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A quantum double construction in Rel[†]

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We study bialgebras and Hopf algebras in the compact closed category **Rel** of sets and binary relations. Various monoidal categories with extra structure arise as the categories of (co)modules of bialgebras and Hopf algebras in **Rel**. In particular, for any group G we derive a ribbon category of crossed G-sets as the category of modules of a Hopf algebra in **Rel** which is obtained by the quantum double construction. This category of crossed G-sets serves as a model of the braided variant of propositional linear logic.

1. Introduction

Many important examples of *traced monoidal categories* (Joyal *et al.* 1996) and *ribbon categories (tortile monoidal categories)* (Shum 1994; Turaev 1994) have emerged over the last two decades in mathematics and theoretical computer science. Ribbon categories of particular interest to mathematicians are those of linear representations of *quantum groups* (*quasi-triangular Hopf algebras*) (Drinfel'd 1987; Kassel 1995). In many of them, we have braidings (Joyal and Street 1993) which are not symmetries: in terms of the graphical presentation (Joyal and Street 1991; Selinger 2011), the braid $c =$  is distinguished from its inverse $c^{-1} =$ , and this is the key property for providing non-trivial invariants (or denotational semantics) of knots, tangles and so on (Freyd and Yetter 1989; Kassel 1995; Turaev 1994; Yetter 2001) as well as solutions of the quantum Yang-Baxter equation (Drinfel'd 1987; Kassel 1995), and 3-dimensional topological quantum field theory (Bakalov and Kirilov 2001). In theoretical computer science, major examples include categories with fixed-point operators used in denotational and algebraic semantics (Bloom and Ésik 1993; Hasegawa 1999; Hasegawa 2009; Ştefănescu 2000), and the category of sets and binary relations and its variations used for models of linear logic (Girard 1987) and game semantics (Joyal 1977; Melliès 2004). Moreover, the Int-construction (Joyal *et al.* 1996) provides a rich class of models of Geometry of Interaction (Girard 1989; Abramsky *et al.* 2002; Haghverdi and Scott 2011) and more generally bi-directional information flow, including (Hildebrandt *et al.* 2004; Katsumata 2008). In most of them, the braiding is a symmetry, hence  is identified with .

[†] This is a revised and expanded version of the work presented at the Conference on the Mathematical Foundations of Programming Semantics (MFPS XXVI) (Hasegawa 2010).

Although it is nice to know that all these examples share a common structure, it is also striking to observe that important examples from mathematics and those from computer science are almost disjoint[†]. Is it just a matter of taste? Or is it the case that categories used in computer science cannot host structures interesting for mathematicians (non-symmetric braidings in particular)?

In this paper we demonstrate that we do have mathematically interesting structures in a category preferred by computer scientists. Specifically, we focus on the category **Rel** of sets and binary relations. **Rel** is a compact closed category (Kelly and Laplaza 1980), that is, a ribbon category in which the braiding is a symmetry. We study bialgebras and Hopf algebras in **Rel**, and show that various monoidal categories with extra structure like traces and autonomy can be derived as the categories of (co)modules of bialgebras in **Rel**. As a most interesting example, for any group G we consider the associated Hopf algebra in **Rel**, and apply the *quantum double construction* (Drinfel'd 1987) to it. The resulting Hopf algebra is equipped with a universal R-matrix as well as a universal twist. We show that the category of its modules is the category of *crossed G-sets* (Freyd and Yetter 1989; Whitehead 1949) and suitable binary relations, featuring non-symmetric braiding and non-trivial twist.

While the results mentioned above are interesting in their own right, we hope that this work serves as a useful introduction to the theory of quantum groups for researchers working on semantics of computation, and helps to connect these two research areas which deserve to interact much more.

Related work

Hopf algebras in connection to quantum groups (Drinfel'd 1987) have been extensively studied: standard references include (Kassel 1995; Majid 1995). The idea of using Hopf algebras for modelling various non-commutative linear logics goes back to Blute (Blute 1996), where the focus is on Hopf algebras in the $*$-autonomous category of topological vector spaces. As far as we know, there is no published result on Hopf algebras in **Rel**. Since Freyd and Yetter's work (Freyd and Yetter 1989), categories of crossed G-sets have appeared frequently as typical examples of braided monoidal categories. In the standard setting of finite-dimensional vector spaces, modules of the quantum double of a Hopf algebra A amount to the crossed A-bimodules (Kassel 1995; Kassel and Turaev 1995), and our result is largely an adaptation of such a standard result to **Rel**. However we are not aware of a characterization of crossed G-sets in terms of a quantum double construction in the literature.

[†] An important exception would be dagger compact closed categories used in the study of quantum information protocols (Abramsky and Coecke 2004), though they do not feature non-symmetric braidings. We shall note that our category of crossed G-sets is actually a dagger tortile category in the sense of Selinger (Selinger 2011).

Organization of this paper

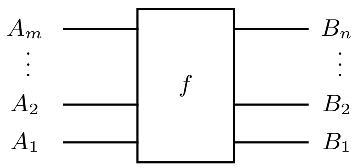
In Section 2, we recall basic notions and facts on monoidal categories and bialgebras. In Section 3, we examine some bialgebras in **Rel** which arise from monoids and groups, and study the categories of (co)modules. Section 4 is devoted to a quantum double construction in **Rel**. In this development, we give a simplified description of the quantum double construction in terms of the Int-construction on traced symmetric monoidal categories. In Section 5 we observe that the ribbon Hopf algebra constructed in the previous section gives rise to a ribbon category of crossed G-sets, and look at some elements of this category. We discuss how this category can be used as a model of braided linear logic in Section 6. Section 7 concludes the paper.

2. Monoidal categories and bialgebras

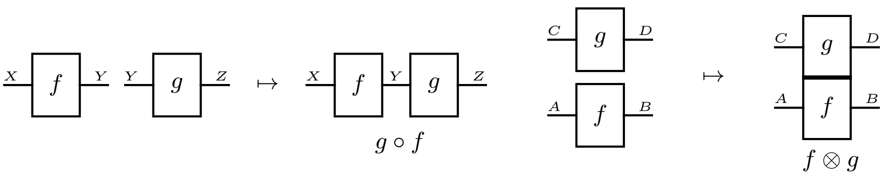
2.1. *Monoidal categories*

A *monoidal category* (*tensor category*) (Mac Lane 1971; Joyal and Street 1993) $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ consists of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathcal{C}$ and natural isomorphisms $a_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$, $l_A : I \otimes A \xrightarrow{\sim} A$ and $r_A : A \otimes I \xrightarrow{\sim} A$ subject to the standard coherence diagrams. It is said to be strict if a, l, r are identity morphisms. For the sake of simplicity, in most places in this paper we pretend that our monoidal categories are strict; Mac Lane's coherence theorem ensures that there is no loss of generality in doing so.

In the sequel, we will make use of the graphical presentation of morphisms in monoidal categories (Joyal and Street 1991; Selinger 2011). A morphism $f : A_1 \otimes A_2 \otimes \ldots \otimes A_m \to B_1 \otimes B_2 \otimes \ldots \otimes B_n$ in a monoidal category will be drawn as (to be read from left to right):



Morphisms can be composed, either sequentially or in parallel:



A *braiding* (Joyal and Street 1993) is a natural isomorphism $c_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$ such that both c and c^{-1} satisfy the following "bilinearity" or "Hexagon Axiom" (the case for c^{-1} is omitted):

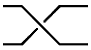
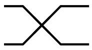
$$\begin{array}{ccccc} (A\otimes B)\otimes C & \xrightarrow{a_{A,B,C}} & A\otimes(B\otimes C) & \xrightarrow{c_{A,B\otimes C}} & (B\otimes C)\otimes A \\ \downarrow{\scriptstyle c_{A,B}\otimes C} & & & & \downarrow{\scriptstyle a_{B,C,A}} \\ (B\otimes A)\otimes C & \xrightarrow[a_{B,A,C}]{} & B\otimes(A\otimes C) & \xrightarrow[B\otimes c_{A,C}]{} & B\otimes(C\otimes A) \end{array}$$

For braidings, we shall use the drawings $c_{A,B} =$ 

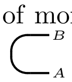
and $c^{-1}_{A,B} =$ 

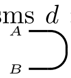
.

A *symmetry* is a braiding such that $c_{A,B} = c^{-1}_{B,A}$. In that case we simply draw  ,

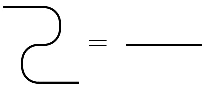
hence  =  =  . A *braided/symmetric monoidal category* is a monoidal category equipped with a braiding/symmetry.

A *twist* or a *balance* for a braided monoidal category is a natural isomorphism $\theta_A : A \xrightarrow{\sim} A$ such that $\theta_{A\otimes B} = c_{B,A} \circ (\theta_B \otimes \theta_A) \circ c_{A,B}$ holds. Twists are drawn as $\theta_A =$  and $\theta^{-1}_A =$  . A *balanced monoidal category* is a braided monoidal category with a twist. Note that a symmetric monoidal category is precisely a balanced monoidal category with $\theta_A = id_A$ for every A.

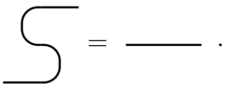
In a monoidal category, a *dual pairing* between two objects A and B is given by a pair of morphisms $d : I \to A \otimes B$, called unit, and $e : B \otimes A \to I$, called counit, drawn as 

and 

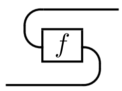
respectively, satisfying



and

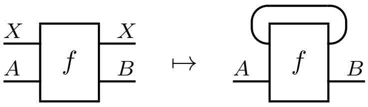


In such a dual pairing, B is called the *left dual* of A, and A is called the *right dual* of B. For an object, its left (or right) dual, if exists, is uniquely determined up to isomorphism. A monoidal category is *left autonomous* or *left rigid* if every object A has a left dual A^* with unit $\eta_A : I \to A \otimes A^*$ and counit $\varepsilon_A : A^* \otimes A \to I$. In a left autonomous category, $I \cong I^*$ as well as $A^* \otimes B^* \cong (B \otimes A)^*$ hold. Also $(-)^*$ extends to a contravariant functor, where, for a morphism $f : A \to B$, its dual $f^* : B^* \to A^*$ is given as:



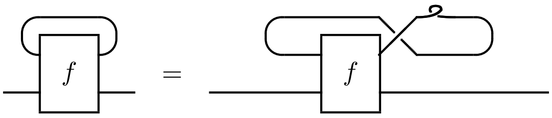
A *ribbon category* (Turaev 1994) (*tortile monoidal category* (Shum 1994)) is a balanced monoidal category which is left autonomous and moreover satisfies $(\theta_A)^* = \theta_{A^*}$. In a ribbon category, $(-)^*$ is a contravariant equivalence, and there is a natural isomorphism $A^{**} \cong A$ (hence the left dual of A and the right dual of A are isomorphic). Note that a ribbon category whose twist is the identity is a *compact closed category* (Kelly and Laplaza 1980).

