Classical Linear Logic of Implications

$$!\sigma = (\sigma \to \bot) \multimap \bot \qquad ?\sigma = (\sigma \multimap \bot) \to \bot$$

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Aim of This Talk

Introduce a *linear lambda calculus DCLL* which is

- for the *multiplicative exponential classical linear logic (MELL)*
- based on the *linear* and *non-linear implications* $\multimap$, $\to$
- very *simple*: just 6 axioms, no commuting conversions
- yet as *expressive* as the conventional systems; can be seen as an extension of Barber and Plotkin's *Dual Intuitionistic Linear Logic*
- and *sound* and *complete* for the categorical models of MELL
- motivated by the study on *linearly used continuations* by Berdine, O'Hearn, Thielecke and Reddy (CW'01)/Hasegawa (FLOPS2002)

. . . and raise some issues on the syntax and semantics of Linear Logic

Why $\rightarrow$ Rather Than $!$

- *Simplicity*: syntactically $\rightarrow$ is far easier to formulate than $!$: simple term expressions, no commuting conversions (and also enjoys simple characterizations in many semantic models, e.g. phase semantics, domain theoretic models)
- *Granularity*: As opposed to the common belief, the proof theory of $\rightarrow$ is no less delicate than that of $!$, up to provable equality: the embedding of the $\{\rightarrow, \multimap\}$-fragment into the $\{!, \multimap\}$-fragment is *fully complete* (Hasegawa, JFP2000 / FLOPS2002)
- *Pragmatics*: $\rightarrow$ is much closer to the standard experience with programming languages (as emphasized in a series of work by Thielecke and his colleagues)
- *Expressiveness and Completeness*: Main technical contribution of this talk

The Dual-Context Formulation

(as used in DILL by Barber and Plotkin)

Our "*dual-context*" typing judgement takes the form

$$\Gamma \;;\; \Delta \vdash M : \tau$$

where Γ represents an *intuitionistic (non-linear)* context
while Δ represents a *linear* context.
Hence

$$\frac{}{\Gamma_1, x : \sigma, \Gamma_2 \;;\; \emptyset \vdash x : \sigma}\ (\text{Int-Ax}) \qquad \frac{}{\Gamma \;;\; x : \sigma \vdash x : \sigma}\ (\text{Lin-Ax})$$

DILL gives an elegant formulation of IMELL:

- logic: natural introduction/elimination rules for I, $\otimes$, $\multimap$, $!$
- natural equational theory: β/η-axioms + commuting conversions
- sound and complete for the categorical models of IMELL

Dual Classical Linear Logic (1) Linear Implication

For linear implication, we just follow the DILL formulation:

$$\frac{\Gamma\ ;\ \Delta, x : \sigma_1 \vdash M : \sigma_2}{\Gamma\ ;\ \Delta \vdash \lambda x^{\sigma_1}.M : \sigma_1 \multimap \sigma_2}\ (\multimap \mathrm{I})$$

$$\frac{\Gamma\ ;\ \Delta_1 \vdash M : \sigma_1 \multimap \sigma_2 \quad \Gamma\ ;\ \Delta_2 \vdash N : \sigma_1}{\Gamma\ ;\ \Delta_1 \sharp \Delta_2 \vdash M\, N : \sigma_2}\ (\multimap \mathrm{E})$$

where $\Delta_1 \sharp \Delta_2$ represents one of possible merges of Δ_1 and Δ_2 (and no variable occurs both in Δ_1 and in Δ_2).

Axioms: Standard β and η axioms.

$$\begin{array}{llcl}
(\beta_{\multimap}) & (\lambda x^{\sigma}.M)\, N & = & M[N/x] \\
(\eta_{\multimap}) & \lambda x^{\sigma}.M\, x & = & M
\end{array}$$

Dual Classical Linear Logic (2) Non-Linear Implication

Usually the non-linear implication $\rightarrow$ is *derived* using $\multimap$ and $!$.

