Planar Lambda Algebras and Semi-closed Operads

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We show how our previous work on *extensional* $\mathbf{BI}(\_)^\bullet$-algebras and internal operads [1] can be extended to (possibly non-extensional) planar lambda algebras which precisely capture the β-theory of the planar lambda calculus.

First, we identify the axioms on $\mathbf{BI}(\_)^\bullet$-terms for making the translations to & from the β-theory of the planar lambda calculus sound and complete. Planar lambda algebras are defined to be $\mathbf{BI}(\_)^\bullet$-algebras satisfying these axioms.

Then we give the construction of the internal operads for planar lambda algebras, which gives an *equivalence* between planar lambda algebras and semi-closed operads. Finally, we discuss the symmetric, braided and cartesian cases.

1 Preliminaries

Planar λ-calculus The *planar λ-calculus* is an untyped linear λ-calculus with no exchange, whose terms are given by the following rules.

$$\frac{}{x \vdash x}\ \text{variable} \qquad \frac{\Gamma, x \vdash M}{\Gamma \vdash \lambda x.M}\ \text{abstraction} \qquad \frac{\Gamma \vdash M \quad \Gamma' \vdash N}{\Gamma, \Gamma' \vdash M\,N}\ \text{application}$$

It is easy to see that planar terms are closed under β- and η-conversion. Typical planar terms include $\mathbf{I} = \lambda f.f$, $\mathbf{B} = \lambda fxy.f\,(x\,y)$, and $M^\bullet = \lambda f.f\,M$ for planar closed term M.

$\mathbf{BI}(\_)^\bullet$-algebras A $\mathbf{BI}(\_)^\bullet$-*algebra* [4] is an applicative structure $\mathcal{A}$ with elements $\mathbf{I}$, $\mathbf{B}$ and $a^\bullet$ for all $a \in \mathcal{A}$ satisfying $\mathbf{B}\,a\,b\,c = a\,(b\,c)$, $\mathbf{I}\,a = a$, and $a^\bullet\,b = b\,a$.

The set of closed terms of the planar lambda calculus modulo the β-equality forms a $\mathbf{BI}(\_)^\bullet$-algebra Λ_0^{planar} with $\mathbf{I} = \lambda f.f$, $\mathbf{B} = \lambda fxy.f\,(x\,y)$, and $M^\bullet = \lambda f.f\,M$.

Semi-closed operads Recall that an *(planar or non-symmetric) operad* $\mathcal{P}$ is a family of sets $(\mathcal{P}(n))_{n\in\mathbf{N}}$ equipped with an identity $id \in \mathcal{P}(1)$ and a composition map sending $f_i \in \mathcal{P}(k_i)$ $(1 \leq i \leq n)$ and $g \in \mathcal{P}(n)$ to the composite $g(f_1, \ldots, f_n) \in \mathcal{P}(k_1 + k_2 + \ldots + k_n)$ which are subject to the unit law and associativity:

$$f(id,\ldots,id) = f = id(f)$$
$$h(g_1(f_{11},\ldots,f_{1j_1}),\ldots,g_k(f_{k1},\ldots,f_{kj_k})) = (h(g_1,\ldots,g_n))(f_{11},\ldots,f_{km_k}).$$

It is *semi-closed* when there is an element $\mathbf{app} \in \mathcal{P}(2)$ and maps $\lambda(-) : \mathcal{P}(n+1) \to \mathcal{P}(n)$ satisfying the β-rule $\mathbf{app}(\lambda(f), id) = f$ and the naturality $\lambda(g(f_1,\ldots,f_n,id)) = (\lambda(g))(f_1,\ldots,f_n)$.

For semi-closed operads $\mathcal{P}$ and $\mathcal{Q}$, a homomorphism $\varphi : \mathcal{P} \to \mathcal{Q}$ is a family of maps $\varphi_n : P(n) \to Q(n)$ (we often omit the subscript n) such that

- $\varphi_1(id) = id$,
- $\varphi_{k_1+\ldots+k_n}(g(f_1,\ldots,f_n)) = \varphi_n(g)(\varphi_{k_1}(f_1)\ldots,\varphi_{k_n}(f_n))$,
- $\varphi_2(\mathbf{app}) = \mathbf{app}$, and
- $\varphi_n(\lambda(f)) = \lambda(\varphi_{n+1}(f))$.

The last condition can be replaced by its single instance $\varphi_1(\lambda(\mathbf{app})) = \lambda(\mathbf{app})$ (cf. [2]), because

$$\begin{array}{rcll}
\varphi_n(\lambda(f)) & = & \varphi_n(\lambda(\mathbf{app}(\lambda(f),id))) & (\beta\text{-rule}) \\
& = & \varphi_n(\lambda(\mathbf{app})(\lambda(f))) & (\text{naturality of } \lambda) \\
& = & \varphi_1(\lambda(\mathbf{app}))(\varphi_n(\lambda(f))) & (\varphi \text{ preserves composition}) \\
& = & \lambda(\mathbf{app})(\varphi_n(\lambda(f))) & (\varphi_1(\lambda(\mathbf{app})) = \lambda(\mathbf{app})) \\
& = & \lambda(\mathbf{app}(\varphi_n(\lambda(f)),id)) & (\text{naturality of } \lambda) \\
& = & \lambda(\varphi_2(\mathbf{app})(\varphi_n(\lambda(f)),\varphi_1(id))) & (\varphi \text{ preserves } id \text{ and } \mathbf{app}) \\
& = & \lambda(\varphi_{n+1}(\mathbf{app}(\lambda(f),id))) & (\varphi \text{ preserves composition}) \\
& = & \lambda(\varphi_{n+1}(f)) & (\beta\text{-rule}).
\end{array}$$

Let us write **SemiClosedOperad** for the category of semi-closed operads and homomorphisms.

Proposition 1 *The term model of the planar lambda calculus modulo β-equality gives an initial object Λ^{planar} of* **SemiClosedOperad**.

Explicitly, $\Lambda^{planar}(n)$ is the set of β-equivalence classes of terms with n free variables $x_1,\ldots,x_n$. id is (the equivalence class of) $x_1 \vdash x_1$, $\mathbf{app}$ is $x_1, x_2 \vdash x_1\, x_2$, while λ sends $x_1,\ldots,x_n,x_{n+1} \vdash M$ to $x_1,\ldots,x_n \vdash \lambda x_{n+1}.M$. Composition is given by the substitution of terms with appropriate renamings. For any semi-closed operad $\mathcal{P}$, the unique homomorphism from Λ^{planar} to $\mathcal{P}$ is determined by the obvious translation given by $[\![x]\!] = id$, $[\![M\,N]\!] = \mathbf{app}([\![M]\!],[\![N]\!])$ and $[\![\lambda x.M]\!] = \lambda([\![M]\!])$.

Corollary 2 *Semi-closed operads are sound and complete for the planar lambda calculus with β-equality.*

2 Axiomatizing Planar Lambda Algebras

Let $\lambda_{\mathsf{planar}}$ be the set of planar lambda terms and $\mathbf{BI}(\_)^\bullet$ be the set of terms generated by variables (each occurring at most once), $\mathbf{B}$, $\mathbf{I}$, application and the $(\_)^\bullet$-operator on closed terms. Let $=_{\mathbf{ext}}$ be the smallest congruence on $\mathbf{BI}(\_)^\bullet$ satisfying the following axioms

$$
\begin{array}{rcll}
\mathbf{B}\,P\,Q\,R & = & P\,(Q\,R) & \text{(B)}\\
\mathbf{I}\,P & = & P & \text{(I)}\\
P^\bullet\,Q & = & Q\,P & (\bullet)\\
\mathbf{B}\,\mathbf{I} & = & \mathbf{I} & \text{(BI)}\\
\mathbf{B}\,\mathbf{B}^\bullet\,(\mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,\mathbf{B}\,\mathbf{B})) & = & \mathbf{B}\,(\mathbf{B}\,\mathbf{B})\,\mathbf{B} & (\text{B}\bullet)\\
\mathbf{B}\,\mathbf{I}^\bullet\,\mathbf{B} & = & \mathbf{B}\,\mathbf{I} & (\text{I}\bullet)\\
\mathbf{B}\,P^{\bullet\bullet}\,\mathbf{B} & = & \mathbf{B}\,(\mathbf{B}\,P^\bullet)\,\mathbf{B} & (\bullet\bullet)\\
(P\,Q)^\bullet & = & \mathbf{B}\,Q^\bullet\,(\mathbf{B}\,P^\bullet\,\mathbf{B}) & (\text{app}\bullet)
\end{array}
$$

It has been shown that this extensional theory $=_{\mathbf{ext}}$ is sound and complete for the $\beta\eta$-theory of the planar lambda calculus and the following translations [1].

