$. Moreover, let $G^{\mathrm{op}} \xrightarrow{\varphi}$

$\mathbf{Gpoid}^H$ *be the functor carrying the unique object* $z \in G$ *to the set* G *with left action of* H *given by multiplication.*

Then the hom-groupoid $\mathrm{Hom}_{\mathbf{Gpoid}^H}(\varphi(z), t)$ *is equivalent to a group. The elements of the group are the functions* θ *on* G *into* N *commuting with the left action of* H*, that is, satisfying* $\theta(hg) = h \cdot \theta(g)$*. Multiplication is pointwise.*

(*Proof*) First we verify that the hom-groupoid is equivalent to a group. It suffices to show every quasi-natural transformation $\varphi(z) \xrightarrow{\nu} t$ is isomorphic to a strict (quasi-)natural transformation $\varphi(z) \xrightarrow{\tilde{\nu}} t$ (strictness means all structural 2-cells are identities), since such a strict one is unique for N has only one object.

Quasi-naturality of ν amounts to giving a family of elements $\nu_h(g)$ in group $t(z)$ for $h \in H$ and $g \in G$, rendering the diagram

$$\begin{tikzcd} * \arrow[rr, "\nu_h(kg)"] \arrow[dr, "\nu_{hk}(g)"'] & & * \arrow[dl, "h\cdot\nu_k(g)"] \\ & * & \end{tikzcd}$$

commutative for $h, k \in H$ and $g \in G$, where $*$ denotes the unique object of group $t(z)$, and $h \cdot (\text{-})$ denotes $t(h)(\text{-})$. Modification $\nu \xrightarrow{\theta} \tilde{\nu}$ amounts to giving a family of $\theta(g) \in t(z)$ for $g \in G$, rendering

$$\begin{tikzcd} * \arrow[rr, "\nu_h(g)"] \arrow[dr, "\theta(hg)"'] & & * \arrow[dl, "h\cdot\theta(g)"] \\ & * & \end{tikzcd}$$

commutative for $h \in H$ and $g \in G$, noticing that $\tilde{\nu}_h(g)$ is identity by assumption of strictness of $\tilde{\nu}$.

Let X be a right transversal of H in G, that is, the set of chosen representatives from the right cosets of H in G. We fix X once and for all. For each $g \in G$, there is a unique $g^\tau \in X$ such that $Hg = Hg^\tau$. We note also $(hg)^\tau = g^\tau$ for $h \in H$. Now let us define $\theta(g)$ by $\nu_{h_0}(g^\tau)$ where $h_0 = g(g^\tau)^{-1}$ is an element of H. Then it is easy to see that the triangle diagram of quasi-naturality of ν implies that of modification. This ends the proof that every quasi-natural transformation is isomorphic to the strict $\tilde{\nu}$.

Furthermore, if $\nu = \tilde{\nu}$, each modification $\tilde{\nu} \xrightarrow{\theta} \tilde{\nu}$ should satisfy $\theta(hg) = h \cdot \theta(g)$ as immediately seen from the diagram above for modification. $\square$

Remark: Let us suppose that the functor t is defined on G. Then a consequence of this lemma (indeed equivalent to it) is that every quasi-natural transformation $\varphi(z) \xrightarrow{\nu} t$ in $\mathbf{Gpoid}^H$ extends to $\mathbf{Gpoid}^G$. Namely every ν defined on subgroup H extends to full domain G. This is proved by defining $\varphi(z) \xrightarrow{\nu^\dagger} t$ in $\mathbf{Gpoid}^G$ with equality $\nu^\dagger_g(g') = (g \cdot \theta(g'))^{-1}\theta(gg')$ for all $g, g' \in G$.

From this lemma, we conclude that the twisted wreath product $N\,\mathrm{Wr}_H\,G$ is equivalent to $\int_{z\in G}\mathrm{Hom}_{\mathbf{Gpoid}^H}(\varphi(z),t)$.

We verify an extension of the Kaloujnine-Krasner theorem to twisted wreath product. We start with the following general lemma without a proof.