A *traced monoidal category* (Joyal *et al.* 1996) is a balanced monoidal category $\mathcal{C}$ equipped with a trace operator $Tr^X_{A,B} : \mathcal{C}(A \otimes X, B \otimes X) \to \mathcal{C}(A, B)$ which will be drawn as a "feedback" operator



satisfying a few coherence axioms. Alternatively, by the structure theorem in *ibid.*, traced monoidal categories are characterized as monoidal full subcategories of ribbon categories. Any ribbon category has a unique trace, called its *canonical trace* (Joyal *et al.* 1996) (for uniqueness see e.g. (Hasegawa 2009)). For a morphism $f : A \otimes X \to B \otimes X$ in a ribbon category, its trace $Tr^X_{A,B} f : A \to B$ is given by

$$Tr^X_{A,B} f \;=\; (id_B \otimes (\varepsilon_X \circ (id_{X^*} \otimes \theta_X) \circ c_{X,X^*})) \circ (f \otimes id_{X^*}) \circ (id_A \otimes \eta_X).$$

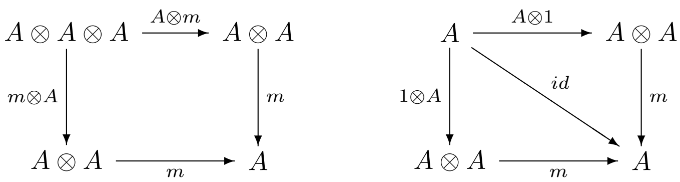


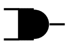
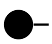
For monoidal categories $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ and $\mathcal{C}' = (\mathcal{C}', \otimes', I', a', l', r')$, a *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a tuple (F, m, m_I) where F is a functor from $\mathcal{C}$ to $\mathcal{C}'$, m is a natural transformation from $F(-) \otimes' F(=)$ to $F(- \otimes =)$ and $m_I : I' \to FI$ is an arrow in $\mathcal{C}'$, satisfying three coherence conditions. It is called *strong* if $m_{A,B}$ and m_I are all isomorphisms, and *strict* if they are all identities. A *balanced monoidal functor* from a balanced $\mathcal{C}$ to another $\mathcal{C}'$ is a monoidal functor (F, m, m_I) which additionally satisfies $m_{B,A} \circ c_{FA,FB} = Fc_{A,B} \circ m_{A.B}$ and $F\theta_A = \theta_{FA}$.

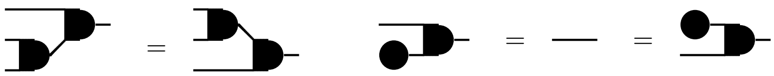
For monoidal functors (F, m, m_I), (G, n, n_I) with the same source and target monoidal categories, a *monoidal natural transformation* from (F, m, m_I) to (G, n, n_I) is a natural transformation $\varphi : F \to G$ such that $\varphi_{A\otimes B} \circ m_{A,B} = n_{A,B} \circ \varphi_A \otimes \varphi_B$ and $\varphi_I \circ m_I = n_I$ hold. A *(balanced/symmetric) monoidal adjunction* between (balanced/symmetric) monoidal categories is an adjunction in which both of the functors are (balanced/symmetric) monoidal and the unit and counit are monoidal natural transformations.

2.2. *Monoids, comonoids and (co)modules*

A *monoid* in a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ is an object A equipped with morphisms $m : A \otimes A \to A$, called the *multiplication*, and $1 : I \to A$, called the *unit*, such that the following diagrams commute.



With notations $m =$  and $1 =$ , these diagrams can be expressed as follows.

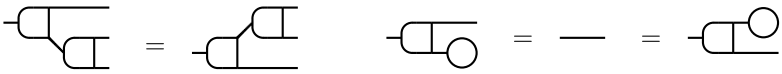


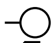
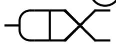
When $\mathcal{C}$ is symmetric and $m \circ c_{A,A} = m$, i.e.,  $=$  holds, we say A is *commutative*.

Dually, a *comonoid* in a monoidal category $\mathcal{C}$ is an object A equipped with morphisms $\Delta : A \to A \otimes A$, called the *comultiplication*, and $\epsilon : A \to I$, called the *counit*, satisfying

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle A\otimes\Delta} \\ A \otimes A & \xrightarrow[\Delta\otimes A]{} & A \otimes A \otimes A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle \Delta}\downarrow & \searrow^{id} & \downarrow{\scriptstyle A\otimes\epsilon} \\ A \otimes A & \xrightarrow[\epsilon\otimes A]{} & A \end{array}$$

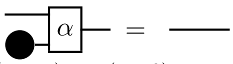
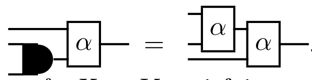
They can be drawn as



where $\Delta =$  and $\epsilon =$  . When $\mathcal{C}$ is symmetric and $c_{A,A} \circ \Delta = \Delta$ holds (graphically:  = ), we say A is *co-commutative*.

Suppose that $A = (A, m, 1)$ is a monoid. A gives rise to a monad $A \otimes (-)$ whose multiplication is $m \otimes X : A \otimes A \otimes X \to A \otimes X$ and unit is $1 \otimes X : X \to A \otimes X$. A (left) A-*module* is an Eilenberg-Moore algebra of this monad. More explicitly, an A-module consists of an object X and a morphism $\alpha : A \otimes X \to X$, called the action, satisfying

$$\begin{array}{ccc} X & \xrightarrow{1\otimes X} & A \otimes X \\ & \searrow^{id} & \downarrow{\scriptstyle \alpha} \\ & & X \end{array} \qquad \begin{array}{ccc} A \otimes A \otimes X & \xrightarrow{A\otimes\alpha} & A \otimes X \\ {\scriptstyle m\otimes X}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A \otimes X & \xrightarrow[\alpha]{} & X \end{array}$$

or, in the graphical presentation,  and .

A morphism of A-modules from (X, α) to (Y, β) is a morphism $f : X \to Y$ satisfying

$$\begin{array}{ccc} A \otimes X & \xrightarrow{A\otimes f} & A \otimes Y \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ X & \xrightarrow[f]{} & Y \end{array}$$

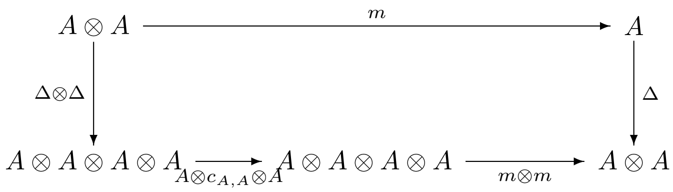
Let us denote the category of A-modules and morphisms by $\mathbf{Mod}(A)$.

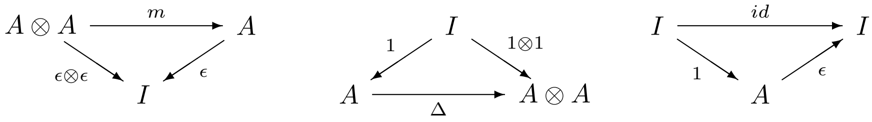
Dually, given a comonoid $A = (A, \Delta, \epsilon)$, a (left) A-*comodule* is an Eilenberg-Moore coalgebra of the comonad $A \otimes (-)$ whose comultiplication is $\Delta \otimes X : A \otimes X \to A \otimes A \otimes X$ and counit is $\epsilon \otimes X : A \otimes X \to X$. Explicitly, an A-comodule consists of an object X and a morphism $\alpha : X \to A \otimes X$, called the coaction, satisfying the axioms dual to those of modules. A morphism of A-comodules from (X, α) to (Y, β) is then a morphism $f : X \to Y$ making the evident diagram commute. We will denote the category of A-comodules and morphisms by $\mathbf{Comod}(A)$.

2.3. *Bialgebras and Hopf algebras*

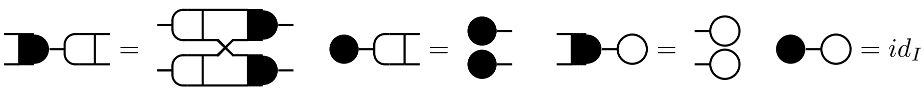
Now suppose that $\mathcal{C}$ is a symmetric monoidal category with a symmetry $c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$. A *bialgebra* in $\mathcal{C}$ is given by a tuple $A = (A, m, 1, \Delta, \epsilon)$ where A is an object of $\mathcal{C}$

and $(A, m, 1)$ is a monoid in $\mathcal{C}$ while (A, Δ, ϵ) is a comonoid in $\mathcal{C}$, satisfying



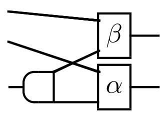


Graphically:



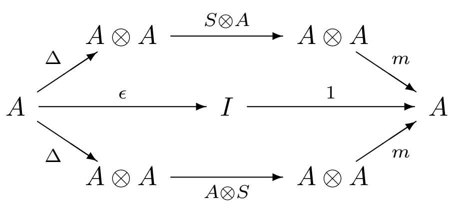
We say A is commutative (resp. co-commutative) when it is commutative (resp. co-commutative) as a monoid (resp. comonoid). For a bialgebra A, we can consider the category of modules $\mathbf{Mod}(A)$ (for A as a monoid) as well as that of comodules $\mathbf{Comod}(A)$ (for A as a comonoid). The functor $A \otimes (-)$ is both monoidal and comonoidal. Moreover, as a monad $A \otimes (-)$ is comonoidal, while as a comonad it is monoidal. It follows that (cf. (Bruguieres and Virelizier 2006; Pastro and Street 2009)) both $\mathbf{Mod}(A)$ and $\mathbf{Comod}(A)$ are monoidal categories. Explicitly, in $\mathbf{Mod}(A)$, the tensor unit is $(I, A \otimes I \cong A \xrightarrow{\epsilon} I)$ and the tensor product of (X, α) and (Y, β) is

$$(X \otimes Y, A \otimes X \otimes Y \xrightarrow{\Delta \otimes X \otimes Y} A \otimes A \otimes X \otimes Y \xrightarrow{A \otimes c_{A,X} \otimes Y} A \otimes X \otimes A \otimes Y \xrightarrow{\alpha \otimes \beta} X \otimes Y).$$

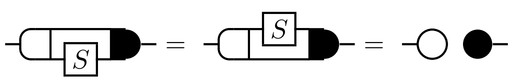


The monoidal structure of $\mathbf{Comod}(A)$ is given by dualizing that of $\mathbf{Mod}(A)$.