Translations Define $(-)^\sharp : \mathbf{BI}(\_)^\bullet \to \lambda_{\mathsf{planar}}$ by

$$
\begin{array}{rcl}
\mathbf{B}^\sharp & \equiv & \lambda xyz.x\,(y\,z)\\
\mathbf{I}^\sharp & \equiv & \lambda x.x\\
(P\,Q)^\sharp & \equiv & P^\sharp\,Q^\sharp\\
(P^\bullet)^\sharp & \equiv & \lambda x.x\,P^\sharp\\
x^\sharp & \equiv & x
\end{array}
$$

and $(-)^\flat : \lambda_{\mathsf{planar}} \to \mathbf{BI}(\_)^\bullet$ by

$$
\begin{array}{rcl}
(\lambda x.M)^\flat & \equiv & \lambda^* x.M^\flat\\
(M\,N)^\flat & \equiv & M^\flat\,N^\flat\\
x^\flat & \equiv & x
\end{array}
$$

where, for P with the rightmost variable x, $\lambda^* x.P$ is given as follows [4].

$$
\begin{array}{rcl}
\lambda^* x.x & = & \mathbf{I}\\
\lambda^* x.P\,Q & = & \begin{cases} \mathbf{B}\,Q^\bullet\,(\lambda^* x.P) & (x \in \mathrm{fv}(P))\\ \mathbf{B}\,P\,(\lambda^* x.Q) & (x \in \mathrm{fv}(Q)) \end{cases}
\end{array}
$$

We want to modify these axioms to be sound and complete with respect to the β-theory of the planar lambda calculus. Among these axioms, only (BI) is unsound for β, as it requires the η-equality:

$$(\mathbf{B}\,\mathbf{I})^\sharp \equiv (\lambda xyz.x\,(y\,z))\,(\lambda x.x) =_\beta \lambda yz.y\,z \neq_\beta \lambda y.y \equiv \mathbf{I}^\sharp$$

Indeed, each $n \geq 0$ should give distinct $\mathbf{B}^n\,\mathbf{I}$ as

$$(\mathbf{B}^n\,\mathbf{I})^\sharp =_\beta \lambda x_0 x_1 \ldots x_n.x_0\,x_1\,\ldots\,x_n$$

while $\mathbf{B}^n\,\mathbf{I}$ is equal to $\mathbf{I}$ in the extensional theory.

For the β-theory, although $\mathbf{B}\,\mathbf{I}\,P = P$ is in general not valid, we still need to validate the following cases.

$$\begin{array}{rcll} \mathbf{B}\,\mathbf{I}\,\mathbf{B} & = & \mathbf{B} & (\mathrm{BI}_{\mathbf{B}_0}) \\ \mathbf{B}\,\mathbf{I}\,\mathbf{I} & = & \mathbf{I} & (\mathrm{BI}_{\mathbf{I}}) \\ \mathbf{B}\,\mathbf{I}\,P^\bullet & = & P^\bullet & (\mathrm{BI}_\bullet) \\ \mathbf{B}\,\mathbf{I}\,(\mathbf{B}\,P) & = & \mathbf{B}\,P & (\mathrm{BI}_{\mathbf{B}_1}) \\ \mathbf{B}\,\mathbf{I}\,(\mathbf{B}\,P\,Q) & = & \mathbf{B}\,P\,Q & (\mathrm{BI}_{\mathbf{B}_2}) \end{array}$$

The last two can be replaced by the following parameter-free axioms.

$$\begin{array}{rcll} \mathbf{B}\,(\mathbf{B}\,\mathbf{I})\,\mathbf{B} & = & \mathbf{B} & (\mathrm{BI}_{\mathbf{B}_1}) \\ \mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\,\mathbf{B} & = & \mathbf{B} & (\mathrm{BI}_{\mathbf{B}_2}) \end{array}$$

It turns out that $(\mathrm{BI}_{\mathbf{B}_0})$ and $(\mathrm{BI}_{\mathbf{B}_1})$ are derivable from other axioms.[1] Hence all we need are

$$\begin{array}{rcll} \mathbf{B}\,\mathbf{I}\,\mathbf{I} & = & \mathbf{I} & (\mathrm{BI}_{\mathbf{I}}) \\ \mathbf{B}\,\mathbf{I}\,P^\bullet & = & P^\bullet & (\mathrm{BI}_\bullet) \\ \mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\,\mathbf{B} & = & \mathbf{B} & (\mathrm{BI}_{\mathbf{B}}) \end{array}$$

(For ease of presentation, in the sequel $(\mathrm{BI}_{\mathbf{B}_2})$ is renamed just $(\mathrm{BI}_{\mathbf{B}})$.) By (BI), we mean these three axioms $(\mathrm{BI}_{\mathbf{I}})$, $(\mathrm{BI}_\bullet)$ and $(\mathrm{BI}_{\mathbf{B}})$ instead of the original unsound $\mathbf{B}\,\mathbf{I} = \mathbf{I}$, and let $=_{\mathbf{BI}(\_)^\bullet}$ be the smallest congruence on $\mathbf{BI}(\_)^\bullet$ satisfying this new set of axioms (summarised in Figure 1). Below we show that they are sound and complete with respect to the β-theory of the planar lambda calculus.

2.1 Proof

Lemma 3 *The following equations are derivable in* $=_{\mathbf{BI}(\_)^\bullet}$*.*

$$\begin{array}{rcll} \mathbf{B}\,(\mathbf{B}\,P\,Q) & = & \mathbf{B}\,(\mathbf{B}\,P)\,(\mathbf{B}\,Q) & (\mathrm{B2}) \\ \mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,P) & = & \mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,P))\,\mathbf{B} & (\mathrm{B3}) \\ \mathbf{B}\,P\,\mathbf{I} & = & \mathbf{B}\,\mathbf{I}\,P & (\mathrm{BI2}) \\ \mathbf{B}\,(\mathbf{B}\,P\,Q)\,R & = & \mathbf{B}\,P\,(\mathbf{B}\,Q\,R) & (\mathit{assoc}) \end{array}$$

[1] We can derive $(\mathrm{BI}_{\mathbf{B}_1})$ from $(\mathrm{BI}_{\mathbf{B}_2})$ and other axioms as follows:

$$\begin{array}{rcll} \mathbf{B}\,(\mathbf{B}\,\mathbf{I})\,\mathbf{B} & = & \mathbf{B}\,(\mathbf{B}\,\mathbf{I})\,(\mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\,\mathbf{B}) & (\mathrm{BI}_{\mathbf{B}_2}) \\ & = & \mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,\mathbf{I})\,(\mathbf{B}\,(\mathbf{B}\,\mathbf{I})))\,\mathbf{B} & (\mathit{assoc}) \\ & = & \mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\,\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\,\mathbf{B} & (\mathrm{B}) \\ & = & \mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\,\mathbf{B} & (\mathrm{BI}_{\mathbf{B}_2}) \\ & = & \mathbf{B} & (\mathrm{BI}_{\mathbf{B}_2}) \end{array}$$

where we use (*assoc*) from Lemma 3 which is derivable without using $(\mathrm{BI}_{\mathbf{B}_0})$ nor $(\mathrm{BI}_{\mathbf{B}_1})$. Similarly, we can derive $(\mathrm{BI}_{\mathbf{B}_0})$ from $(\mathrm{BI}_{\mathbf{B}_1})$ — replace $(\mathbf{B}\,\mathbf{I})$ in the proof above by $\mathbf{I}$.

Proof (B3) is derivable from (B), ($\bullet$) and (B$\bullet$):

$$\begin{array}{rcll}
\mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,P) & = & \mathbf{B}\,(\mathbf{B}\,\mathbf{B})\,\mathbf{B}\,P & \text{(B)} \\
& = & \mathbf{B}\,\mathbf{B}^{\bullet}\,(\mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,\mathbf{B}\,\mathbf{B}))\,P & \text{(B}\bullet\text{)} \\
& = & \mathbf{B}^{\bullet}\,(\mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,\mathbf{B}\,\mathbf{B})\,P) & \text{(B)} \\
& = & \mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,\mathbf{B}\,\mathbf{B})\,P\,\mathbf{B} & (\bullet) \\
& = & \mathbf{B}\,(\mathbf{B}\,\mathbf{B}\,\mathbf{B}\,P)\,\mathbf{B} & \text{(B)} \\
& = & \mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,P))\,\mathbf{B} & \text{(B)}
\end{array}$$

(B2) is derivable from (B) and (B3):

$$\begin{array}{rcll}
\mathbf{B}\,(\mathbf{B}\,P\,Q) & = & \mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,P)\,Q & \text{(B)} \\
& = & \mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,P))\,\mathbf{B}\,Q & \text{(B3)} \\
& = & \mathbf{B}\,(\mathbf{B}\,P)\,(\mathbf{B}\,Q) & \text{(B)}
\end{array}$$

(BI2) follows from (B), (I$\bullet$), ($\bullet$):

$$\begin{array}{rcll}
\mathbf{B}\,P\,\mathbf{I} & = & \mathbf{I}^{\bullet}\,(\mathbf{B}\,P) & (\bullet) \\
& = & \mathbf{B}\,\mathbf{I}^{\bullet}\,\mathbf{B}\,P & \text{(B)} \\
& = & \mathbf{B}\,\mathbf{I}\,P & \text{(I}\bullet\text{)}
\end{array}$$

From (BI2) we have $\mathbf{B}\,(\lambda^* x.P)\,\mathbf{I} = \mathbf{B}\,\mathbf{I}\,(\lambda^* x.P) = \lambda^* x.P$ because $\lambda^* x.P$ is either $\mathbf{I}$ or $\mathbf{B}\,Q\,R$ for some Q and R.