Here we take $\rightarrow$ as a primitive construct (cf. Thielecke's work).

$$\frac{\Gamma, x : \sigma_1 \ ; \ \Delta \vdash M : \sigma_2}{\Gamma \ ; \ \Delta \vdash \boldsymbol{\lambda} x^{\sigma_1}.M : \sigma_1 \rightarrow \sigma_2} \ (\rightarrow \mathrm{I})$$

$$\frac{\Gamma \ ; \ \Delta \vdash M : \sigma_1 \rightarrow \sigma_2 \quad \Gamma \ ; \ \emptyset \vdash N : \sigma_1}{\Gamma \ ; \ \Delta \vdash M \,\texttt{@}\, N : \sigma_2} \ (\rightarrow \mathrm{E})$$

Axioms: Standard β and η axioms.

$$\begin{array}{llcll}
(\beta_{\rightarrow}) & (\boldsymbol{\lambda} x^{\sigma}.M) \,\texttt{@}\, N & = & M[N/x] & \\
(\eta_{\rightarrow}) & \boldsymbol{\lambda} x^{\sigma}.M \,\texttt{@}\, x & = & M & (x \notin FV(M))
\end{array}$$

Dual Classical Linear Logic (3) Duality

We use the *falsity* $\bot$ for expressing the *duality* in Classical Linear Logic:
$(\sigma \multimap \bot) \multimap \bot \;\simeq\; \sigma$.
There are a number of ways to do this (e.g. the $\lambda\mu$-formulation);
in this talk we simply introduce a combinator C_σ for the
"double-negation elimination".

$$\frac{}{\Gamma \,;\, \emptyset \vdash \mathsf{C}_\sigma : ((\sigma \multimap \bot) \multimap \bot) \multimap \sigma}\ (\mathsf{C})$$

Axioms: Two axioms for ensuring that C_σ is an isomorphism.

$$\begin{array}{llcll}
(\mathsf{C}_1) & L\,(\mathsf{C}_\sigma\,M) & = & M\,L & (L : \sigma \multimap \bot) \\
(\mathsf{C}_2) & \mathsf{C}_\sigma\,(\lambda k^{\sigma \multimap \bot}.k\,M) & = & M &
\end{array}$$

Note: if there is no ground type we could do without C's
(and with just a single axiom instead of $(\mathsf{C}_1)(\mathsf{C}_2)$ – see the paper)

Dual Classical Linear Logic – Summary

That's it: *Dual Classical Linear Logic (DCLL)* has

- 3 type constructors: $\multimap$, $\to$ and $\perp$
- term expressions for variables, linear/non-linear lambda abstractions and applications, and a combinator for duality
- and just 6 axioms: 4 standard $\beta\eta$ axioms and 2 axioms for duality

as given in the last slides. No extra material, e.g. commuting conversion.

Is it Good Enough?

Well, DCLL certainly seems to be a sound system, but isn't it too naive?

- Is it *expressive* (both on types and proofs)? — Yes!
- Is it *complete* for standard semantic models? — Yes!

Expressiveness (1) Types

For types, not much surprise:

$$\begin{array}{lcl} I & \equiv & \bot \multimap \bot \\ \sigma_1 \otimes \sigma_2 & \equiv & (\sigma_1 \multimap \sigma_2 \multimap \bot) \multimap \bot \\ \sigma_1 \parr \sigma_2 & \equiv & (\sigma_1 \multimap \bot) \multimap (\sigma_2 \multimap \bot) \multimap \bot \\ !\,\sigma & \equiv & (\sigma \to \bot) \multimap \bot \\ ?\,\sigma & \equiv & (\sigma \multimap \bot) \to \bot \end{array}$$

The encoding of the modalities $!$ and $?$ seems less popular: they are motivated by the recent study on linearity in *continuation-passing* (Berdine et al, CW'01 / Hasegawa, FLOPS2002) where constructions

$(\sigma \to R) \multimap R$ *linearly used continuations*

$(\sigma \multimap R) \to R$ *linearly implemented continuations*

play essential roles (and we obtain $!\sigma$ and $?\sigma$ by letting R be $\bot$)

Expressiveness (2) Proofs (Terms)