3.3 Lemma

Let $\mathbf{C}\xrightarrow{F}\mathbf{D}$ be a pseudo-functor between bicategories.

If we are given a family of objects $G(X)\in\mathbf{D}$ satisfying $F(X)\cong G(X)$ for all objects $X\in\mathbf{C}$, then G gives rise to a pseudo-functor, and F and G turn out to be quasi-naturally equivalent.

3.4 Theorem

Let F be a group extension of N by G. Let us assume that F splits on a subgroup $H\leq G$, that is, there is an injective group homomorphism $H\xrightarrow{\sigma}F$ yielding an identity on H by composing the canonical projection $F\to G$.

Under these conditions, F embeds into the twisted wreath product $N\,\mathrm{Wr}_H\,G$ where the left action of H on N is given by $h\cdot n=\sigma(h)n\sigma(h)^{-1}$.

(*Proof*) By Thm. 2.4, the canonical projection $F\xrightarrow{p}G$ corresponds to the functor $t=\int_{x\in F}G(p(x),\text{-})$ in $\mathbf{Gpoid}^G$. We can construct a quasi-natural transformation ν in $\mathbf{Gpoid}^G$ such that

$$\nu_z\ :\ t(z)\ \longrightarrow\ \mathrm{Hom}_{\mathbf{Gpoid}^H}(\varphi(z),t)$$

where $G^{\mathrm{op}}\xrightarrow{\varphi}\mathbf{Gpoid}^H$ is defined as in Lem. 3.2. This construction is given as a slight modification of the Yoneda lemma. Indeed, if $H=G$ then ν is nothing but the equivalence appearing in the Yoneda lemma, observing that φ is the Yoneda embedding: $\varphi(z)=G(z,\text{-})$. For general subgroup $H\leq G$, there is still a quasi-natural transformation ν. Each ν_z is faithful as a groupoid homomorphism.

We note that $t(z)=\int_{x\in F}G(p(x),z)$ is equivalent to the group N. So, from the preceding lemma, t is quasi-naturally equivalent to a pseudo-functor $t'\in\mathbf{Gpoid}^H$ satisfying that $t'(z)=N$. Hence, without loss of generality, we can replace t in the codomain of ν_z by t' satisfying $t'(z)=N$. Now, returning to the world of $\mathbf{Gpoid}/G$ by the Grothendieck construction using Thm. 2.4, we have a groupoid homomorphism $F\xrightarrow{e}\int_{z\in G}\mathrm{Hom}_{\mathbf{Gpoid}^H}(\varphi(z),t')$. It is easy to prove that e is faithful.

This appears the end of the proof: $\int_{z\in G}\mathrm{Hom}_{\mathbf{Gpoid}^H}(\varphi(z),t')$ looks equivalent to the twisted wreath product $N\,\mathrm{Wr}_H\,G$. This is not true, however, since t' has no assurance to be a strict functor. The definition of twisted wreath product requires that H act on N in an ordinary sense, that is, the corresponding pseudo-functor $t'\in\mathbf{Gpoid}^H$ to be strict. By analysis of $t\cong t'$, the automorphism

$t'(h) : N \to N$ carries n to $h^\tau n(h^\tau)^{-1}$ where h^τ is a chosen element in $p^{-1}(h)$. Hence, if there is a splitting on H, we can choose h^τ in a functorial way (that is $(hk)^\tau = h^\tau k^\tau$), yielding the strictness of t'. $\square$

3.5 Corollary

Let F be a group extension of N by G.

Then F embeds into the standard wreath product $N \operatorname{Wr} G$.

(*Proof*) Set $H = \{1\}$. We note that the last paragraph in the proof of the theorem turns out to be unnecessary, since t' becomes a strict functor trivially. $\square$

The direct proof of the theorem is found in [6]. See also [7]. We emphasize that the proof above uses only simple general facts in the theory of groupoids and bicategories. Moreover the use of (a slight modification of) the Yoneda lemma in the proof suggests, someways, why the wreath product arises as the target of the embedding.

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