The associativity law (*assoc*), to be frequently used below, is derivable as

$$\mathbf{B}\,(\mathbf{B}\,P\,Q)\,R \overset{\text{B2}}{=} \mathbf{B}\,(\mathbf{B}\,P)\,(\mathbf{B}\,Q)\,R \overset{\text{B}}{=} \mathbf{B}\,P\,(\mathbf{B}\,Q\,R).$$

Lemma 4 *$P =_{\mathbf{BI}(\_)^{\bullet}} Q$ implies $P^{\sharp} =_{\beta} Q^{\sharp}$.*

Proof Just to check that $P^{\sharp} =_{\beta} Q^{\sharp}$ holds for each axiom $P = Q$.

Lemma 5 (Crucial) *$P =_{\mathbf{BI}(\_)^{\bullet}} Q$ implies $\lambda^* x.P =_{\mathbf{BI}(\_)^{\bullet}} \lambda^* x.Q$.*

Proof For each axiom $P = Q$ with free (rightmost) x we show $\lambda^* x.P =_{\mathbf{BI}(\_)^{\bullet}} \lambda^* x.Q$. The relevant cases are (B), (I) and ($\bullet$), as other axioms are variable-free.

1. (B) $\lambda^* x.\mathbf{B}\,P\,Q\,R = \lambda^* x.P\,(Q\,R)$ with $x \in fv(P)$:

$$\begin{array}{rcll}
\lambda^* x.\mathbf{B}\,P\,Q\,R & \equiv & \mathbf{B}\,R^{\bullet}\,(\mathbf{B}\,Q^{\bullet}\,(\mathbf{B}\,\mathbf{B}\,(\lambda^* x.P))) & \\
& = & \mathbf{B}\,R^{\bullet}\,(\mathbf{B}\,(\mathbf{B}\,Q^{\bullet}\,\mathbf{B})\,(\lambda^* x.P)) & (\mathit{assoc}) \\
& = & \mathbf{B}\,(\mathbf{B}\,R^{\bullet}\,(\mathbf{B}\,Q^{\bullet}\,\mathbf{B}))\,(\lambda^* x.P) & (\mathit{assoc}) \\
& = & \mathbf{B}\,(Q\,R)^{\bullet}\,(\lambda^* x.P) & \text{(app}\bullet\text{)} \\
& \equiv & \lambda^* x.P\,(Q\,R) &
\end{array}$$

2. (B) $\lambda^* x.\mathbf{B}\,P\,Q\,R = \lambda^* x.P\,(Q\,R)$ with $x \in fv(Q)$:

$$\begin{array}{rcll}
\lambda^* x.\mathbf{B}\,P\,Q\,R & \equiv & \mathbf{B}\,R^{\bullet}\,(\mathbf{B}\,(\mathbf{B}\,P)\,(\lambda^* x.Q)) & \\
& = & \mathbf{B}\,(\mathbf{B}\,R^{\bullet}\,(\mathbf{B}\,P))\,(\lambda^* x.Q) & (\mathit{assoc}) \\
& = & \mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,R^{\bullet})\,\mathbf{B}\,P)\,(\lambda^* x.Q) & \text{(B)} \\
& = & \mathbf{B}\,(\mathbf{B}\,R^{\bullet\bullet}\,\mathbf{B}\,P)\,(\lambda^* x.Q) & (\bullet\bullet) \\
& = & \mathbf{B}\,(R^{\bullet\bullet}\,(\mathbf{B}\,P))\,(\lambda^* x.Q) & \text{(B)} \\
& = & \mathbf{B}\,(\mathbf{B}\,P\,R^{\bullet})\,(\lambda^* x.Q) & (\bullet) \\
& = & \mathbf{B}\,P\,(\mathbf{B}\,R^{\bullet}\,(\lambda^* x.Q)) & (\mathit{assoc}) \\
& \equiv & \lambda^* x.P\,(Q\,R) &
\end{array}$$

3. (B) $\lambda^* x.\mathbf{B}\, P\, Q\, R = \lambda^* x.P\,(Q\, R)$ with $x \in fv(R)$:

$$\begin{array}{rcll} \lambda^* x.\mathbf{B}\, P\, Q\, R & \equiv & \mathbf{B}\,(\mathbf{B}\, P\, Q)\,(\lambda^* x.R) & \\ & = & (\mathbf{B}\, P)\,(\mathbf{B}\, Q\,(\lambda^* x.R)) & (assoc) \\ & \equiv & \lambda^* x.P\,(Q\, R) & \end{array}$$

4. (I) $\lambda^* x.\mathbf{I}\, P = \lambda^* x.P$ with $x \in fv(P)$:

$$\begin{array}{rcll} \lambda^* x.\mathbf{I}\, P & \equiv & \mathbf{B}\,\mathbf{I}\,(\lambda^* x.P) & \\ & = & \lambda^* x.P & \text{(BI)} \end{array}$$

5. ($\bullet$) $\lambda^* x.P^\bullet\, Q = \lambda^* x.Q\, P$ with $x \in fv(Q)$:

$$\begin{array}{rcl} \lambda^* x.P^\bullet\, Q & \equiv & \mathbf{B}\, P^\bullet\,(\lambda^* x.Q) \\ & \equiv & \lambda^* x.Q\, P \end{array}$$

Lemma 6 $(M[x := N])^\flat \equiv M^\flat[x := N^\flat]$.

<u>Proof</u> Easy induction on M.

Lemma 7 $(\lambda^* x.P)\, Q =_{\mathbf{BI}(\_)^\bullet} P[x := Q]$.

<u>Proof</u> Easy induction on P, using axioms (B), (I) and ($\bullet$).

Lemma 8 *$M =_\beta N$ implies $M^\flat =_{\mathbf{BI}(\_)^\bullet} N^\flat$.*

<u>Proof</u> Induction on the derivation of $M =_\beta N$.
The case of β-axiom $(\lambda x.M)\, N =_\beta M[x := N]$ follows from Lemma 6 and 7. Most other cases are obvious, except the case of the compatibility with lambda abstraction (the ξ-rule)

$$\frac{M =_\beta N}{\lambda x.M =_\beta \lambda x.N}$$

For this assume $M =_\beta N$. By induction hypothesis we have $M^\flat =_{\mathbf{BI}(\_)^\bullet} N^\flat$. By Lemma 5 we obtain $\lambda^* x.M^\flat =_{\mathbf{BI}(\_)^\bullet} \lambda^* x.N^\flat$, hence $(\lambda x.M)^\flat =_{\mathbf{BI}(\_)^\bullet} (\lambda x.N)^\flat$.

Lemma 9 $(P^\sharp)^\flat =_{\mathbf{BI}(\_)^\bullet} P$.

<u>Proof</u> Induction on P. The cases of variables, applications and $\mathbf{I}$ are obvious.