We can encode the term constructs for I, $\otimes$ and $!$ of DILL as follows:

$$
\begin{array}{lcl}
* & \equiv & \lambda x^{\perp}.x \\
\mathsf{let}\ *\ \mathsf{be}\ M^{I}\ \mathsf{in}\ N^{\tau} & \equiv & \mathsf{C}_{\tau}\,(\lambda k^{\tau\multimap\perp}.M\,(k\,N)) \\
M^{\sigma_1}\otimes N^{\sigma_2} & \equiv & \lambda k^{\sigma_1\multimap\sigma_2\multimap\perp}.k\,M\,N \\
\mathsf{let}\ x^{\sigma_1}\otimes y^{\sigma_2}\ \mathsf{be}\ M^{\sigma_1\otimes\sigma_2}\ \mathsf{in}\ N^{\tau} & \equiv & \mathsf{C}_{\tau}\,(\lambda k^{\tau\multimap\perp}.M\,(\lambda x^{\sigma_1}.\lambda y^{\sigma_2}.k\,N)) \\
!M^{\sigma} & \equiv & \lambda h^{\sigma\to\perp}.h\,@\,M \\
\mathsf{let}\ !x^{\sigma}\ \mathsf{be}\ M^{!\sigma}\ \mathsf{in}\ N^{\tau} & \equiv & \mathsf{C}_{\tau}\,(\lambda k^{\tau\multimap\perp}.M\,(\boldsymbol{\lambda} x^{\sigma}.k\,N))
\end{array}
$$

Here is an unexpected (but very pleasant) observation:
all axioms (including 30 instances of commuting conversions) of DILL are derivable from the 6 axioms of DCLL!

Theorem 1. All the equality axioms in DILL are admissible in DCLL (via these encoding of types and terms).

Proof of Theorem 1

By direct calculations. The β-axioms are very easy. For example:

$$\begin{array}{rcl}
\mathsf{let}\ !x^\sigma\ \mathsf{be}\ !M^\sigma\ \mathsf{in}\ N^\tau & \equiv & \mathsf{C}_\tau\,(\lambda k^{\tau\multimap\bot}.(\lambda h^{\sigma\to\bot}.h\,@\,M)\,(\boldsymbol{\lambda} x^\sigma.k\,N)) \\
& \overset{\beta_\multimap}{=} & \mathsf{C}_\tau\,(\lambda k^{\tau\multimap\bot}.(\boldsymbol{\lambda} x^\sigma.k\,N)\,@\,M) \\
& \overset{\beta_\to}{=} & \mathsf{C}_\tau\,(\lambda k^{\tau\multimap\bot}.k\,N[M/x]) \\
& \overset{\mathsf{C}_2}{=} & N[M/x]
\end{array}$$

For the η-axioms and commuting conversions, it is better to use a few lemmas given in the paper.

Summary on Expressiveness

DCLL can express all the type constructions of MELL, and
DCLL can be regarded as an extension of DILL so that
the encoding validates all the equations valid in DILL.

Completeness (1)

Theorem 1 suggests that there is no missing axiom for DCLL
– the 6 axioms seem to be *complete*.
In fact this is the case, for the categorical models of MELL.

By a *categorical model of IMELL*, we mean a *symmetric monoidal closed category* equipped with a monoidal comonad enjoying certain coherence properties (*linear exponential comonad* in the sense of Hyland and Shalk).

By a *categorical model of MELL*, we mean a categorical model of IMELL such that the symmetric monoidal closed structure is $*$-*autonomous*, thus with a *dualizing object* $\perp$ such that the canonical arrow from X to $(X \multimap \perp) \multimap \perp$ is an isomorphism for each X.

Completeness (2)

Observation: All equalities in DCLL are valid for any interpretation in models of MELL. Hence *soundness* follows (straightforward).

Observation: Theorem 1 implies that the term model of DCLL has the structure of a categorical model of IMELL (since it is a model of DILL).

Observation: Because of the presence of duality, this term model of DCLL is actually a model of MELL.

Observation: Two terms are provably equal in DCLL iff their interpretations in the term model (regarded as a categorical model of MELL) are the same. Hence *completeness* follows.

Thus we conclude:

Theorem 2. The equational theory of DCLL is sound and complete for the categorical models of MELL.

Summary on Completeness

The 6 axioms of DCLL precisely capture all the desirable
($=$ semantically justified) equations on the proofs of MELL.

Additives

It is routine to enrich DCLL with additives and establish the semantic completeness for this extension (see the paper).

Issue (1) Treatment of Duality

Some people complain: "DCLL is not quite faithful to CLL, because the types σ and $\sigma^{\perp\perp} = (\sigma \multimap \perp) \multimap \perp$ are *not* identical (as opposed to proof-nets formulations) – they are only isomorphic!"

This comment actually addresses to the following coherence problem:

Is a free $*$-autonomous category strictly equivalent to a one in which $(-)^{\perp\perp}$ is the identity functor and the canonical iso $X \xrightarrow{\simeq} X^{\perp\perp}$ is an identity arrow?