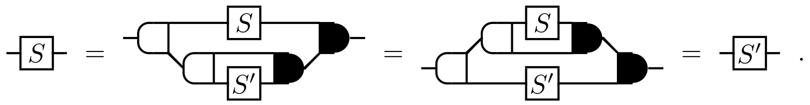
A *Hopf algebra* is a bialgebra $A = (A, m, 1, \Delta, \epsilon)$ equipped with a morphism $S : A \to A$, called an *antipode*, such that



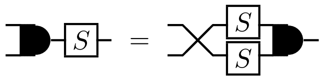
commutes (see the picture below).



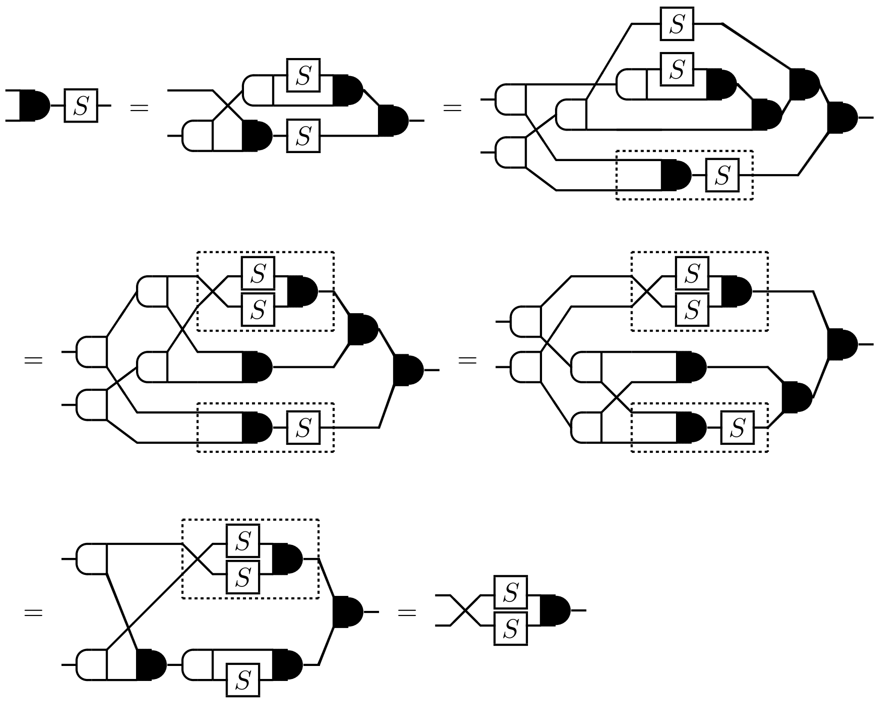
We shall recall some basic results on Hopf algebras. First, the antipode of a Hopf algebra is unique — if S and S' are both antipodes, we have $S = S'$ because



Lemma 2.1. For any Hopf algebra $A = (A, m, 1, \Delta, \epsilon, S)$, the equation $S \circ m = m \circ (S \otimes S) \circ c_{A,A}$ holds.



Proof. We give a graphical proof, in which each step follows from the axioms of bialgebras and antipode.

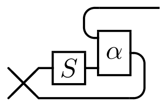


(Readers familiar with group theory might notice that this is just a graphical reworking of the proof of $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$, cf. Example 2.7.) □

From Lemma 2.1, we can easily derive the well-known fact that the antipode S of any commutative or co-commutative Hopf algebra satisfies $S \circ S = id$, hence is invertible. In general, an antipode does not have to be invertible; see (Takeuchi 1971) for some examples. It is also known that any Hopf algebra in a compact closed category with equalizers has an invertible antipode (Takeuchi 1999), and this is the case for the category of finite dimensional vector spaces. All concrete examples considered below have an invertible antipode (see also Remark 3.5).

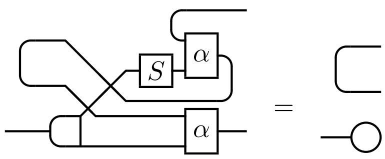
Lemma 2.2. If $\mathcal{C}$ is a compact closed category and A is a Hopf algebra in $\mathcal{C}$, then $\mathbf{Mod}(A)$ is left autonomous, where a left dual of a module (X, α) is

$$A \otimes X^* \xrightarrow{c} X^* \otimes A \xrightarrow{X^* \otimes S \otimes \eta} X^* \otimes A \otimes X \otimes X^* \xrightarrow{X^* \otimes \alpha \otimes X^*} X^* \otimes X \otimes X^* \xrightarrow{\varepsilon \otimes X^*} X^*.$$



The unit and counit of the dual pairing are given by the unit and counit of the dual pairing of X and X^* in $\mathcal{C}$.

Proof. It suffices to show that the unit $\eta_X : I \to X \otimes X^*$ is a morphism of modules from (I, ϵ) to $(X, \alpha) \otimes (X, \alpha)^*$ and that the counit $\varepsilon_X : X^* \otimes X \to I$ is a morphism of modules of $(X, \alpha)^* \otimes (X, \alpha) \to (I, \epsilon)$. The former amounts to the equation



which follows from the axioms of duality, modules and antipode. The latter also follows in a similar way. □

Remark 2.3. In this paper we only consider bialgebras and Hopf algebras in *symmetric* monoidal categories. However, it completely makes sense to think about bialgebras and Hopf algebras in *braided* monoidal categories, and this is the central topic in (Majid 1994).

Remark 2.4. As noted in (Cockett and Seely 1997), the category of modules of a bialgebra in a symmetric or braided linearly distributive category is a linearly distributive category. Similarly, the category of modules of a Hopf algebra in a symmetric or braided $*$-autonomous category is a $*$-autonomous category.

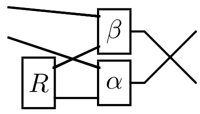
2.4. *Braiding and twist on modules of a bialgebra*

If a bialgebra A is co-commutative, the monoidal category $\mathbf{Mod}(A)$ has a symmetry inherited from the base symmetric monoidal category. However, (whether A is co-commutative or not) there can be some non-trivial braiding and twist on $\mathbf{Mod}(A)$. Suppose that $\mathbf{Mod}(A)$ is braided with a braiding σ (while we use c for the symmetry of the base symmetric monoidal category). Since $A = (A, m)$ is an A-module, we have $\sigma_{A,A} : A \otimes A \to A \otimes A$, and $c_{A,A} \circ \sigma_{A,A} \circ (1 \otimes 1) : I \to A \otimes A$ which we shall denote by R. Conversely, for this $R : I \to A \otimes A$ one can see that

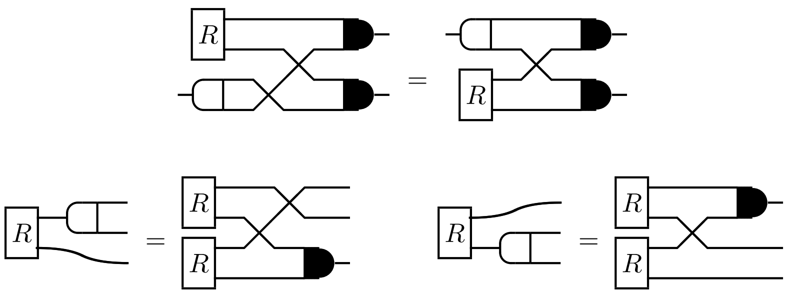
$$\sigma_{X,Y} \circ (f \otimes g) = c_{X,Y} \circ (\alpha \otimes \beta) \circ (A \otimes c_{A,X} \otimes Y) \circ (R \otimes X \otimes Y) \circ (f \otimes g)$$

holds for modules $X = (X, \alpha)$ and $Y = (Y, \beta)$ and morphisms $f : I \to X$ and $g : I \to Y$ in $\mathcal{C}$. So from R, we can recover $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ as

$$\sigma_{X,Y} = c_{X,Y} \circ (\alpha \otimes \beta) \circ (A \otimes c_{A,X} \otimes Y) \circ (R \otimes X \otimes Y)$$

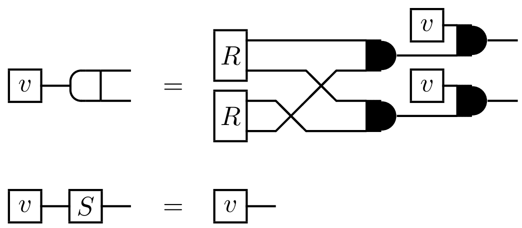


provided the base symmetric monoidal category $\mathcal{C}$ is closed and the global section functor $\mathcal{C}(I, -) : \mathcal{C} \to \mathbf{Set}$ is faithful; this is the case for all commonly used examples, including the category of vector spaces and linear maps, as well as **Rel**. In such cases there is a bijective correspondence between braidings on $\mathbf{Mod}(A)$ and morphisms of $I \to A \otimes A$ satisfying certain equations (Kassel 1995; Majid 1995; Street 2007). Such a morphism of $I \to A \otimes A$ is called a *universal R-matrix* or a *braiding element*. Explicitly, a universal R-matrix is a morphism $R : I \to A \otimes A$ which is convolution-invertible (there exists $R^\circ : I \to A \otimes A$ satisfying $(m \otimes m) \circ (A \otimes c_{A,A} \otimes A) \circ (R \otimes R^\circ) = (m \otimes m) \circ (A \otimes c_{A,A} \otimes A) \circ (R^\circ \otimes R) = 1 \otimes 1$) and satisfies the following three equations.