For $\mathbf{B}$:

$$\begin{array}{rcll}
(\mathbf{B}^\sharp)^\flat & \equiv & \mathbf{B}\,(\mathbf{B}\,\mathbf{I}^\bullet\,(\mathbf{B}\,\mathbf{B}\,\mathbf{I}))^\bullet\,(\mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,\mathbf{B}\,\mathbf{I})) & \\
& = & \mathbf{B}\,(\mathbf{B}\,\mathbf{I}^\bullet\,(\mathbf{B}\,\mathbf{I}\,\mathbf{B}))^\bullet\,(\mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,\mathbf{I}\,\mathbf{B})) & (\mathrm{BI2}) \\
& = & \mathbf{B}\,(\mathbf{B}\,\mathbf{I}^\bullet\,\mathbf{B})^\bullet\,(\mathbf{B}\,\mathbf{B}\,\mathbf{B}) & (\mathrm{BI}_{\mathbf{B}_0}) \\
& = & \mathbf{B}\,(\mathbf{B}\,\mathbf{I})^\bullet\,(\mathbf{B}\,\mathbf{B}\,\mathbf{B}) & (\mathrm{I}\bullet) \\
& = & \mathbf{B}\,(\mathbf{B}\,\mathbf{I}^\bullet\,(\mathbf{B}\,\mathbf{B}^\bullet\,\mathbf{B}))\,(\mathbf{B}\,\mathbf{B}\,\mathbf{B}) & (\mathrm{app}\bullet) \\
& = & \mathbf{B}\,\mathbf{I}^\bullet\,(\mathbf{B}\,\mathbf{B}^\bullet\,(\mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,\mathbf{B}\,\mathbf{B}))) & (\mathit{assoc}) \\
& = & \mathbf{B}\,\mathbf{I}^\bullet\,(\mathbf{B}\,(\mathbf{B}\,\mathbf{B})\,\mathbf{B}) & (\mathrm{B}\bullet) \\
& = & \mathbf{B}\,(\mathbf{B}\,\mathbf{I}^\bullet\,(\mathbf{B}\,\mathbf{B}))\,\mathbf{B} & (\mathit{assoc}) \\
& = & \mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}^\bullet)\,\mathbf{B}\,\mathbf{B})\,\mathbf{B} & (\mathrm{B}) \\
& = & \mathbf{B}\,(\mathbf{B}\,\mathbf{I}^{\bullet\bullet}\,\mathbf{B}\,\mathbf{B})\,\mathbf{B} & (\bullet\bullet) \\
& = & \mathbf{B}\,(\mathbf{I}^{\bullet\bullet}\,(\mathbf{B}\,\mathbf{B}))\,\mathbf{B} & (\mathrm{B}) \\
& = & \mathbf{B}\,(\mathbf{B}\,\mathbf{B}\,\mathbf{I}^\bullet)\,\mathbf{B} & (\bullet) \\
& = & \mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,\mathbf{I}^\bullet\,\mathbf{B}) & (\mathit{assoc}) \\
& = & \mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,\mathbf{I}) & (\mathrm{I}\bullet) \\
& = & \mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\,\mathbf{B} & (\mathrm{B3}) \\
& = & \mathbf{B} & (\mathrm{BI}_{\mathbf{B}})
\end{array}$$

For $P^\bullet$:

$$\begin{array}{rcll}
((P^\bullet)^\sharp)^\flat & \equiv & \mathbf{B}\,((P^\sharp)^\flat)^\bullet\,\mathbf{I} & \\
& = & \mathbf{B}\,\mathbf{I}\,((P^\sharp)^\flat)^\bullet & (\mathrm{BI2}) \\
& = & ((P^\sharp)^\flat)^\bullet & (\mathrm{BI}_\bullet) \\
& = & P^\bullet & \text{ind. hyp.}
\end{array}$$

Lemma 10 $(\lambda^* x.P)^\sharp =_\beta \lambda x.P^\sharp$.

<u>Proof</u> Induction on P.

- $P \equiv x$:

$$(\lambda^* x.x)^\sharp \equiv \mathbf{I}^\sharp \equiv \lambda x.x \equiv \lambda x.x^\sharp.$$

- $P \equiv Q\,R$ with $x \in fv(Q)$:

$$\begin{array}{rcll}
(\lambda^* x.Q\,R)^\sharp & \equiv & (\mathbf{B}\,R^\bullet\,(\lambda^* x.Q))^\sharp & \\
& \equiv & \mathbf{B}^\sharp\,(R^\bullet)^\sharp\,(\lambda^* x.Q)^\sharp & \\
& \equiv & \mathbf{B}^\sharp\,(\lambda u.u\,R^\sharp)\,(\lambda^* x.Q)^\sharp & \\
& =_\beta & \mathbf{B}^\sharp\,(\lambda u.u\,R^\sharp)\,(\lambda x.Q^\sharp) & \text{i.h.} \\
& \equiv & (\lambda xyz.x\,(y\,z))\,(\lambda u.u\,R^\sharp)\,(\lambda x.Q^\sharp) & \\
& =_\beta & \lambda z.(\lambda u.u\,R^\sharp)\,((\lambda x.Q^\sharp)\,z) & \\
& =_\beta & \lambda x.(\lambda u.u\,R^\sharp)\,Q^\sharp & \\
& =_\beta & \lambda x.(Q^\sharp\,R^\sharp) & \\
& \equiv & \lambda x.(Q\,R)^\sharp &
\end{array}$$

- $P \equiv Q\, R$ with $x \in fv(R)$:

$$\begin{array}{rcll}
(\lambda^* x.Q\, R)^\sharp & \equiv & (\mathbf{B}\, Q\, (\lambda^* x.R))^\sharp & \\
& \equiv & \mathbf{B}^\sharp\, Q^\sharp\, (\lambda^* x.R)^\sharp & \\
& =_\beta & \mathbf{B}^\sharp\, Q^\sharp\, (\lambda x.R^\sharp) & \text{i.h.} \\
& \equiv & (\lambda xyz.x\,(y\,z))\, Q^\sharp\, (\lambda x.R^\sharp) & \\
& =_\beta & \lambda z.Q^\sharp\, ((\lambda x.R^\sharp)\, z) & \\
& =_\beta & \lambda x.Q^\sharp\, R^\sharp & \\
& \equiv & \lambda x.(Q\, R)^\sharp &
\end{array}$$

Lemma 11 $(M^\flat)^\sharp =_\beta M$.

<u>Proof</u> Induction on M. Only the case of lambda abstraction is nontrivial, in which we use Lemma 10.

Proposition 12 $P =_{\mathbf{BI}(\_)^\bullet} Q$ *iff* $P^\sharp =_\beta Q^\sharp$.

<u>Proof</u> $P^\sharp =_\beta Q^\sharp$ implies $P =_{\mathbf{BI}(\_)^\bullet} (P^\sharp)^\flat =_{\mathbf{BI}(\_)^\bullet} (Q^\sharp)^\flat =_{\mathbf{BI}(\_)^\bullet} Q$ by Lemma 9 and 8.

Proposition 13 $M =_\beta N$ *iff* $M^\flat =_{\mathbf{BI}(\_)^\bullet} N^\flat$.

<u>Proof</u> $M^\flat =_{\mathbf{BI}(\_)^\bullet} N^\flat$ implies $M =_\beta (M^\flat)^\sharp =_\beta (N^\flat)^\sharp =_\beta N$ by Lemma 11 and 4.

In summary, we have shown that the axioms in Figure 1 are sound and complete for the β-theory of the planar lambda calculus.

2.2 Planar lambda algebras

A $\mathbf{BI}(\_)^\bullet$-algebra satisfying the axioms of Figure 1 will be called a *planar lambda algebra*. Note that our (BI) axioms are similar to Selinger's axioms for lambda algebras in the classical case (**SK**-algebras) [3], where $\mathbf{1} = \mathbf{S}\,(\mathbf{K}\,\mathbf{I})$ plays the role of $\mathbf{B}\,\mathbf{I}$. Any extensional $\mathbf{BI}(\_)^\bullet$-algebra is a planar lambda algebra, as all (BI) axioms follow from the axiom $\mathbf{B}\,\mathbf{I} = \mathbf{I}$ of extensional $\mathbf{BI}(\_)^\bullet$-algebras.

For planar lambda algebras $\mathcal{A}$ and $\mathcal{B}$, a homomorphism $h : \mathcal{A} \to \mathcal{B}$ is a map h from $\mathcal{A}$ to $\mathcal{B}$ satisfying $h(\mathbf{I}) = \mathbf{I}$, $h(\mathbf{B}) = \mathbf{B}$, $h(a^\bullet) = (h(a))^\bullet$ and $h(a\, b) = h(a)\, h(b)$.

The category of planar lambda algebras and homomorphisms will be denoted by **PlanarLamAlg**. The closed term model of the planar lambda calculus (modulo β-equality) gives an initial object Λ_0^{planar} of **PlanarLamAlg**.

3 Internal Operads

We expect that Hyland's approach to the lambda calculus using semi-closed cartesian operads [2] and our previous approach to extensional $\mathbf{BI}(\_)^\bullet$-algebras

$\mathbf{B}\,a\,b\,c$	$=$	$a\,(b\,c)$	(B)
$\mathbf{I}\,a$	$=$	a	(I)
$a^\bullet\,b$	$=$	$b\,a$	($\bullet$)
$\mathbf{B}\,(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\,\mathbf{B}$	$=$	$\mathbf{B}$	(BI$_\mathbf{B}$)
$\mathbf{B}\,\mathbf{I}\,\mathbf{I}$	$=$	$\mathbf{I}$	(BI$_\mathbf{I}$)
$\mathbf{B}\,\mathbf{I}\,a^\bullet$	$=$	$a^\bullet$	(BI$_\bullet$)
$\mathbf{B}\,\mathbf{B}^\bullet\,(\mathbf{B}\,\mathbf{B}\,(\mathbf{B}\,\mathbf{B}\,\mathbf{B}))$	$=$	$\mathbf{B}\,(\mathbf{B}\,\mathbf{B})\,\mathbf{B}$	(B$\bullet$)
$\mathbf{B}\,\mathbf{I}^\bullet\,\mathbf{B}$	$=$	$\mathbf{B}\,\mathbf{I}$	(I$\bullet$)
$\mathbf{B}\,a^{\bullet\bullet}\,\mathbf{B}$	$=$	$\mathbf{B}\,(\mathbf{B}\,a^\bullet)\,\mathbf{B}$	($\bullet\bullet$)
$(a\,b)^\bullet$	$=$	$\mathbf{B}\,b^\bullet\,(\mathbf{B}\,a^\bullet\,\mathbf{B})$	(app$\bullet$)

If we write $a \circ b$ for the composition $\mathbf{B}\,a\,b$, they can be rewritten as follows.