If the answer is positive, there is no gap between
the up-to-identity approach (like proof-nets) and
the up-to-iso approach (like DCLL) for the duality.
Good News (Aug.'02): Solved positively (with help of Cockett and Seely)

Issue (2) ILL vs. CLL

I think that the relationship between ILL and CLL at the level of proofs (rather than provability) has not been sufficiently sorted out:

Conservativity Question: Is the equational theory of DCLL conservative over DILL?

More strongly:

Full-Completeness Question: Is the embedding of DILL into DCLL fully complete?

(Note: The embedding from MILL into MLL and also
the embedding from MILL into DILL are known to be fully complete.)

I conjecture that the answers are positive for both questions,
but not sure at all for extensions with additives.

Further Issues (more specific to DCLL)

Decidability Question: Is the equational theory of DCLL decidable?

Note: For DILL, the answer is positive (Barber, 1997).
Also positive for MLL (e.g. Koh and Ong, CTCS'99).

More Direct Categorical Semantics? (Not Just Completeness)
Perhaps fibration-based models provide a better correspondence, e.g. DCLL as an internal language (cf. Meietti, de Paiva and Ritter, 2000)

Relation to 2nd-order $\{\to, \multimap\}$-Calculus (cf. Plotkin, LICS'93)
Very similar constructions work, e.g. $!\sigma = \forall X.(\sigma \to X) \multimap X$
— but also very different semantic nature

Summary on These Issues

There still remain many fundamental questions on Linear Logic which deserve to be attacked seriously.
(And I expect that DCLL can be a good tool —
for its simplicity, expressiveness and completeness.)

(End of Talk - Thank You)

Typing of DCLL

$$\frac{}{\Gamma_1, x : \sigma, \Gamma_2 \; ; \; \emptyset \vdash x : \sigma} \; (\textsf{Int-Ax}) \qquad \frac{}{\Gamma \; ; \; x : \sigma \vdash x : \sigma} \; (\textsf{Lin-Ax})$$

$$\frac{\Gamma \; ; \; \Delta, x : \sigma_1 \vdash M : \sigma_2}{\Gamma \; ; \; \Delta \vdash \lambda x^{\sigma_1}.M : \sigma_1 \multimap \sigma_2} \; (\multimap \mathrm{I}) \qquad \frac{\Gamma \; ; \; \Delta_1 \vdash M : \sigma_1 \multimap \sigma_2 \quad \Gamma \; ; \; \Delta_2 \vdash N : \sigma_1}{\Gamma \; ; \; \Delta_1 \sharp \Delta_2 \vdash M \; N : \sigma_2} \; (\multimap \mathrm{E})$$

$$\frac{\Gamma, x : \sigma_1 \; ; \; \Delta \vdash M : \sigma_2}{\Gamma \; ; \; \Delta \vdash \boldsymbol{\lambda} x^{\sigma_1}.M : \sigma_1 \to \sigma_2} \; (\to \mathrm{I}) \qquad \frac{\Gamma \; ; \; \Delta \vdash M : \sigma_1 \to \sigma_2 \quad \Gamma \; ; \; \emptyset \vdash N : \sigma_1}{\Gamma \; ; \; \Delta \vdash M \,@\, N : \sigma_2} \; (\to \mathrm{E})$$

$$\frac{}{\Gamma \; ; \; \emptyset \vdash \mathsf{C}_\sigma : ((\sigma \multimap \bot) \multimap \bot) \multimap \sigma} \; (\mathsf{C})$$

Axioms of DCLL

$(\beta_{\multimap})$	$(\lambda x^\sigma.M)\; N$	$=$	$M[N/x]$	
$(\eta_{\multimap})$	$\lambda x^\sigma.M\; x$	$=$	M	
$(\beta_{\to})$	$(\boldsymbol{\lambda} x^\sigma.M) \,@\, N$	$=$	$M[N/x]$	
$(\eta_{\to})$	$\boldsymbol{\lambda} x^\sigma.M \,@\, x$	$=$	M	$(x \notin FV(M))$
(C_1)	$L\; (\mathsf{C}_\sigma\; M)$	$=$	$M\; L$	$(L : \sigma \multimap \bot)$
(C_2)	$\mathsf{C}_\sigma\; (\lambda k^{\sigma \multimap \bot}.k\; M)$	$=$	M	