The convolution-invertibility ensures the invertibility of the braid σ induced from R. These three graphically presented equations imply that σ is a morphism of modules, that σ is bilinear, and that σ^{-1} is bilinear, respectively. A bialgebra equipped with a universal R-matrix is called a *quasi-triangular bialgebra*.

Next, let A be a quasi-triangular Hopf algebra in a compact closed category $\mathcal{C}$ and suppose that $\mathbf{Mod}(A)$ is a ribbon category, i.e., not just braided but also with a twist θ. We then have a morphism $v = \theta_A \circ 1 : I \to A$, which satisfies $\theta_X \circ f = \alpha \circ (v \otimes X) \circ f$ for a module $X = (X, \alpha)$ and a morphism $f : I \to X$ in $\mathcal{C}$. Thus from this v we can recover θ_X as $\theta_X = \alpha \circ (v \otimes X)$, provided the global section functor $\mathcal{C}(I, -)$ is faithful. In such cases we have a bijective correspondence between twists on $\mathbf{Mod}(A)$ and certain morphisms $v : I \to A$ satisfying a few axioms (Kassel 1995; Majid 1995; Turaev 1994). Such a v is called a *universal twist* or a *twist element*. Explicitly, a universal twist is a morphism $v : I \to A$ which is convolution-invertible (there exists $v^\circ : I \to A$ such that $m \circ (v \otimes v^\circ) = 1$), central ($m \circ (A \otimes v) = m \circ (v \otimes A)$) and satisfies the following two equations.



The convolution-invertibility implies that θ induced from v is invertible, and centrality says that θ is a morphism of modules. The first equation amounts to the axiom for twists, while the second one is required for the axiom $(\theta_X)^* = \theta_{X^*}$. A quasi-triangular Hopf algebra equipped with a universal twist is called a *ribbon Hopf algebra*. In summary, we have the following results:

Proposition 2.5. (Turaev 1994; Kassel 1995; Yetter 2001)

1 If A is a quasi-triangular bialgebra in a symmetric monoidal category $\mathcal{C}$, then $\mathbf{Mod}(A)$ is a braided monoidal category.
2 If A is a ribbon Hopf algebra in a compact closed category $\mathcal{C}$, then $\mathbf{Mod}(A)$ is a ribbon category.

Note that every co-commutative Hopf algebra is equipped with a universal R-matrix $R = 1 \otimes 1$ and a universal twist $v = 1$, giving rise to the symmetry and trivial twist on the category of modules. We will give a non-commutative non-co-commutative ribbon Hopf algebra in **Rel** in Section 4.

2.5. *Examples*

We shall look at a few basic cases.

Example 2.6. As a classical example, let us consider the category $\mathbf{Vect}_k$ of vector spaces over a field k and linear maps. $\mathbf{Vect}_k$ is a symmetric monoidal category whose monoidal product is given by the tensor product of vector spaces, and k (the 1-dimensional space) serves as the tensor unit. Its full subcategory $\mathbf{Vect}_k^{\mathrm{fin}}$ of finite dimensional vector spaces is a compact closed category; for a finite dimensional V, its left (and right) dual is the dual vector space $V^* = \hom(V, k)$ of linear maps from V to k, with unit given by the dual basis and counit the evaluation map. A monoid in $\mathbf{Vect}_k$ is nothing but an algebra in the standard sense. Similarly, a comonoid in $\mathbf{Vect}_k$ is what is normally called a coalgebra. Modules, comodules, bialgebras and Hopf algebras in $\mathbf{Vect}_k$ and $\mathbf{Vect}_k^{\mathrm{fin}}$ are exactly those in the classical sense; a detailed account can be found in (Kassel 1995).

Example 2.7. Let **Set** be the category of sets and functions. By taking finite products as tensor products, **Set** forms a symmetric monoidal category. A monoid in **Set** is just a monoid in the usual sense. For any set X, the diagonal map $X \to X \times X$ and the terminal map $X \to 1$ give a commutative comonoid structure on X — and this is the unique comonoid structure on X. Given a monoid M, its modules are just the M-sets, i.e., sets on which M acts, and $\mathbf{Mod}(M)$ is isomorphic to the category M-**Set** of M-sets and functions respecting M-actions. For any set X, a comodule $(A, \alpha : A \to X \times A)$ of the unique comonoid $X = (X, \Delta, \epsilon)$ on X is determined by the function $\pi \circ \alpha : A \to X$, and $\mathbf{Comod}(X)$ is isomorphic to the slice category $\mathbf{Set}/X$. A bialgebra in **Set** is a monoid equipped with the unique comonoid structure. A Hopf algebra in **Set** is then a group G with the unique comonoid structure, where the antipode is given by the inverse $g \mapsto g^{-1} : G \to G$.

3. Bialgebras in Rel

Now let us turn our attention to the category **Rel** of sets and binary relations. **Rel** is a compact closed (hence ribbon) category, where the tensor product of sets X and Y is given by the direct product $X \times Y$ of sets and the unit object is a singleton set $I = \{*\}$. For a set X, its left dual X^* is X itself, with unit and counit given by

$$\begin{array}{rcll} \eta_X & = & \{(*, (x, x)) \mid x \in X\} & : I \to X \times X, \\ \varepsilon_X & = & \{((x, x), *) \mid x \in X\} & : X \times X \to I. \end{array}$$

3.1. *Bialgebras and Hopf algebras inherited from* **Set**

The easiest cases of bialgebras and Hopf algebras in **Rel** are those arising from monoids and groups in **Set**, respectively. First, we shall note that there is an identity-on-objects, strict symmetric monoidal functor $J : \mathbf{Set} \to \mathbf{Rel}$ sending a set to itself and a function $f : X \to Y$ to a binary relation $\{(x, f(x)) \mid x \in X\}$ from X to Y, and recall a standard result:

Lemma 3.1. A strong symmetric monoidal functor preserves the structure of monoids, comonoids, bialgebras and Hopf algebras.

From this and Example 2.7, it follows that a monoid $M = (M, \cdot, e)$ (in **Set**) gives rise to a co-commutative bialgebra $\overline{M} = (M, m, 1, \Delta, \epsilon)$ in **Rel**, with

$$\begin{array}{rclcl} m & = & J((a_1, a_2) \mapsto a_1 \cdot a_2) & = & \{((a_1, a_2), a_1 \cdot a_2) \mid a_1, a_2 \in M\} \\ 1 & = & J(* \mapsto e) & = & \{(*, e)\} \\ \Delta & = & J(a \mapsto (a, a)) & = & \{(a, (a, a)) \mid a \in M\} \\ \epsilon & = & J(a \mapsto *) & = & \{(a, *) \mid a \in M\}. \end{array}$$

$\overline{M}$ is commutative if M is commutative. Similarly, a group $G = (G, \cdot, e, (-)^{-1})$ gives rise to a co-commutative Hopf algebra $\overline{G} = (G, m, 1, \Delta, \epsilon, S)$ in **Rel**, with an antipode $S = \{(g, g^{-1}) \mid g \in G\} : G \to G$.

Let us examine the category $\mathbf{Mod}(\overline{G})$ for a group $G = (G, \cdot, e, (-)^{-1})$ (it makes sense to think about $\mathbf{Mod}(\overline{M})$ for a monoid M, but when M is not a group the description of $\mathbf{Mod}(\overline{M})$ can be rather complicated). A module of $\overline{G}$ is a set X equipped with a binary relation $\alpha : G \times X \to X$ subject to the two axioms given before. It is not hard to see that α is actually a function, because, for each $g \in G$, the relation $\alpha \circ (g \times X) = \{(x, x') \mid ((g, x), x') \in \alpha\} : X \to X$ is an isomorphism in **Rel** with the inverse $\alpha \circ (g^{-1} \times X)$, hence a bijective function. In fact α is a G-action on X: for $g \in G$ and $x \in X$, by letting $g \bullet x$ be the unique $x' \in X$ such that $((g, x), x') \in \alpha$, we have $e \bullet x = x$ and

$(g \cdot h) \bullet x = g \bullet (h \bullet x)$. Therefore we can identify objects of $\mathbf{Mod}(\overline{G})$ with G-sets: a morphism from a G-set $(X, \bullet)$ to $(Y, \bullet)$ is then a binary relation $r : X \to Y$ such that $(x, y) \in r$ implies $(g \bullet x, g \bullet y) \in r$. Since $\overline{G}$ is a co-commutative Hopf algebra, $\mathbf{Mod}(\overline{G})$ is a compact closed category which is actually very similar to **Rel**. Explicitly, the tensor of $(X, \bullet)$ and $(Y, \bullet)$ is $(X \times Y,\ (g, (x, y)) \mapsto (g \bullet x, g \bullet y))$, while the tensor unit is $(\{*\},\ (g, *) \mapsto *)$. A left dual of $(X, \bullet)$ is $(X, \bullet)$ itself.

Next, we shall look at $\mathbf{Comod}(\overline{M})$ for a monoid $M = (M, \cdot, e)$. A comodule of $\overline{M}$ is a set X with a binary relation $\alpha : X \to M \times X$ subject to the comodule axioms — but the axioms imply that α is a function whose second component is the identity on X. Hence an object of $\mathbf{Comod}(\overline{M})$ can be identified with a set X equipped with a function $|\_| : X \to M$; a morphism from $(X, |\_|)$ to $(Y, |\_|)$ is then a binary relation $r : X \to Y$ such that $(x, y) \in r$ implies $|x| = |y|$. $\mathbf{Comod}(\overline{M})$ is a monoidal category, with $(X, |\_|) \otimes (Y, |\_|) = (X \times Y,\ (x, y) \mapsto |x| \cdot |y|)$ and $I = (\{*\},\ * \mapsto e)$. Note that the function $|\_|$ does not have to respect the monoid structure in any way; indeed the only place where the monoid structure of M is used is in the definition of $\otimes$ and I.

Proposition 3.2.

1 If M is a commutative monoid, $\mathbf{Comod}(\overline{M})$ is symmetric monoidal.

2 If M is a left (resp. right)-cancellable monoid, $\mathbf{Comod}(\overline{M})$ has a left (resp. right) trace in the sense of Selinger (Selinger 2011).