$\mathbf{B}\,a\,b\,c$	$=$	$a\,(b\,c)$	(B)
$\mathbf{I}\,a$	$=$	a	(I)
$a^\bullet\,b$	$=$	$b\,a$	($\bullet$)
$(\mathbf{B}\,(\mathbf{B}\,\mathbf{I})) \circ \mathbf{B}$	$=$	$\mathbf{B}$	(BI$_\mathbf{B}$)
$\mathbf{I} \circ \mathbf{I}$	$=$	$\mathbf{I}$	(BI$_\mathbf{I}$)
$\mathbf{I} \circ a^\bullet$	$=$	$a^\bullet$	(BI$_\bullet$)
$\mathbf{B}^\bullet \circ (\mathbf{B} \circ (\mathbf{B} \circ \mathbf{B}))$	$=$	$(\mathbf{B}\,\mathbf{B}) \circ \mathbf{B}$	(B$\bullet$)
$\mathbf{I}^\bullet \circ \mathbf{B}$	$=$	$\mathbf{B}\,\mathbf{I}$	(I$\bullet$)
$a^{\bullet\bullet} \circ \mathbf{B}$	$=$	$(\mathbf{B}\,a^\bullet) \circ \mathbf{B}$	($\bullet\bullet$)
$(a\,b)^\bullet$	$=$	$b^\bullet \circ (a^\bullet \circ \mathbf{B})$	(app$\bullet$)

Figure 1: Axioms of planar lambda algebras

using closed operads and the internal operad construction [1] can be applied to the planar lambda calculus (with the β-equality) and planar lambda algebras. Below we shall spell out some of the basic concepts and preliminary results towards this direction.

3.1 From semi-closed operads to planar lambda algebras

Every semi-closed operad $\mathcal{P}$ gives rise to a planar lambda algebra $\mathcal{P}(0)$:

Proposition 14 *For any semi-closed operad P with* $\mathbf{app} \in \mathcal{P}(2)$ *and* $\lambda : \mathcal{P}(n+1) \to \mathcal{P}(n)$*,* $\mathcal{P}(0)$ *is a planar lambda algebra with* $a \cdot b = \mathbf{app}(a, b)$*,* $\mathbf{I} = \lambda(id)$*,* $\mathbf{B} = \lambda(\lambda(\lambda(\mathbf{app}(id, \mathbf{app}))))$ *and* $a^\bullet = \lambda(\mathbf{app}(id, a))$*. This map* $\mathcal{P} \mapsto \mathcal{P}(0)$ *extends to a functor* $\mathcal{U}$: **SemiClosedOperad** $\to$ **PlanarLamAlg** *sending* $\varphi : \mathcal{P} \to \mathcal{Q}$ *to* $\varphi_0 : \mathcal{P}(0) \to \mathcal{Q}(0)$*.*

Proof Verifying that $\mathcal{P}(0)$ is a planar lambda algebra is routine, and essentially amounts to the soundness of the translation $(-)^\sharp$ into the planar lambda calculus (Lemma 4). Seeing that $\varphi_0 : \mathcal{P}(0) \to \mathcal{Q}(0)$ is a homomorphism of planar lambda algebras is immediate as φ preserves all the constructs of the planar lambda algebras by definition.

Proposition 15 *Let $\mathcal{P}$ and $\mathcal{Q}$ be semi-closed operads. Suppose that there is a homomorphism $h : \mathcal{P}(0) \to \mathcal{Q}(0)$ between the planar lambda algebras $\mathcal{P}(0)$ and $\mathcal{Q}(0)$ given as the previous proposition. Then there exists a homomorphism of operads $\varphi : \mathcal{P} \to \mathcal{Q}$ such that $\varphi_0 = h$ holds.*

Proof Define $\varphi_n : \mathcal{P}(n) \to \mathcal{Q}(n)$ by $\varphi_0 = h$ and $\varphi_{n+1}(f) = \mathbf{app}(\varphi_n(\lambda(f)), id)$. We shall verify that φ is a homomorphism of semi-closed operads.

- $\varphi_1(id) = id$:

$$\begin{array}{rcll} \varphi_1(id) & = & \mathbf{app}(h(\lambda(id)), id) & \\ & = & \mathbf{app}(h(\mathbf{I}), id) & (\mathbf{I} = \lambda(id)) \\ & = & \mathbf{app}(\mathbf{I}, id) & (h(\mathbf{I}) = \mathbf{I}) \\ & = & \mathbf{app}(\lambda(id), id) & (\mathbf{I} = \lambda(id)) \\ & = & id & \end{array}$$

- $\varphi_{k_1+\ldots+k_n}(g(f_1, \ldots, f_n)) = \varphi_n(g)(\varphi_{k_1}(f_1) \ldots, \varphi_{k_n}(f_n))$:

  For $f \in \mathcal{P}(n)$ let $\lceil f \rceil = \overbrace{\lambda(\ldots\lambda}^{n \text{ times}}(f)\ldots) \in \mathcal{P}(0)$. We shall note that $\varphi_n(f) = \overbrace{\mathbf{app}(\ldots\mathbf{app}}^{n \text{ times}}(h(\lceil f \rceil), id) \ldots, id)$ holds. Then

$$\lceil g(f_1, \ldots, f_n) \rceil = F \, \lceil g \rceil \, \lceil f_1 \rceil \, \ldots \, \lceil f_n \rceil$$

  holds, where $F \in \mathcal{P}(0)$ is given by

$$\lambda^* p q_1 \ldots q_n x_{11} \ldots x_{nk_n}.p \, (q_1 \, x_{11} \, \ldots \, x_{1k_1}) \, \ldots \, (q_n \, x_{n1} \, \ldots \, x_{nk_n}).$$

Since h is a homomorphism of planar lambda algebras, we have

$$\begin{array}{rl} & h(\lceil g(f_1,\ldots,f_n)\rceil) \\ = & h(F\,\lceil g\rceil\,\lceil f_1\rceil\,\ldots\,\lceil f_n\rceil) \\ = & F\,(h(\lceil g\rceil))\,(h(\lceil f_1\rceil))\,\ldots\,(h(\lceil f_n\rceil)) \\ = & \lambda^* x_{11}\ldots x_{nk_n}.h(\lceil g\rceil)\,(h(\lceil f_1\rceil)\,x_{11}\,\ldots\,x_{1k_1})\,\ldots\,(h(\lceil f_n\rceil)\,x_{n1}\,\ldots\,x_{nk_n}) \end{array}$$

and $\varphi_{k_1+\ldots+k_n}(g(f_1,\ldots,f_n)) = \varphi_n(g)(\varphi_{k_1}(f_1)\ldots,\varphi_{k_n}(f_n))$.[2]

- $\varphi_2(\mathbf{app}) = \mathbf{app}$:

$$\begin{array}{rcll} \varphi_2(\mathbf{app}) & = & \mathbf{app}(\mathbf{app}(h(\lambda(\lambda(\mathbf{app}))), id), id) & \\ & = & \mathbf{app}(\mathbf{app}(h(\mathbf{B}\,\mathbf{I}), id), id) & (\lambda(\lambda(\mathbf{app})) = \mathbf{B}\,\mathbf{I}) \\ & = & \mathbf{app}(\mathbf{app}(\mathbf{B}\,\mathbf{I}, id), id) & (h(\mathbf{B}\,\mathbf{I}) = \mathbf{B}\,\mathbf{I}) \\ & = & \mathbf{app}(\mathbf{app}(\lambda(\lambda(\mathbf{app})), id), id) & (\lambda(\lambda(\mathbf{app})) = \mathbf{B}\,\mathbf{I}) \\ & = & \mathbf{app}. & \end{array}$$

- $\varphi_1(\lambda(\mathbf{app})) = \lambda(\mathbf{app})$:

$$\begin{array}{rcll} \varphi_1(\lambda(\mathbf{app})) & = & \mathbf{app}(h(\lambda(\lambda(\mathbf{app}))), id) & \\ & = & \mathbf{app}(\lambda(\lambda(\mathbf{app})), id) & \text{just as the above} \\ & = & \lambda(\mathbf{app}). & \end{array}$$

Corollary 16 *Let $\mathcal{P}$ and $\mathcal{Q}$ be semi-closed operads such that $\mathcal{P}(0)$ and $\mathcal{Q}(0)$ are isomorphic as planar lambda algebras. Then $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic as semi-closed operads.*

Thus, for any planar combinatory algebra $\mathcal{A}$, up to isomorphism there is at most one semi-closed operad $\mathcal{P}$ such that $\mathcal{P}(0) \cong \mathcal{A}$. This applies to extensional $\mathbf{BI}(\_)^\bullet$-algebras too, and the claim in [1] that there can be many non-isomorphic closed operads giving rise to the same extensional $\mathbf{BI}(\_)^\bullet$-algebra is invalid. The adjunction between closed operads and extensional $\mathbf{BI}(\_)^\bullet$-algebras is actually an equivalence.

