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A note on the biadjunction between 2-categories of traced monoidal categories and tortile monoidal categories

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Abstract

We illustrate a minor error in the biadjointness result for 2-categories of traced monoidal categories and tortile monoidal categories stated by Joyal, Street and Verity. We also show that the biadjointness holds after suitably changing the definition of 2-cells.

In the seminal paper "Traced Monoidal Categories" by Joyal, Street and Verity [**4**], it is claimed that the Int-construction gives a left biadjoint of the inclusion of the 2-category **TortMon** of tortile monoidal categories, balanced strong monoidal functors and monoidal natural transformations in the 2-category **TraMon** of traced monoidal categories, traced strong monoidal functors and monoidal natural transformations [**4**, proposition 5·2]. However, this statement is not correct. We shall give a simple counterexample below.

Notation. We follow notations and conventions used in [**4**]. We write Int $\mathcal{V}$ for the tortile monoidal category obtained by the Int-construction on a traced monoidal category $\mathcal{V}$, and $N : \mathcal{V} \to \text{Int } \mathcal{V}$ for the canonical functor defined by $N(X) = (X, I)$ and $N(f) = f$.

Example 1. Let $\mathbf{N} = (\mathbf{N}, 0, +, \leqslant)$ be the traced symmetric monoidal partially ordered set of natural numbers. Then the compact closed preordered set Int $\mathbf{N}$ is equivalent to the compact closed partially ordered set $\mathbf{Z} = (\mathbf{Z}, 0, +, -, \leqslant)$ of integers. The biadjointness would imply that **TraMon**($\mathbf{N}$, $\mathbf{Z}$) is equivalent to **TortMon**(Int $\mathbf{N}$, $\mathbf{Z}$), which in turn is equivalent to **TortMon**($\mathbf{Z}$, $\mathbf{Z}$). However, some calculation shows that **TraMon**($\mathbf{N}$, $\mathbf{Z}$) is isomorphic to the partially ordered set of natural numbers, while **TortMon**($\mathbf{Z}$, $\mathbf{Z}$) is isomorphic to a discrete category with countably many objects.

It is possible to recover the biadjointness, by introducing the 2-category $\mathbf{TraMon}_g$ of traced monoidal categories, traced strong monoidal functors and *invertible* monoidal natural transformations. Note that the 2-cells of **TortMon** are invertible because of the presence of duals [**3**, **5**], and the inclusion of **TortMon** in **TraMon** factors through $\mathbf{TraMon}_g$.

PROPOSITION 1. *The inclusion of the* 2*-category* **TortMon** *in the* 2*-category* $\mathbf{TraMon}_g$ *has a left biadjoint with unit having component at a traced monoidal category* $\mathcal{V}$ *by* $N: \mathcal{V} \to$ Int $\mathcal{V}$.

Proof. What we need to show is, for each traced monoidal category $\mathcal{V}$ and tortile monoidal category $\mathcal{W}$, composition with N induces an equivalence of categories from

TortMon(Int $\mathcal{V}$, $\mathcal{W}$) to **TraMon**$_g$($\mathcal{V}$, $\mathcal{W}$). We prove that this induced functor is essentially surjective on objects, and is fully faithful.

For showing that it is essentially surjective, the proof of [**4**, proposition 5·2] is sufficient. For a traced monoidal functor $F\colon \mathcal{V} \to \mathcal{W}$, let $K\colon \mathrm{Int}\,\mathcal{V} \to \mathcal{W}$ be the balanced strong monoidal functor sending (X, U) to $FX \otimes (FU)^\vee$ and $f\colon (X, U) \to (Y, V)$ to

$$FX \otimes (FU)^\vee \xrightarrow{1\otimes\eta\otimes 1} FX \otimes FV \otimes (FV)^\vee \otimes (FU)^\vee \xrightarrow{Ff\otimes 1} FY \otimes FU \otimes (FV)^\vee \otimes (FU)^\vee$$
$$\xrightarrow{1\otimes c^{-1}\otimes 1} FY \otimes (FV)^\vee \otimes FU \otimes (FU)^\vee \xrightarrow{1\otimes\varepsilon'} FY \otimes (FV)^\vee.$$

That K is a balanced strong monoidal functor is shown exactly in the same manner as in the proof of [**4**, proposition 5·2]. Clearly $KN \simeq F$ holds.