3 If M is a commutative cancellable monoid, $\mathbf{Comod}(\overline{M})$ is a traced symmetric monoidal category.

4 If G is a group, every object $(X, |\_|)$ of $\mathbf{Comod}(\overline{G})$ has a left dual $(X, |\_|^{-1})$ (and $\mathbf{Comod}(\overline{G})$ is pivotal (Freyd and Yetter 1989)).

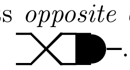
5 If G is an Abelian group, $\mathbf{Comod}(\overline{G})$ is a compact closed category.

This proposition makes a connection between structures on a monoid (commutativity, left/right cancellability, inverses) and the structures respectively induced on the monoidal category of comodules (symmetry, left/right trace, pivotal structure).

Thus we can derive a number of monoidal categories with symmetry, duals, and trace as categories of (co)modules of (the associated bialgebra of) a monoid or a group. However, none of them have a non-symmetric braiding; in Section 4 we give a Hopf algebra in **Rel** whose category of modules has a non-symmetric braiding and a non-trivial twist.

3.2. *Some constructions*

There are a number of ways of constructing bialgebras and Hopf algebras in **Rel** from the existing ones. Here we shall look at some basic constructions, which make sense not only for **Rel** but also for general symmetric monoidal categories and compact closed categories.

Opposite bialgebras and Hopf algebras Given a bialgebra $A = (A, m, 1, \Delta, \epsilon)$ in a symmetric monoidal category, its *opposite bialgebra* is the bialgebra $A^{\mathrm{op}} = (A, m^{\mathrm{op}}, 1, \Delta, \epsilon)$ where $m^{\mathrm{op}} = m \circ c_{A,A}$, i.e., . If A is a Hopf algebra with invertible antipode

S, then A^{op} is a Hopf algebra with antipode S^{-1}. That S^{-1} is an antipode of A^{op} is an immediate consequence of Lemma 2.1.

This opposite construction makes sense in **Rel**. Concretely, given a bialgebra $A = (A, m, 1, \Delta, \epsilon)$ in **Rel**, its *opposite bialgebra* is the bialgebra $A^{\mathrm{op}} = (A, m^{\mathrm{op}}, 1, \Delta, \epsilon)$ where

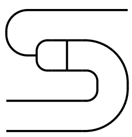
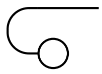
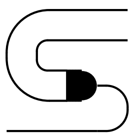
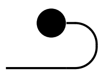
$$m^{\mathrm{op}} = m \circ c_{A,A} = \{((x_2, x_1), y) \mid ((x_1, x_2), y) \in m\}.$$

If A is a Hopf algebra with invertible antipode S, then A^{op} is a Hopf algebra with antipode

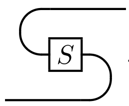
$$S^{-1} = \{(y, x) \mid (x, y) \in S\}.$$

For a group G, the Hopf algebra $\overline{G}^{\mathrm{op}}$ is isomorphic to $\overline{G^{\mathrm{op}}}$ where G^{op} is the group obtained by reverting the multiplication of G.

Dual bialgebras and Hopf algebras Given a bialgebra $A = (A, m, 1, \Delta, \epsilon)$ in a compact closed category, its *dual bialgebra* is the bialgebra $A^* = (A^*, \Delta^*, \epsilon^*, m^*, 1^*)$ where[‡]

$\Delta^* =$  , $\epsilon^* =$  , $m^* =$  , $1^* =$  .

If A is a Hopf algebra with antipode S, then A^* is a Hopf algebra with antipode

$S^* =$ 

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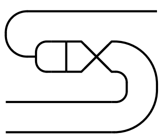
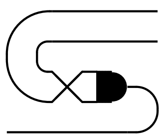
In the case of **Rel**, for a bialgebra $A = (A, m, 1, \Delta, \epsilon)$, its *dual bialgebra* is the bialgebra $A^* = (A, \Delta^*, \epsilon^*, m^*, 1^*)$ where

$$\begin{array}{rcl} \Delta^* & = & \{((y_2, y_1), x) \mid (x, (y_1, y_2)) \in \Delta\} \\ \epsilon^* & = & \{(*, x) \mid (x, *) \in \epsilon\} \\ m^* & = & \{(y, (x_2, x_1)) \mid ((x_1, x_2), y) \in m\} \\ 1^* & = & \{(y, *) \mid (*, y) \in 1\}. \end{array}$$

If A is a Hopf algebra with antipode S, then A^* is a Hopf algebra with antipode $S^* = \{(y, x) \mid (x, y) \in S\}$.

Remark 3.3. Some authors define the dual bialgebra (Hopf algebra) A^* as our $((A^{\mathrm{op}})^*)^{\mathrm{op}}$, whose multiplication and comultiplication are given by

 and  .

See for example (Kassel 1995); our definition agrees with that of (Majid 1994).

‡ Strictly speaking, we need to put the isomorphisms $X^* \otimes Y^* \cong (Y \otimes X)^*$ and $I \cong I^*$ in some appropriate places in this definition.

Tensor products When $A_1 = (A_1, m_1, 1_1, \Delta_1, \epsilon_1)$ and $A_2 = (A_2, m_2, 1_2, \Delta_2, \epsilon_2)$ are bialgebras in a symmetric monoidal category, their *tensor product* is the bialgebra $A_1 \otimes A_2 = (A_1 \otimes A_2, m_{12}, 1_{12}, \Delta_{12}, \epsilon_{12})$ where

$$\begin{array}{lcl} m_{12} & = & (m_1 \otimes m_2) \circ (A_1 \otimes c_{A_2,A_1} \otimes A_2) \\ 1_{12} & = & 1_1 \otimes 1_2 \\ \Delta_{12} & = & (A_1 \otimes c_{A_1,A_2} \otimes A_2) \circ (\Delta_1 \otimes \Delta_2) \\ \epsilon_{12} & = & \epsilon_1 \otimes \epsilon_2 \end{array}$$

We shall note that the bialgebra $A_1 \otimes A_2$ is isomorphic to $A_2 \otimes A_1$, and $A_1^{\rm op} \otimes A_2^{\rm op}$ is isomorphic to $(A_1 \otimes A_2)^{\rm op}$. In the case of compact closed categories, the bialgebra $A_1^* \otimes A_2^*$ is isomorphic to $(A_2 \otimes A_1)^*$. When both A_1 and A_2 are Hopf algebras with antipode S_1 and S_2 respectively, then $A_1 \otimes A_2$ is a Hopf algebra with antipode $S_{12} = S_1 \otimes S_2$. In **Rel**, they are

$$\begin{array}{lcl} m_{12} & = & \{(((x_1,x_2),(y_1,y_2)),(z_1,z_2)) \mid ((x_i,y_i),z_i) \in m_i\} \\ 1_{12} & = & \{(*,(x_1,x_2)) \mid x_i \in 1_i\} \\ \Delta_{12} & = & \{((x_1,x_2),((y_1,y_2),(z_1,z_2))) \mid (x_i,(y_i,z_i)) \in \Delta_i\} \\ \epsilon_{12} & = & \{((x_1,x_2),*) \mid (x_i,*) \in \epsilon_i\} \\ S_{12} & = & \{((x_1,x_2),(y_1,y_2)) \mid (x_i,y_i) \in S_i\}. \end{array}$$

For groups G_1 and G_2, it is not hard to see that $\overline{G_1} \otimes \overline{G_2}$ is isomorphic to $\overline{G_1 \times G_2}$.

By these constructions, one can construct a non-commutative non-co-commutative bialgebras and Hopf algebras in **Rel**. For example, for a non-Abelian group G, $\overline{G} \otimes \overline{G}^*$ is a Hopf algebra which is neither commutative nor co-commutative. However, this Hopf algebra does not have an R-matrix — for which we need a more sophisticated construction, and it is the topic of the next section.

Remark 3.4. Of course, there are lots of bialgebras and Hopf algebras in **Rel** which cannot be obtained by these constructions on $\overline{M}$'s or $\overline{G}$'s. (In fact, bialgebras derived in this way are isomorphic to $\overline{M_1} \otimes \overline{M_2}^*$ for some monoids M_1 and M_2.) For an easy example, let X be a set and $(MX, \oplus, 0)$ be the free commutative monoid on X; or, equivalently, let MX be the set of finite multisets of elements of X, $\oplus$ the union of multisets, and 0 the empty multiset. Then there is a bialgebra $MX = (MX, m, 1, \Delta, \epsilon)$ in **Rel** where $m = \{((x_1,x_2), x_1 \oplus x_2) \mid x_i \in MX\}$, $1 = \{(*,0)\}$, $\Delta = \{(x_1 \oplus x_2, (x_1,x_2)) \mid x_i \in MX\}$ and $\epsilon = \{(0,*)\}$. Obviously $MX^{\rm op}$ and MX^* are isomorphic to MX.

Remark 3.5. As of writing this paper, we do not know if all Hopf algebras in **Rel** have an invertible antipode. Note that **Rel** does not have all equalizers, so the result in (Takeuchi 1999) cannot be applied to **Rel**. On the other hand, it is not clear if the construction of a Hopf algebra with a non-invertible antipode in (Takeuchi 1971) can be carried out in **Rel**.

4. A quantum double construction in Rel

In the previous section, we have observed that every group $G = (G, \cdot, e, (-)^{-1})$ gives rise to a co-commutative Hopf algebra $\overline{G} = (G, m, 1, \Delta, \epsilon, S)$ in **Rel**. For obtaining a quasi-triangular Hopf algebra, in this section we shall apply Drinfel'd's *quantum double construction* (Drinfel'd 1987; Majid 1990) to $\overline{G}$.

4.1. *Quantum double construction in compact and traced categories*

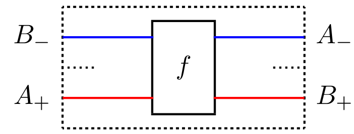
We shall use the quantum double construction given in terms of Hopf algebras in compact closed categories:

Proposition 4.1. (cf. (Chen 2000; Kassel 1995; Kassel and Turaev 1995)) Suppose that $\mathcal{C}$ is a compact closed category and $A = (A, m, 1, \Delta, \epsilon, S)$ is a Hopf algebra in $\mathcal{C}$, where the antipode S is invertible. Then there exists a quasi-triangular Hopf algebra $D(A)$ on $A \otimes A^*$.