3.2 Internal operads of planar lambda algebras

The internal operad construction [1] can be carried out on any planar lambda algebra and the construction gives an equivalence between **SemiClosedOperad** and **PlanarLamAlg**. We shall spell out the expected construction, which is largely the same as the extensional case [1], though the lack of extensionality calls for some extra care.

Definition 17 *An element a of a planar lambda algebra $\mathcal{A}$ is said to be of arity $m \to n$ when*

$$a^\bullet \circ \mathbf{B}^{m+1} = (\mathbf{B}\,a) \circ \mathbf{B}^n \quad \text{and} \quad (\mathbf{B}^m\,\mathbf{I}) \circ a = a$$

hold.

[2] This part is hard to follow, largely because the notations of operads and those of combinatory algebras are badly mixed. A better presentation would be desirable.

Note that, in the extensional case [1], only the first equation in Definition 17 is required; the second equation is always valid in the extensional case.

For the basic constructs of planar lambda algebras, we have

- $\mathbf{B}$ is of arity $2 \to 1$ by the axioms (B•) and ($\mathrm{BI}_{\mathbf{B}}$);
- $\mathbf{I}$ is of arity $0 \to 0$ by the axioms (I•) and ($\mathrm{BI}_{\mathbf{I}}$); and
- $a^\bullet$ is of arity $0 \to 1$ by the axioms (••) and ($\mathrm{BI}_\bullet$).

Thus six among the ten axioms of planar lambda algebras are directly related to the notion of arity. Assuming the first equation $a^\bullet \circ \mathbf{B}^{m+1} = (\mathbf{B}\, a) \circ \mathbf{B}^n$, the second equation $(\mathbf{B}^m\, \mathbf{I}) \circ a = a$ is equivalent to $a \circ (\mathbf{B}^n \mathbf{I}) = a$. It follows that the composition respects the arities, and $\mathbf{B}^m\, \mathbf{I} : m \to m$ serves as the identity on m. We shall note that, when $n = 1$, a is of arity $m \to 1$ if and only if the equation

$$(a\, \mathbf{I})^\bullet \circ \mathbf{B}^m = a$$

holds.[3] This is the same condition as the one used for the extensional case [1]. So, as long as we are to define internal operads (where only the case of $n = 1$ is needed), we can re-use the same characterization from the extensional case. However, for handling the internal PRO, we do need an extra axiom $(\mathbf{B}^m\, \mathbf{I}) \circ a = a$.

Tensor products are given using the composition $\circ$ and the following "adding lower/upper strands" constructions [1]: for $a : m \to n$,

$$k + a = \mathbf{B}^k\, a : k + m \to k + n$$

and

$$a + k = (\mathbf{B}^{m+k}\, \mathbf{I}) \circ a = a \circ (\mathbf{B}^{n+k}\, \mathbf{I}) : m + k \to n + k.$$

Then, for $a : m \to n$ and $a' : m' \to n'$, their tensor $a + a' : m + m' \to n + n'$ is $(a + m') \circ (n + a') = (m + a') \circ (a + n')$.

These data determine a PRO $\mathcal{C}_\mathcal{A}$ with $\mathcal{C}_\mathcal{A}(m, n) = \{a \in \mathcal{A} \mid a : m \to n\}$ and an operad — the internal operad — $\mathcal{P}_\mathcal{A}$ with $\mathcal{P}_\mathcal{A}(m) = \mathcal{C}_\mathcal{A}(m, 1)$.

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[3] This might not be entirely obvious. Assuming $a : m \to 1$, we have

$$\begin{array}{rcll}
(a\, \mathbf{I})^\bullet \circ \mathbf{B}^m & = & \mathbf{I}^\bullet \circ a^\bullet \circ \mathbf{B}^{m+1} & (\mathrm{app}\bullet) \\
& = & \mathbf{I}^\bullet \circ (\mathbf{B}\, a) \circ \mathbf{B} & (a : m \to 1) \\
& = & ((\mathbf{B}\, (\mathbf{I}^\bullet) \circ \mathbf{B})\, a) \circ \mathbf{B} & (B) \\
& = & ((\mathbf{I}^{\bullet\bullet} \circ \mathbf{B})\, a) \circ \mathbf{B} & (\bullet\bullet) \\
& = & a \circ \mathbf{I}^\bullet \circ \mathbf{B} & (B, \bullet) \\
& = & a \circ \mathbf{B}\, \mathbf{I} & (I\bullet) \\
& = & a & (a : m \to 1).
\end{array}$$

Conversely, assuming $a = b^\bullet \circ \mathbf{B}^m$, we have $a : m \to 1$ from $b^\bullet : 0 \to 1$ and $\mathbf{B} : 2 \to 1$ using the argument for adding upper strands: whenever $a : l \to m$ and $b : m + k \to n$, $a \circ b = a \circ (\mathbf{B}^{m+k}\, \mathbf{I}) \circ b = (a + k) \circ b$ is of arity $l + k \to n$.

For $a \in \mathcal{P}_{\mathcal{A}}(m+1)$, let $\lambda(a) = (a\,\mathbf{I})^\bullet \circ \mathbf{B}^m \in \mathcal{P}_{\mathcal{A}}(m)$. Let $\mathbf{app} = \mathbf{B} \in \mathcal{P}_{\mathcal{A}}(2)$. Then

$$\begin{array}{rcl}
\mathbf{app}(\lambda(a), id) & = & (\lambda(a)+1) \circ \mathbf{app} \\
& = & (\mathbf{B}^{m+1}\,\mathbf{I}) \circ (a\,\mathbf{I})^\bullet \circ \mathbf{B}^m \circ \mathbf{B} \\
& = & (\mathbf{B}^{m+1}\,\mathbf{I}) \circ \mathbf{I}^\bullet \circ a^\bullet \circ \mathbf{B} \circ \mathbf{B}^m \circ \mathbf{B} \\
& = & (\mathbf{B}^{m+1}\,\mathbf{I}) \circ \mathbf{I}^\bullet \circ (\mathbf{B}\,a) \circ \mathbf{B} \\
& = & (\mathbf{B}^{m+1}\,\mathbf{I}) \circ a \circ \mathbf{I}^\bullet \circ \mathbf{B} \\
& = & a \circ (\mathbf{B}\,\mathbf{I}) \\
& = & a
\end{array}$$

as expected. Moreover, the naturality $b \circ \lambda(a) = \lambda((b+1) \circ a) \in \mathcal{P}_{\mathcal{A}}(m')$ holds for $a \in \mathcal{P}_{\mathcal{A}}(m)$ and $b : m' \to m$. Hence $\mathcal{P}_{\mathcal{A}}$ is a semi-closed operad. On the other hand, the η-equality is not valid: for $a \in \mathcal{P}_{\mathcal{A}}(m)$, $\lambda(\mathbf{app}(a, id)) = \lambda((a+1) \circ \mathbf{B}) = a \circ \lambda(\mathbf{B}) = a \circ (\mathbf{B}\,\mathbf{I})^\bullet \circ \mathbf{B}$ may not be equal to a.

It follows that $\mathcal{P}_{\mathcal{A}}(0) = \mathcal{A}^\bullet \equiv \{a^\bullet \mid a \in \mathcal{A}\}$. The axioms $(\bullet\bullet)$ and $(\mathrm{BI}_\bullet)$ say that $a^\bullet$ is of arity $0 \to 1$, hence $a^\bullet \in \mathcal{P}_{\mathcal{A}}(0)$. Conversely, if b is of arity $0 \to 1$,

$$\begin{array}{rcll}
b & = & b \circ (\mathbf{B}\,\mathbf{I}) & (b : 0 \to 1) \\
& = & b \circ \mathbf{I}^\bullet \circ \mathbf{B} & (\mathrm{I}\bullet) \\
& = & (\mathbf{B}\,\mathbf{I}^{\bullet\bullet}\,\mathbf{B}\,b) \circ \mathbf{B} & (B) \\
& = & (\mathbf{B}\,(\mathbf{B}\,\mathbf{I}^\bullet)\,\mathbf{B}\,b) \circ \mathbf{B} & (\bullet\bullet) \\
& = & \mathbf{I}^\bullet \circ (\mathbf{B}\,b) \circ \mathbf{B} & (B) \\
& = & \mathbf{I}^\bullet \circ b^\bullet \circ \mathbf{B} & (b : 0 \to 1) \\
& = & (b\,\mathbf{I})^\bullet & (\mathrm{app}\bullet)
\end{array}$$

hence $b = (b\,\mathbf{I})^\bullet \in \mathcal{A}^\bullet$. As in the extensional case, $\mathcal{A}^\bullet \cong \mathcal{A}$ holds, via $(\_)\,\mathbf{I} : \mathcal{A}^\bullet \to \mathcal{A}$ and $(\_)^\bullet : \mathcal{A} \to \mathcal{A}^\bullet$. Thus $\mathcal{P}_{\mathcal{A}}$ is a semi-closed operad such that $\mathcal{P}_{\mathcal{A}}(0)$ is isomorphic to $\mathcal{A}$.