Typing of DILL

$$\frac{}{\Gamma_1, x : \sigma, \Gamma_2 ;\ \emptyset \vdash x : \sigma}\ (\mathsf{Int\text{-}Ax}) \qquad \frac{}{\Gamma ;\ x : \sigma \vdash x : \sigma}\ (\mathsf{Lin\text{-}Ax})$$

$$\frac{}{\Gamma ;\ \emptyset \vdash * : I}\ (I\ \mathrm{I}) \qquad \frac{\Gamma ;\ \Delta_1 \vdash M : I \quad \Gamma ;\ \Delta_2 \vdash N : \sigma}{\Gamma ;\ \Delta_1 \sharp \Delta_2 \vdash \mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ N : \sigma}\ (I\ \mathrm{E})$$

$$\frac{\Gamma ;\ \Delta_1 \vdash M : \sigma_1 \quad \Gamma ;\ \Delta_2 \vdash N : \sigma_2}{\Gamma ;\ \Delta_1 \sharp \Delta_2 \vdash M \otimes N : \sigma_1 \otimes \sigma_2}\ (\otimes\ \mathrm{I}) \qquad \frac{\Gamma ;\ \Delta_1 \vdash M : \sigma_1 \otimes \sigma_2 \quad \Gamma ;\ \Delta_2, x : \sigma_1, y : \sigma_2 \vdash N : \tau}{\Gamma ;\ \Delta_1 \sharp \Delta_2 \vdash \mathsf{let}\ x^{\sigma_1} \otimes y^{\sigma_2}\ \mathsf{be}\ M\ \mathsf{in}\ N : \tau}\ (\otimes\ \mathrm{E})$$

$$\frac{\Gamma ;\ \Delta, x : \sigma_1 \vdash M : \sigma_2}{\Gamma ;\ \Delta \vdash \lambda x^{\sigma_1}.M : \sigma_1 \multimap \sigma_2}\ (\multimap \mathrm{I}) \qquad \frac{\Gamma ;\ \Delta_1 \vdash M : \sigma_1 \multimap \sigma_2 \quad \Gamma ;\ \Delta_2 \vdash N : \sigma_1}{\Gamma ;\ \Delta_1 \sharp \Delta_2 \vdash M\ N : \sigma_2}\ (\multimap \mathrm{E})$$

$$\frac{\Gamma ;\ \emptyset \vdash M : \sigma}{\Gamma ;\ \emptyset \vdash !M : !\sigma}\ (!\ \mathrm{I}) \qquad \frac{\Gamma ;\ \Delta_1 \vdash M : !\sigma \quad \Gamma, x : \sigma ;\ \Delta_2 \vdash N : \tau}{\Gamma ;\ \Delta_1 \sharp \Delta_2 \vdash \mathsf{let}\ !x\ \mathsf{be}\ M\ \mathsf{in}\ N : \tau}\ (!\ \mathrm{E})$$

Axioms of DILL

$$\begin{array}{rcl@{\qquad}rcl}
\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ M & = & M & \mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ * & = & M \\
\mathsf{let}\ x \otimes y\ \mathsf{be}\ M \otimes N\ \mathsf{in}\ L & = & L[M/x, N/y] & \mathsf{let}\ x \otimes y\ \mathsf{be}\ M\ \mathsf{in}\ x \otimes y & = & M \\
(\lambda x.M)\ N & = & M[N/x] & \lambda x.M\ x & = & M \\
\mathsf{let}\ !x\ \mathsf{be}\ !M\ \mathsf{in}\ N & = & N[M/x] & \mathsf{let}\ !x\ \mathsf{be}\ M\ \mathsf{in}\ !x & = & M
\end{array}$$

$$\begin{array}{rcl}
C[\mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ N] & = & \mathsf{let}\ *\ \mathsf{be}\ M\ \mathsf{in}\ C[N] \\
C[\mathsf{let}\ x \otimes y\ \mathsf{be}\ M\ \mathsf{in}\ N] & = & \mathsf{let}\ x \otimes y\ \mathsf{be}\ M\ \mathsf{in}\ C[N] \\
C[\mathsf{let}\ !x\ \mathsf{be}\ M\ \mathsf{in}\ N] & = & \mathsf{let}\ !x\ \mathsf{be}\ M\ \mathsf{in}\ C[N]
\end{array}$$

where $C[-]$ is a linear context (no ! binds $[-]$).