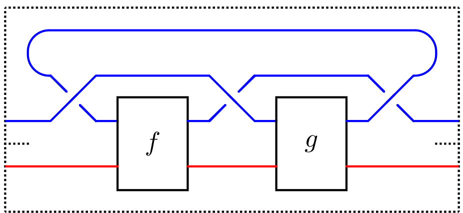
Before going into the technical details, let us first explain an outline of the construction and give some informal remarks. Given a Hopf algebra $A = (A, m, 1, \Delta, \epsilon, S)$ with S invertible, let $A^{\mathrm{op}*} = (A^*, \Delta^*, \epsilon^*, (m^{\mathrm{op}})^*, 1^*, (S^{-1})^*)$ be the dual opposite Hopf algebra. There are suitable actions of A on $A^{\mathrm{op}*}$ and $A^{\mathrm{op}*}$ on A, and with them we can form a bicrossed product (Majid 1990; Majid 1995) of A with $A^{\mathrm{op}*}$ which is the Hopf algebra $D(A)$. We shall note that $D(A)$ is almost like a tensor product of A and $A^{\mathrm{op}*}$ — except some clever adjustment on the multiplication and antipode. Also let us remark that $\mathbf{Mod}(A^{\mathrm{op}*})$ is isomorphic to $\mathbf{Comod}(A)$ as a monoidal category, and $\mathbf{Mod}(D(A))$ can be regarded as a combination of $\mathbf{Mod}(A)$ and $\mathbf{Comod}(A)$, as we will soon see for the case of $\overline{G}$ in $\mathbf{Rel}$ below.

Unfortunately, a direct description of $D(A)$ is rather complicated; see (Chen 2000) for instance. Instead, we shall give an alternative, simpler description using the *Int-construction* of Joyal, Street and Verity (Joyal *et al.* 1996).

Recall that, for a traced monoidal category $\mathcal{C}$, one can construct a ribbon category $\mathbf{Int}(\mathcal{C})$ whose objects are pairs of those of $\mathcal{C}$, and a morphism $f : (A_+, A_-) \to (B_+, B_-)$ in $\mathbf{Int}(\mathcal{C})$ is a morphism from $A_+ \otimes B_-$ to $B_+ \otimes A_-$ in $\mathcal{C}$ which can be drawn as

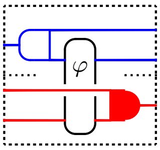


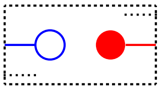
The composition of $f : (A_+, A_-) \to (B_+, B_-)$ and $g : (B_+, B_-) \to (C_+, C_-)$ is

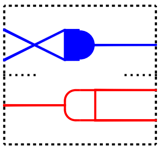


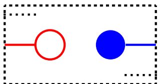
The tensor product of (A_+, A_-) and (B_+, B_-) is $(A_+ \otimes B_+, B_- \otimes A_-)$, while the unit object is (I, I); see (Joyal *et al.* 1996; Hasegawa 2009) for further details of the structure of $\mathbf{Int}(\mathcal{C})$.

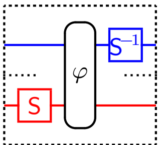
Proposition 4.2. For a Hopf algebra $A = (A, m, 1, \Delta, \epsilon, S)$ with an invertible antipode S in a traced symmetric monoidal category $\mathcal{C}$, there is a quasi-triangular Hopf algebra $((A,A), m^{\mathsf{d}}, 1^{\mathsf{d}}, \Delta^{\mathsf{d}}, \epsilon^{\mathsf{d}}, S^{\mathsf{d}})$ with a universal R-matrix R in $\mathbf{Int}(\mathcal{C})$ given as follows.

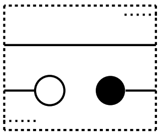
$m^{\mathsf{d}} : (A,A) \otimes (A,A) \to (A,A)$ 

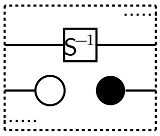
$1^{\mathsf{d}} : I \to (A,A)$ 

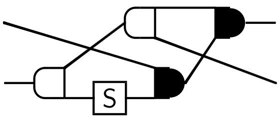
$\Delta^{\mathsf{d}} : (A,A) \to (A,A) \otimes (A,A)$ 

$\epsilon^{\mathsf{d}} : (A,A) \to I$ 

$S^{\mathsf{d}} : (A,A) \to (A,A)$ 

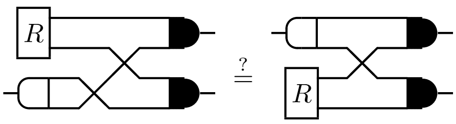
$R : I \to (A,A) \otimes (A,A)$ 

$R^{\circ} : I \to (A,A) \otimes (A,A)$ 

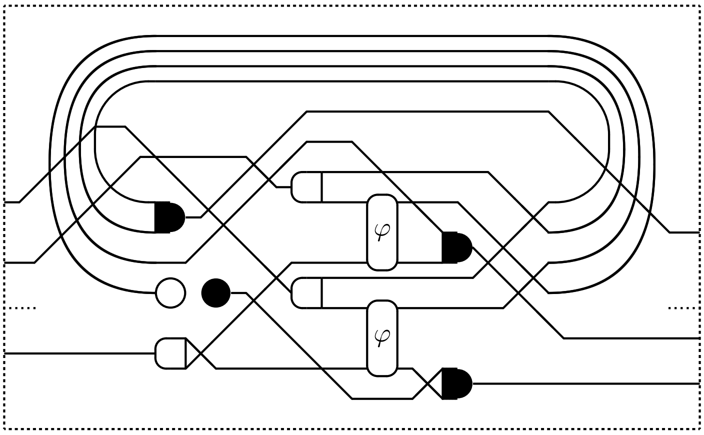
where $\varphi : A \otimes A \to A \otimes A$ is 

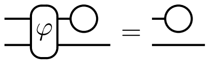
.

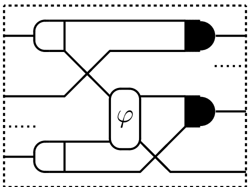
Proof (outline).
It suffices to check that all axioms of quasi-triangular Hopf algebras hold — perhaps best by the equational reasoning on the graphical presentations on sufficiently large sheets. Checking axioms which do not involve m^d and S^d are fairly straightforward, as there is no interaction between the first (positive, lower) component and the second (negative, upper) component. Cases with m^d or S^d, hence φ, do need some work. Here we shall see one of the most complex cases: the first axiom for universal R-matrices.



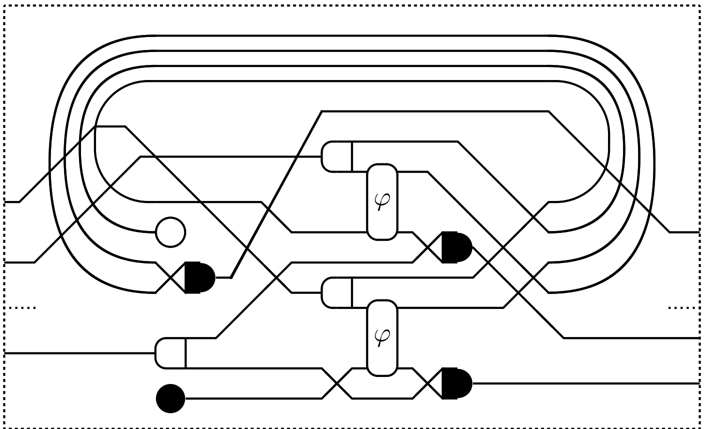
From the definition, the left hand side of this axiom is equal to

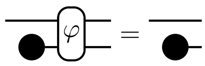


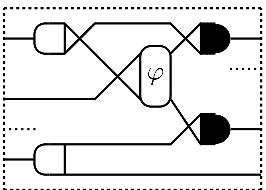
After some simplifications (using an easily derivable equation ), it is equal to



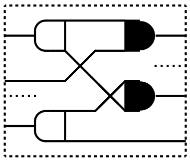
Similarly, the right hand side is



which turns out to be equal to (with making use of an easy equation )



By expanding the definition of φ and by some further simplifications, both of them finally agrees with



□

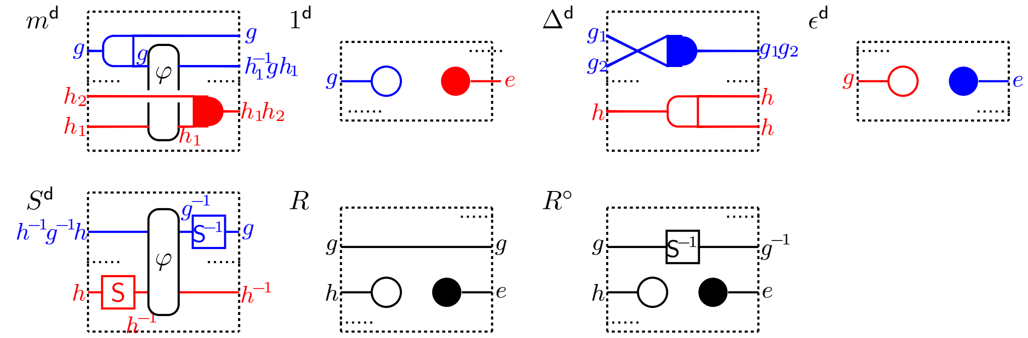
When $\mathcal{C}$ itself is a compact closed category, there is a strong symmetric monoidal equivalence $F : \mathbf{Int}(\mathcal{C}) \to \mathcal{C}$ sending (A_+, A_-) to $F(A_+, A_-) = A_-^* \otimes A_+$, with the obvious isomorphism from $F(A_+, A_-) \otimes F(B_+, B_-) = A_-^* \otimes A_+ \otimes B_-^* \otimes B_+$ to $F((A_+, A_-) \otimes (B_+, B_-)) = (B_- \otimes A_-)^* \otimes A_+ \otimes B_-$. Via this equivalence, this quasi-triangular Hopf algebra on (A, A) in $\mathbf{Int}(\mathcal{C})$ is sent to a quasi-triangular Hopf algebra on $A \otimes A^*$ in $\mathcal{C}$, as claimed in Proposition 4.1.