3.3 Example: the internal operad of the planar lambda calculus

Consider the planar lambda algebra Λ_0^{planar} of the β-equivalence classes of closed planar lambda terms. Then an element a is of arity $m \to n$ if and only if a is the equivalence class of a β-normal form

$$\lambda f x_1 \ldots x_m.f\,M_1\ \ldots\ M_n$$

with no free f in M_i's.[4] In particular, an element of arity $a \to 1$ is of the form

$$\lambda f x_1 \ldots x_m.f\,M$$

which encodes

$$x_1, \ldots\, x_m \vdash M$$

in $\Lambda^{planar}(n)$ of the semi-closed operad Λ^{planar}. For instance:

[4]This claim is far from obvious; we even think that this is one of the most difficult results in our study.

- $\mathbf{B} = \lambda f x_1 x_2 . f\,(x_1\,x_2)$ encodes the application $\mathbf{app} \;=\; x_1, x_2 \vdash x_1\,x_2$.
- $\mathbf{B}\,\mathbf{I} = \lambda f x_1 . f\,x_1$ encodes the identity $id \;=\; x_1 \vdash x_1$.
- For closed M, $M^\bullet = \lambda f . f\,M$ encodes $\vdash M$.

For $a = \lambda f x_1 \ldots x_m . f\,M_1\;\ldots\;M_n$, adding lower k strands gives

$$k + a = \mathbf{B}^k\,a = \lambda f y_1 \ldots y_k x_1 \ldots x_m . f\;y_1\;\ldots\;y_k\,M_1\;\ldots\;M_n$$

whereas adding upper k strands gives

$$a + k = (\mathbf{B}^{m+k}\,\mathbf{I}) \circ a = \lambda f x_1 \ldots x_m z_1 \ldots z_k . f\,M_1\;\ldots\;M_n\;z_1\;\ldots\;z_k.$$

For $a = \lambda f x_1 \ldots x_m . f\,M_1\;\ldots\;M_n$ and $a' = \lambda f y_1 \ldots y_{m'} . f\,M'_1\;\ldots\;M'_{n'}$, their tensor product $a + a'$ is

$$\lambda f x_1 \ldots x_m, y_1 \ldots y_{m'} . f\,M_1\;\ldots\;M_n\,M'_1\;\ldots\;M'_{n'}.$$

For

$$a = \lambda f x_1 \ldots x_l . f\,M_1\;\ldots\;M_m : l \to m$$

and

$$b = \lambda f y_1 \ldots y_m z_1 \ldots z_k . f\,N_1\;\ldots\;N_n : m + k \to n,$$

$a \circ b : l + k \to n$ is

$$\lambda f x_1 \ldots x_l z_1 \ldots z_k . (f\,N_1 \ldots N_n)[y_1 := M_1, \ldots, y_m := M_m].$$

For $a = \lambda f x_1 \ldots x_n x_{n+1} . f\,M$, let $\lambda(a) = \lambda f x_1 \ldots x_n . f\,(\lambda x_{n+1} . M)$. For $a = \lambda f x_1 \ldots x_m . f\,M$ and $b = \lambda f y_1 \ldots y_n . f\,N$, $\mathbf{app}(a, b) = a \circ (\mathbf{B}\,b) \circ \mathbf{B}$ gives

$$\lambda f x_1 \ldots x_m y_1 \ldots y_n . f\,(M\,N)$$

as expected.

All these can be given in the graphical language of the planar lambda calculus as done in [1]. The crucial difference from *ibid.* is that now adding upper stands is not free at all as we do not assume the η-equality.

4 Extensions

We sketch three extensions of planar lambda algebras: the symmetric, braided, and cartesian lambda algebras.

Adding Symmetry A *symmetry* in a planar lambda algebra is an element $\mathbf{C}$ which is subject to the following conditions.

- $\mathbf{C}$ is of arity $2 \to 2$, i.e., satisfies

$$\mathbf{C}^\bullet \circ \mathbf{B}^3 = (\mathbf{B}\,\mathbf{C}) \circ \mathbf{B}^2 \quad \text{and} \quad (\mathbf{B}^2\,\mathbf{I}) \circ \mathbf{C} = \mathbf{C}.$$

- The Coxter relations (or Reidemeister moves)

$$\mathbf{C} \circ \mathbf{C} = \mathbf{B}^2\,\mathbf{I} \quad \text{and} \quad (\mathbf{B}\,\mathbf{C}) \circ \mathbf{C} \circ (\mathbf{B}\,\mathbf{C}) = \mathbf{C} \circ (\mathbf{B}\,\mathbf{C}) \circ \mathbf{C}$$

  hold.

- Naturality with respect to $\mathbf{B} : 2 \to 1$ and $a^\bullet : 0 \to 1$

$$(\mathbf{B}\,\mathbf{B}) \circ \mathbf{C} = \mathbf{C} \circ (\mathbf{B}\,\mathbf{C}) \circ \mathbf{B} \quad \text{and} \quad a^\bullet \circ \mathbf{C} = \mathbf{B}\,a^\bullet$$

  hold.

From the naturality with respect to $a^\bullet$, we can derive

$$\mathbf{C}\,a\,b\,c = a\,c\,b.$$

It follows that $a^\bullet = \mathbf{C}\,\mathbf{I}\,a$ holds, and more generally $a^\bullet \circ b = \mathbf{C}\,b\,a$ is derivable. So it is possible to axiomatize planar lambda algebras with a symmetry as **BCI**-algebras where $(\_)^\bullet$ is not a primitive construct but a derived operator $\mathbf{C}\,\mathbf{I}\,(\_)$. For instance, the arity condition for $a : m \to n$ can be replaced by

$$(\mathbf{C}\,\mathbf{B}\,a) \circ \mathbf{B}^m = (\mathbf{B}\,a) \circ \mathbf{B}^n \quad \text{and} \quad (\mathbf{B}^m\,\mathbf{I}) \circ a = a.$$

Such an axiomatization is given in Figure 2. In this axiomatization,

- $(\mathrm{BI}_{\mathbf{B}})$ and (α) say $\mathbf{B} : 2 \to 1$;
- $(\mathrm{BI}_{\mathbf{C}})$ and (cox_2) say $\mathbf{C} : 2 \to 2$;
- $(\mathrm{BI}_{\mathbf{I}})$ and (ρ) say $\mathbf{I} : 0 \to 0$;
- (cox_1) and (cox_3) are the Coxter relations: and
- (bc) is the naturality of $\mathbf{C}$ with respect to $\mathbf{B} : 2 \to 1$.

Let us call such algebras *symmetric lambda algebras* (or *linear lambda algebras* if we want to emphasize linearity). Symmetric lambda algebras satisfying $\mathbf{B}\,\mathbf{I} = \mathbf{I}$ are precisely the extensional **BCI**-algebras in [1]. The internal operad of a symmetric lambda algebra is a semi-closed symmetric operad.

Adding Braiding A *braiding* in a planar lambda algebra is a pair of elements $\mathbf{C}^+$ and $\mathbf{C}^-$ which are subject to the following conditions.