For showing the full faithfulness, for an invertible monoidal natural transformation $\beta\colon KN \to K'N$ with balanced strong monoidal functors $K, K'\colon \mathrm{Int}\,\mathcal{V} \to \mathcal{W}$, let $\overline{\beta}\colon K \to K'$ be the monoidal natural transformation whose (X, U)-component is given by

$$K(X, U) \xrightarrow{\simeq} KNX \otimes (KNU)^\vee \xrightarrow{\beta_X \otimes (\beta_U^{-1})^\vee} K'NX \otimes (K'NU)^\vee \xrightarrow{\simeq} K'(X, U).$$

That $\overline{\beta}$ is a monoidal natural transformation is verified by direct calculation. We have $\overline{\alpha N} = \alpha$ for a monoidal natural transformation $\alpha\colon K \to K'$, as

$$\begin{aligned}
&K(X, U) \xrightarrow{\overline{\alpha N}_{(X,U)}} K'(X, U)\\
&= K(X, U) \xrightarrow{\simeq} KNX \otimes (KNU)^\vee \xrightarrow{(\alpha N)_X \otimes ((\alpha N)_U^{-1})^\vee} K'NX \otimes (K'NU)^\vee \xrightarrow{\simeq} K'(X, U)\\
&= K(X, U) \xrightarrow{\simeq} KNX \otimes (KNU)^\vee \xrightarrow{\alpha_{NX} \otimes (\alpha_{NU}^{-1})^\vee} K'NX \otimes (K'NU)^\vee \xrightarrow{\simeq} K'(X, U)\\
&= K(X, U) \xrightarrow{\simeq} KNX \otimes (KNU)^\vee \xrightarrow{\simeq} KNX \otimes (K((NU)^\vee))^{\vee\vee}\\
&\quad \xrightarrow{\alpha_{NX} \otimes \alpha_{(NU)^\vee}^{\vee\vee}} K'NX \otimes (K'((NU)^\vee))^{\vee\vee} \xrightarrow{\simeq} K'NX \otimes (K'NU)^\vee \xrightarrow{\simeq} K'(X, U)\\
&= K(X, U) \xrightarrow{\simeq} KNX \otimes K((NU)^\vee) \xrightarrow{\alpha_{NX} \otimes \alpha_{(NU)^\vee}} K'NX \otimes K'((NU)^\vee) \xrightarrow{\simeq} K'(X, U)\\
&= K(X, U) \xrightarrow{\alpha_{(X,U)}} K'(X, U)
\end{aligned}$$

where we have omitted some details on the structural isomorphisms. Note the isomorphism $(X, U) \simeq (X, I) \otimes (I, U) = NX \otimes (NU)^\vee$; also note that, for a 2-cell $\alpha : K \to K'$ in **TortMon**, its inverse $\alpha^{-1} : K' \to K$ is given by (cf. [**3**, proposition 7·1], [**5**, corollary 2·2])

$$K'C \xrightarrow{\simeq} (K'(C^\vee))^\vee \xrightarrow{(\alpha_{C^\vee})^\vee} (K(C^\vee))^\vee \xrightarrow{\simeq} KC.$$

On the other hand, it is easy to see that $\overline{\beta}N = \beta$ holds. Hence the mapping $\alpha \mapsto \alpha N$ is a bijection, and the functor induced by composition with N is full and faithful.

Remark. This biadjointness result has been frequently quoted in the literature, often with no mention of 2-cells. However, there are some cases where the incorrect statement in [**4**] is inherited, with explicit mention of 2-cells. For example, in [**2**], the biadjunction is incorrectly stated for non-invertible 2-cells [**2**, section 5·1], although the technical development there does not depend on the choice of 2-cells and the error has no effect on the results. Another case is [**1**] in which the biadjointness of a variant of the Int-construction for linearly distributive categories is stated [**1**, proposition 27]; it contains the same problem as [**4**, proposition 5·2], and we expect that a similar change in the definition of 2-cells will make the claim correct. Again, this error has no effect on the other results in [**1**].

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