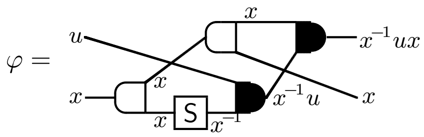
4.2. *Quantum double of* $\overline{G}$ *in* **Rel**

Now we turn our attention to the Hopf algebra $\overline{G}$ in **Rel**. Since the antipode S of $\overline{G}$ is invertible, we can apply the quantum double construction to $\overline{G}$, and we obtain a quasi-triangular (in fact, ribbon) Hopf algebra $D(\overline{G})$.

By Proposition 4.2, the quantum double of $\overline{G}$ in $\mathbf{Int}(\mathbf{Rel})$ is $((G, G), m^{\mathsf{d}}, 1^{\mathsf{d}}, \Delta^{\mathsf{d}}, \epsilon^{\mathsf{d}}, S^{\mathsf{d}}, R)$ where

$$\begin{aligned}
m^{\mathsf{d}} &= \{(((h_1, h_2), g), (h_1 h_2, (h_1^{-1} g h_1, g))) \mid g, h_1, h_2 \in G\} \\
1^{\mathsf{d}} &= \{((*, g), (e, *)) \mid g \in G\} \\
\Delta^{\mathsf{d}} &= \{((h, (g_2, g_1)), ((h, h), g_1 g_2)) \mid g_1, g_2, h \in G\} \\
\epsilon^{\mathsf{d}} &= \{((g, *), (*, e)) \mid g \in G\} \\
S^{\mathsf{d}} &= \{((h, h^{-1} g^{-1} h), (h^{-1}, g)) \mid g, h \in G\} \\
R &= \{((*, (h, g)), ((e, g), *)) \mid g, h \in G\} \\
R^{\circ} &= \{((*, (h, g^{-1})), ((e, g), *)) \mid g, h \in G\}
\end{aligned}$$





Moreover, $\overline{G}$ has a universal twist $\{((*, g), (g, *)) \mid g \in G\} : I \to (G, G)$, so it is a ribbon Hopf algebra. Via the strong symmetric monoidal equivalence from **Int**(**Rel**) to **Rel**, we obtain:

Theorem 4.3. Suppose that $G = (G, \cdot, e, (-)^{-1})$ is a group. There is a ribbon Hopf algebra $D(\overline{G}) = (G \times G, m^{\mathrm{d}}, 1^{\mathrm{d}}, \Delta^{\mathrm{d}}, \epsilon^{\mathrm{d}}, S^{\mathrm{d}}, R, v)$ in **Rel**, with

$$\begin{array}{rcl} m^{\mathrm{d}} & = & \{(((g, h_1), (h_1^{-1} g h_1, h_2)), (g, h_1 h_2)) \mid g, h_1, h_2 \in G\} \\ 1^{\mathrm{d}} & = & \{(*, (g, e)) \mid g \in G\} \\ \Delta^{\mathrm{d}} & = & \{((g_1 g_2, h), ((g_1, h), (g_2, h)) \mid g_1, g_2, h \in G\} \\ \epsilon^{\mathrm{d}} & = & \{((e, g), *) \mid g \in G\} \\ S^{\mathrm{d}} & = & \{((g, h), (h^{-1} g^{-1} h, h^{-1})) \mid g, h \in G\} \\ R & = & \{(*, ((g, e), (h, g))) \mid g, h \in G\} \\ v & = & \{(*, (g, g)) \mid g \in G\} \end{array}$$

where R is the universal R-matrix and v is the universal twist.

When G is not Abelian, $D(\overline{G})$ is neither commutative nor co-commutative. Below we shall observe that modules of $D(\overline{G})$ can be identified with the *crossed G-sets* (Freyd and Yetter 1989; Whitehead 1949).

5. A ribbon category of crossed G-sets

5.1. *Crossed G-sets*

Let $G = (G, \cdot, e, (-)^{-1})$ be a group. A *crossed G-set* $X = (X, \bullet, |\_|)$ is given by a set X together with a group action $\bullet : G \times X \to X$ and a function $|\_|$ from X to G such that, for any $g \in G$ and $x \in X$, $|g \bullet x| = g \cdot |x| \cdot g^{-1}$ holds. For instance, G itself can be seen a crossed G-set with $g \bullet h = g \cdot h \cdot g^{-1}$ and $|h| = h$. Another trivial example is a G-set with $|x| = e$.

Proposition 5.1. For any set X, there is a bijective correspondence between $D(\overline{G})$-modules on X and crossed G-sets on X.

Proof. If $\alpha : G \times G \times X \to X$ is a $D(\overline{G})$-module, for any $g \in G$ and $x \in X$ there are unique $h \in G$ and $y \in X$ such that $(((h, g), x), y) \in \alpha$, and X carries the structure of crossed G-set where $g \bullet x$ is this uniquely determined y and $|x|$ is the unique h such that $(((h, e), x), x) \in \alpha$. Conversely, a crossed G-set $(X, \bullet, |\_|)$ gives rise to a module $\{(((|g \bullet x|, g), x), g \bullet x) \mid g \in G, x \in X\} : G \times G \times X \to X$. It is not hard to see that this is a bijective correspondence. □

A morphism of crossed G-sets from $(X, \bullet, |\_|)$ to $(Y, \bullet, |\_|)$, corresponding to the morphism of $D(\overline{G})$-modules, is a binary relation $r : X \to Y$ such that $(x, y) \in r$ implies $(g \bullet x, g \bullet y) \in r$ as well as $|x| = |y|$. The identity and composition of morphisms are just the same as those of binary relations. Let us denote the category of crossed G-sets and morphisms by $\mathbf{XRel}(G)$ which is isomorphic to $\mathbf{Mod}(D(\overline{G}))$. We note that the category G-$\mathbb{XSf}$ of crossed G-sets of Freyd and Yetter (Freyd and Yetter 1989) is the subcategory of $\mathbf{XRel}(G)$ whose morphisms are restricted to functions and objects are restricted to finite ones. A variant of $\mathbf{XRel}(G)$ where G is not a group but a commutative monoid has appeared in (Abramsky *et al.* 1999).

For any set X, the free crossed G-set over X is given by $\mathcal{F}(X) = (G \times G \times X, \bullet, |\_|)$ with $g \bullet (h_1, h_2, x) = (g \cdot h_1 \cdot g^{-1}, g \cdot h_2, x)$ and $|(h_1, h_2, x)| = h_1$. $\mathcal{F}$ extends to a functor from $\mathbf{Rel}$ to $\mathbf{XRel}(G)$ which is left adjoint to the forgetful functor $\mathcal{U} : \mathbf{XRel}(G) \to \mathbf{Rel}$ which sends $(X, \bullet, |\_|)$ to X.

5.2. *The ribbon structure on* $\mathbf{XRel}(G)$

By Proposition 2.5, $\mathbf{Mod}(D(\overline{G}))$, hence $\mathbf{XRel}(G)$, is a ribbon category. In $\mathbf{XRel}(G)$, the tensor unit is $I = (\{*\},\ (g, *) \mapsto *,\ * \mapsto e)$, and the tensor product of $X = (X, \bullet, |\_|)$ and $Y = (Y, \bullet, |\_|)$ is

$$X \otimes Y = (X \times Y,\ (g, (x, y)) \mapsto (g \bullet x, g \bullet y),\ (x, y) \mapsto |x| \cdot |y|).$$

The tensor product of morphisms, as well as the coherence isomorphisms a, l, r, are inherited from $\mathbf{Rel}$. For this monoidal structure we have a braiding $\sigma_{X,Y} : X \otimes Y \overset{\cong}{\to} Y \otimes X$ induced by the universal R-matrix R as

$$\sigma_{X,Y} = \{((x, y), (|x| \bullet y, x)) \mid x \in X, y \in Y\}.$$

There is a twist $\theta_X : X \overset{\cong}{\to} X$ induced by the universal twist v:

$$\theta_X = \{(x, |x| \bullet x) \mid x \in X\}.$$

For a crossed G-set $X = (X, \bullet, |\_|)$, its left dual is $X^* = (X, \bullet, |\_|^{-1})$, with unit $\eta_X = \{(*, (x, x)) \mid x \in X\} : I \to X \otimes X^*$ and counit $\varepsilon_X = \{((x, x), *) \mid x \in X\} : X^* \otimes X \to I$. We note that the canonical trace on $\mathbf{XRel}(G)$ is given just like that on $\mathbf{Rel}$: for $f : A \otimes X \to B \otimes X$, its trace $Tr^X_{A,B} f : A \to B$ is

$$Tr^X_{A,B} f = \{(a, b) \in A \times B \mid \exists x \in X\ ((a, x), (b, x)) \in f\}.$$

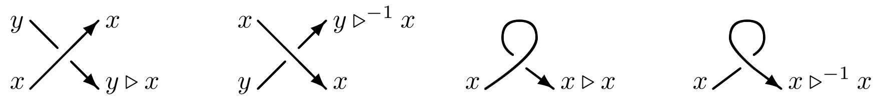
5.3. *Interpreting tangles in* $\mathbf{XRel}(G)$

Since the category of (oriented, framed) tangles is equivalent to the ribbon category freely generated by a single object (Shum 1994), by specifying a ribbon category and an object, we always obtain a structure-preserving functor from the category of tangles to the ribbon category, which determines an invariant of tangles (Yetter 2001). This is also the case for $\mathbf{XRel}(G)$.