- $\mathbf{C}^+$ and $\mathbf{C}^-$ are of arity $2 \to 2$, i.e.,

$$\mathbf{C}^{\pm\bullet} \circ \mathbf{B}^3 = (\mathbf{B}\,\mathbf{C}^\pm) \circ \mathbf{B}^2 \quad \text{and} \quad (\mathbf{B}^2\,\mathbf{I}) \circ \mathbf{C}^\pm = \mathbf{C}^\pm.$$

- The Coxter relations (or Reidemeister moves):

$$\mathbf{C}^\pm \circ \mathbf{C}^\mp = \mathbf{B}^2\,\mathbf{I} \quad \text{and} \quad (\mathbf{B}\,\mathbf{C}^\pm) \circ \mathbf{C}^\pm \circ (\mathbf{B}\,\mathbf{C}^\pm) = \mathbf{C}^\pm \circ (\mathbf{B}\,\mathbf{C}^\pm) \circ \mathbf{C}^\pm.$$

$\mathbf{B}\,a\,b\,c$	$=$	$a\,(b\,c)$	(B)
$\mathbf{C}\,a\,b\,c$	$=$	$a\,c\,b$	(C)
$\mathbf{I}\,a$	$=$	a	(I)
$(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\circ\mathbf{B}$	$=$	$\mathbf{B}$	$(\mathrm{BI}_{\mathbf{B}})$
$(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\circ\mathbf{C}$	$=$	$\mathbf{C}$	$(\mathrm{BI}_{\mathbf{C}})$
$\mathbf{I}\circ\mathbf{I}$	$=$	$\mathbf{I}$	$(\mathrm{BI}_{\mathbf{I}})$
$\mathbf{C}\,\mathbf{B}\,\mathbf{I}$	$=$	$\mathbf{B}\,\mathbf{I}$	(ρ)
$(\mathbf{B}\,\mathbf{B})\circ\mathbf{B}$	$=$	$(\mathbf{C}\,\mathbf{B}\,\mathbf{B})\circ(\mathbf{B}\circ\mathbf{B})$	(α)
$\mathbf{C}\circ\mathbf{C}$	$=$	$\mathbf{B}\,(\mathbf{B}\,\mathbf{I})$	(cox_1)
$(\mathbf{B}\,\mathbf{C})\circ(\mathbf{B}\circ\mathbf{B})$	$=$	$(\mathbf{C}\,\mathbf{B}\,\mathbf{C})\circ(\mathbf{B}\circ\mathbf{B})$	(cox_2)
$(\mathbf{B}\,\mathbf{C})\circ(\mathbf{C}\circ(\mathbf{B}\,\mathbf{C}))$	$=$	$\mathbf{C}\circ((\mathbf{B}\,\mathbf{C})\circ\mathbf{C})$	(cox_3)
$(\mathbf{B}\,\mathbf{B})\circ\mathbf{C}$	$=$	$\mathbf{C}\circ((\mathbf{B}\,\mathbf{C})\circ\mathbf{B})$	(bc)

Figure 2: Axioms of symmetric lambda algebras

- Naturality with respect to $\mathbf{B} : 2 \to 1$ and $a^{\bullet} : 0 \to 1$:

$$(\mathbf{B}\,\mathbf{B})\circ\mathbf{C}^{\pm} = \mathbf{C}^{\pm}\circ(\mathbf{B}\,\mathbf{C}^{\pm})\circ\mathbf{B} \quad \text{and} \quad a^{\bullet}\circ\mathbf{C}^{\pm} = \mathbf{B}\,a^{\bullet}.$$

From the naturality condition we can defrive

$$\mathbf{C}^{\pm}\,a\,b\,c = a\,c\,b \quad \text{and} \quad \mathbf{C}^{+}\,a\,b = \mathbf{C}^{-}\,a\,b.$$

We shall call a planar lambda algebra with a braiding a *braided lambda algebra*. Similarly to the case of symmetry, it is possible to axiomatize braided lambda algebras in terms of $\mathbf{B}$, $\mathbf{C}^{+}$, $\mathbf{C}^{-}$ and $\mathbf{I}$; see Figure 3.

The internal operad of a braided lambda algebra is a semi-closed braided operad.

Adding Comonoid Structure A *cartesian lambda algebra* is a symmetric lambda algebra with elements $\mathbf{W}$ and $\mathbf{K}$ subject to the axioms saying

- $\mathbf{W} : 1 \to 2$ and $\mathbf{K} : 1 \to 0$,
- $\mathbf{W}$ and $\mathbf{K}$ form a co-commutative comonoid, and
- $\mathbf{B}$ and $a^{\bullet}$ are comonoid morphisms (the latter implies $\mathbf{W}\,a\,b = a\,b\,b$ and $\mathbf{K}\,a\,b = a$).

Explicitly, these axioms can be given as Figure 4. Cartesian lambda algebras are precisely the lambda algebras in the sense of [3], and their internal operads are semi-closed cartesian operads.

References

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$\mathbf{B}\,a\,b\,c$	$=$	$a\,(b\,c)$	(B)
$\mathbf{C}^{\star}\,a\,b\,c$	$=$	$a\,c\,b$	(C)
$\mathbf{I}\,a$	$=$	a	(I)
$\mathbf{C}^{+}\,a\,b$	$=$	$\mathbf{C}^{-}\,a\,b$	$(C2)$
$(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\circ\mathbf{B}$	$=$	$\mathbf{B}$	$(\mathrm{BI}_{\mathbf{B}})$
$(\mathbf{B}\,(\mathbf{B}\,\mathbf{I}))\circ\mathbf{C}^{\pm}$	$=$	$\mathbf{C}^{\pm}$	$(\mathrm{BI}_{\mathbf{C}^{\pm}})$
$\mathbf{I}\circ\mathbf{I}$	$=$	$\mathbf{I}$	$(\mathrm{BI}_{\mathbf{I}})$
$\mathbf{C}^{\star}\,\mathbf{B}\,\mathbf{I}$	$=$	$\mathbf{I}$	(ρ)
$(\mathbf{B}\,\mathbf{B})\circ\mathbf{B}$	$=$	$(\mathbf{C}^{\star}\,\mathbf{B}\,\mathbf{B})\circ(\mathbf{B}\circ\mathbf{B})$	(α)
$\mathbf{C}^{\pm}\circ\mathbf{C}^{\mp}$	$=$	$\mathbf{B}\,(\mathbf{B}\,\mathbf{I})$	(cox_1)
$(\mathbf{B}\,\mathbf{C}^{\pm})\circ(\mathbf{B}\circ\mathbf{B})$	$=$	$(\mathbf{C}^{\star}\,\mathbf{B}\,\mathbf{C}^{\pm})\circ(\mathbf{B}\circ\mathbf{B})$	(cox_2)
$(\mathbf{B}\,\mathbf{C}^{\pm})\circ(\mathbf{C}^{\pm}\circ(\mathbf{B}\,\mathbf{C}^{\pm}))$	$=$	$\mathbf{C}^{\pm}\circ((\mathbf{B}\,\mathbf{C}^{\pm})\circ\mathbf{C}^{\pm})$	(cox_3)
$(\mathbf{B}\,\mathbf{B})\circ\mathbf{C}^{\pm}$	$=$	$\mathbf{C}^{\pm}\circ((\mathbf{B}\,\mathbf{C}^{\pm})\circ\mathbf{B})$	(bc)

The double signs $\pm$ and $\mp$ in an equation should be taken as appropriately linked, while $\star$ indicates an arbitrary choice of $+$ or $-$.

Figure 3: Axioms of braided lambda algebras

$\mathbf{W}^{\bullet}\circ\mathbf{B}\circ\mathbf{B}$	$=$	$(\mathbf{B}\,\mathbf{W})\circ\mathbf{B}\circ\mathbf{B}$	$(\mathbf{W}:1\to 2)$
$\mathbf{K}^{\bullet}\circ\mathbf{B}\circ\mathbf{B}$	$=$	$\mathbf{B}\,\mathbf{K}$	$(\mathbf{K}:1\to 0)$
$(\mathbf{B}\,\mathbf{I})\circ\mathbf{W}$	$=$	$\mathbf{W}$	$(\mathbf{W}:1\to 2)$
$(\mathbf{B}\,\mathbf{I})\circ\mathbf{K}$	$=$	$\mathbf{K}$	$(\mathbf{W}:1\to 0)$
$\mathbf{W}\circ\mathbf{K}$	$=$	$\mathbf{B}\,\mathbf{I}$	(*co-unit*)
$\mathbf{W}\circ\mathbf{W}$	$=$	$\mathbf{W}\circ(\mathbf{B}\,\mathbf{W})$	(*co-associativity*)
$\mathbf{W}\circ\mathbf{C}$	$=$	$\mathbf{W}$	(*co-commutativity*)
$\mathbf{B}\circ\mathbf{W}$	$=$	$(\mathbf{B}\,\mathbf{W})\circ\mathbf{W}\circ(\mathbf{B}\,\mathbf{C})\circ\mathbf{B}\circ(\mathbf{B}\,\mathbf{B})$	($\mathbf{B}$ *comonoid hom*)
$\mathbf{B}\circ\mathbf{K}$	$=$	$\mathbf{K}\circ\mathbf{K}$	($\mathbf{B}$ *comonoid hom*)
$a^{\bullet}\circ\mathbf{W}$	$=$	$a^{\bullet}\circ a^{\bullet}$	($a^{\bullet}$ *comonoid hom*)
$a^{\bullet}\circ\mathbf{K}$	$=$	$\mathbf{I}$	($a^{\bullet}$ *comonoid hom*)

Figure 4: Axioms of cartesian lambda algebras (only those for **W** and **K**)

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