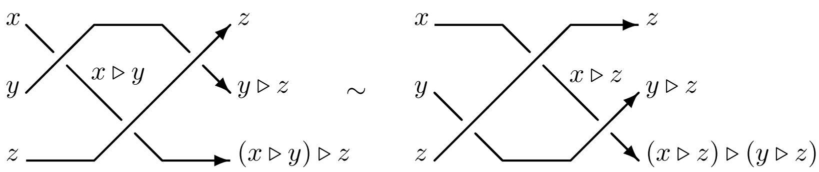
For understanding how a crossed G-set gives rise to an invariant of tangles, it is helpful to consider the *rack* (Fenn and Rourke 1992) associated to the crossed G-set[§]. Given a crossed G-set $(X, \bullet, |\_|)$, let us define operators $\triangleright, \triangleright^{-1} : X \times X \to X$ as $x \triangleright y = |y| \bullet x$ and $x \triangleright^{-1} y = |y|^{-1} \bullet x$. Then $(X, \triangleright, \triangleright^{-1})$ forms a rack; that is, the following equations hold[¶].

$$(x \triangleright y) \triangleright^{-1} y \;=\; x \;=\; (x \triangleright^{-1} y) \triangleright y \qquad \text{(bijectivity of } (-) \triangleright y)$$
$$(x \triangleright y) \triangleright z \;=\; (x \triangleright z) \triangleright (y \triangleright z) \qquad \text{(self-distributivity)}$$

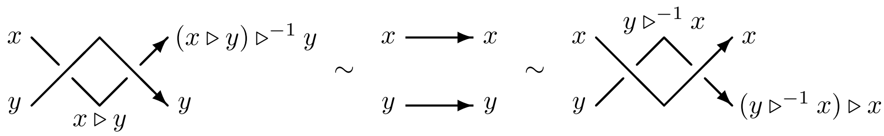
Now the braiding and twist can be described in terms of this rack: $\sigma_{X,Y} = \{((x, y), (y \triangleright x, x)) \mid x \in X, y \in Y\}$ and $\theta_X = \{(x, x \triangleright x) \mid x \in X\}$. The interpretation of a tangle diagram in $\mathbf{XRel}(G)$ with a crossed G-set X is then determined by all possible X-labellings of the segments from an underpass to the next underpass satisfying "y under x from left gives $y \triangleright x$" and "y under x from right gives $y \triangleright^{-1} x$".



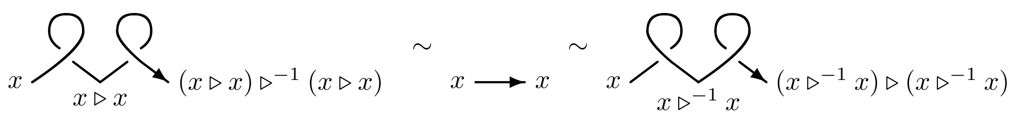
For instance, the self-distributivity justifies the Reidemeister move III:



Similarly, the Reidemeister move II is justified by the bijectivity:



The framed version of Reidemeister move I is also justified by the self-distributivity and bijectivity:



[§] Indeed, another name for crossed G-sets coined by Fenn and Rourke is *augmented racks*. They have shown that every rack arises from an augmented rack, hence a crossed G-set.

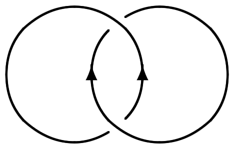
[¶] However, this does not have to be a *quandle* in the sense of Joyce (Joyce 1982), since the idempotency $x \triangleright x = x$ does not hold in general.

In the above, the equation $(x \triangleright x) \triangleright^{-1} (x \triangleright x) = x$ is derivable as

$$\begin{aligned}
(x \triangleright x) \triangleright^{-1} (x \triangleright x) &= (((x \triangleright^{-1} x) \triangleright x) \triangleright x) \triangleright^{-1} (x \triangleright x) && \text{bijectivity} \\
&= (((x \triangleright^{-1} x) \triangleright x) \triangleright (x \triangleright x)) \triangleright^{-1} (x \triangleright x) && \text{self-distributivity} \\
&= (x \triangleright^{-1} x) \triangleright x && \text{bijectivity} \\
&= x && \text{bijectivity.}
\end{aligned}$$

The equation $(x \triangleright^{-1} x) \triangleright (x \triangleright^{-1} x) = x$ also follows from a similar reasoning (consider $(((x \triangleright^{-1} x) \triangleright (x \triangleright^{-1} x)) \triangleright x) \triangleright^{-1} x)$.

Example 5.2. Consider of the following link:



Its interpretation in $\mathbf{XRel}(G)$ with a crossed G-set X takes a value in $\mathbf{XRel}(G)(I, I) = \{id_I, \emptyset\}$, and it is the identity relation id_I if there exist $x, y \in X$ such that $x = x \triangleright y$ and $y = y \triangleright x$ hold; otherwise it is the empty relation $\emptyset$.

These invariants are far from complete. For example, the links   and  always have the same interpretation for any crossed G-set.

6. A model of braided linear logic

In this section, we outline the notion of models of (fragments of) braided linear logic, and see how $\mathbf{XRel}(G)$ in the previous section gives such a model. For a detailed exposition on categorical models of linear logic, see (Melliès 2009). Some considerations on the proof theory of braided linear logic are found in (Bellin and Fleury 1998).

6.1. *Models of braided linear logic*

By a model of braided multiplicative linear logic (braided MLL), we mean a braided $*$-autonomous category (Barr 1995); note that a ribbon category is braided $*$-autonomous, hence is a model of braided MLL. A model of braided multiplicative additive linear logic (braided MALL) is then a braided $*$-autonomous category with finite products.

For exponential, we employ the following generalization of the notion of linear exponential comonads (Hyland and Schalk 2003) on symmetric monoidal categories: by a linear exponential comonad on a braided monoidal category we mean a braided monoidal comonad whose category of coalgebras is a category of commutative comonoids. A model of braided MELL is then a braided $*$-autonomous category with a linear exponential comonad. (An implication of this definition is that braiding becomes symmetry on exponential objects: $\sigma^{-1}_{!X,!Y} = \sigma_{!Y,!X}$.) A model of braided LL is a model of braided MALL with a linear exponential comonad (or a model of MELL with finite products).

6.2. **XRel**(G) *as a model of braided linear logic*

XRel(G) is a ribbon category with finite products, hence is a model of braided MALL.

There is a strict balanced monoidal functor $F : \mathbf{Rel} \to \mathbf{XRel}(G)$ which sends a set X to $FX = (X, (g,x) \mapsto x, x \mapsto e)$. F has a right adjoint $U : \mathbf{XRel}(G) \to \mathbf{Rel}$ which sends $X = (X, \bullet, |\_|)$ to $UX = \{x \in X \mid |x| = e\}/_\sim$ where $x \sim y$ iff $g \bullet x = y$ for some g. By composing F and U with a linear exponential comonad ! on **Rel** (e.g. the finite multiset comonad), we obtain a linear exponential comonad $F!U$ on **XRel**(G) whose category of coalgebras is equivalent to that of !. Hence **XRel**(G) is a model of braided LL.

As a result, there exists a linear fixed-point operator on **XRel**(G) derived from the trace and the linear exponential comonad on **XRel**(G) as given in (Hasegawa 2009), which can be used for interpreting a linear fixed-point combinator $\mathsf{Y}_X : !(X \multimap X) \multimap X$. (In *ibid.*, we construct such a linear fixed-point operator on traced *symmetric* monoidal categories with a linear exponential comonad. While the braiding of **XRel**(G) is not symmetric, the construction given there works without any change, essentially because braiding becomes symmetry on exponential objects, as noted above.)

XRel(G) is degenerate as a model of LL in the sense that it cannot distinguish tensor from par. As an easy remedy, one may apply the simple self-dualization construction (Hyland and Schalk 2003) for obtaining a "non-compact" model. For a braided monoidal closed category $\mathcal{C}$ with finite products, there is a braided $*$-autonomous structure on $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ whose tensor unit is $(I, 1)$ (where 1 is a terminal object and should not be confused with the unit element of a monoid) and tensor product is given by $(U, X) \otimes (V, Y) = (U \otimes V, U \multimap Y \times V \multimap X)$, while the duality is given by $(U, X)^\perp = (X, U)$. By applying the simple self-dualization construction to **XRel**(G) we obtain a "non-compact" model $\mathbf{XRel}(G) \times \mathbf{XRel}(G)^{\mathrm{op}}$ of braided LL. Alternatively, $\mathbf{XRel}(G) \times \mathbf{XRel}(G)^{\mathrm{op}}$ arises as the category of modules of $D(\overline{G})$ (or $(D(\overline{G}), \emptyset)$ to be more precise) in the $*$-autonomous category $\mathbf{Rel} \times \mathbf{Rel}^{\mathrm{op}}$ obtained by the simple self-dualization on **Rel**.

7. Concluding remarks

We have demonstrated that there are many non-trivial Hopf algebras in the category of sets and binary relations. In particular, by applying the quantum double construction we have constructed a non-commutative non-co-commutative Hopf algebra with a universal R-matrix and a universal twist, and the ribbon category of its modules turns out to be a category of crossed G-sets.

Technically, most of our results are variations or instances of the already established theory of quantum groups, and we do not claim much novelty in this regard. What is much more important in this work, we believe, is that our results show that it is indeed possible to carry out a substantial part of quantum group theory in a category used for semantics of computation and logic. Although we have spelled out just a particular case of **Rel**, we expect that the same can be done meaningfully in various other settings, including

— the $*$-autonomous category of coherent spaces and linear maps (Girard 1987), and its variations used as models of linear logic,

— various categories of games, in particular the compact closed category of Conway games (Joyal 1977; Melliès 2004), and
— the category of sets (or presheaves on discrete categories) and linear normal functors (Hasegawa 2002), as well as the bicategory of small categories and profunctors.

The first two would lead to models of braided linear logic and some braided variants of game semantics. The third should be a direct refinement of our work on **Rel**, in that we replace binary relations $X \times Y \to 2$ with **Set**-valued functors $X \times Y \to \mathbf{Set}$ (which amount to linear normal functors from $\mathbf{Set}^X$ to $\mathbf{Set}^Y$).

Finally, we must admit that the computational significance of braided monoidal structure is yet to be examined. As far as we know, $\mathbf{XRel}(G)$ is the first non-symmetric ribbon category featuring a linear exponential comonad, allowing non-trivial interpretations of braidings as well as recursive programs at the same time. If we are to develop a sort of braided variant of denotational semantics in future, $\mathbf{XRel}(G)$ might be a good starting point. A potentially related direction would be the area of *topological quantum computation* (Freedman *et al.* 2002; Kitaev 2003; Wang 2010; Panangaden and Paquette 2011), in which modular tensor categories (semisimple ribbon categories with finite simple objects satisfying an extra condition) (Turaev 1994; Bakalov and Kirilov 2001) play the central role. Although $\mathbf{XRel}(G)$ is not modular, it might be possible to develop a toy (and suitably simplified) model of topological quantum computation in it.

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