Parameterizations and Fixed-Point Operators on Control Categories

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Abstract. The $\lambda\mu$-calculus features both variables and names, together with their binding mechanisms. This means that constructions on open terms are necessarily parameterized in two different ways for both variables and names. Semantically, such a construction must be modeled by a bi-parameterized family of operators. In this paper, we study these bi-parameterized operators on Selinger's categorical models of the $\lambda\mu$-calculus called control categories. The overall development is analogous to that of Lambek's functional completeness of cartesian closed categories via polynomial categories. As a particular and important case, we study parameterizations of uniform fixed-point operators on control categories, and show bijective correspondences between parameterized fixed-point operators and non-parameterized ones under uniformity conditions.

1 Introduction

1.1 Parameterization on Models of $\lambda\mu$-Calculus

The simply typed $\lambda\mu$-calculus introduced by Parigot [9] is an extension of the simply typed λ-calculus with first-class continuations. In the $\lambda\mu$-calculus, every judgment has two kinds of type declarations: one is for variables and the other is for continuation variables, which are often called names. So, it is natural that an operator $(-)^\dagger$ on the $\lambda\mu$-calculus takes the following form:

$$\frac{\Gamma \vdash M : A \mid \Delta}{\Gamma \vdash M^\dagger : B \mid \Delta}$$

The typed call-by-name $\lambda\mu$-calculus (with classical disjunctions) has a sound and complete class of models called control categories [11]. An interpretation of a judgment $x_1 : B_1, \ldots, x_n : B_n \vdash M : A \mid \alpha_1 : A_1, \ldots, \alpha_n : A_n$ in a control category $\mathcal{P}$ is a morphism $f \in \mathcal{P}([\![B_1]\!] \times \cdots \times [\![B_n]\!], [\![A]\!] \mathbin{⅋} [\![A_1]\!] \mathbin{⅋} \cdots \mathbin{⅋} [\![A_n]\!])$. So, the above syntactic construction is modeled in a control category $\mathcal{P}$ like the following:

$$\frac{f \in \mathcal{P}(X, [\![A]\!] \mathbin{⅋} Y)}{f^\dagger \in \mathcal{P}(X, [\![B]\!] \mathbin{⅋} Y)}$$

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Therefore, from the semantic point of view, operators should be understood as parameterized construction with two parameters X and Y. We call such a parameterization *bi-parameterization*. The study of this sort of parameterization on cartesian categories was initiated by Lambek [7], and its significance in modeling parameterized constructs associated to algebraic data-types has been studied by Jacobs [5]. The aim of this work is to derive analogous results for bi-parameterization on control categories.

1.2 Fixed-Point Operators and Parameterizations

Our motivation to study parameterization on control categories comes from our previous work about fixed-point operators on the $\lambda\mu$-calculi in [4] and [6]. The equational theories of fixed-point operators in call-by-name λ-calculi have been studied extensively, and now there are some canonical axiomatizations including iteration theories [1] and Conway theories, equivalently traced cartesian categories [3] (see [12] for recent results). Because the $\lambda\mu$-calculus is an extension of the simply typed call-by-name λ-calculus, it looks straightforward to consider fixed-point operators in the $\lambda\mu$-calculus and indeed we have considered an appropriate model of the $\lambda\mu$-calculus with a fixed-point operator. In a control category, however, the possible forms of parameterized fixed-point operators are various and their relation is sensitive. To understand our problem, let us recall a folklore construction.

In a cartesian closed category, it is possible to derive a parameterized fixed-point operator from a non-parameterized one:

$$\dfrac{\dfrac{X \times A \xrightarrow{f} A}{A \xrightarrow{\mathrm{cur}(f)} A^X}\ \text{Currying}}{A^X \xrightarrow{(\cdot)^X} (A^X)^X \xrightarrow{A^{\Delta}} A^X}$$

$$\dfrac{\dfrac{A^X \xrightarrow{(\cdot)^X} (A^X)^X \xrightarrow{A^{\Delta}} A^X}{1 \xrightarrow{(\cdot)^*} A^X}\ \text{Fixed point}}{X \xrightarrow{f^{\dagger}} A}\ \text{Uncurrying}$$

If we use the simply typed λ-calculus as an internal language of cartesian closed categories, this construction amounts to taking the fixed-point of $k\colon X \to A \vdash \lambda x^X.f(x,kx)\colon X \to A$ for $x\colon X, a\colon A \vdash f(x,a)\colon A$. So, by letting f be $\lambda(f^{A\to A}, x^A).fx$, we obtain a fixed-point combinator $\mathtt{fix}_A : (A \to A) \to A$. It is routine to see that this $\mathtt{fix}_A$ is indeed a fixed-point combinator. In a cartesian closed category, to give a parameterized fixed-point operator is to give a fixed-point combinator. Thus, we have a parameterized fixed-point operator from a non-parameterized one.

In this paper, we investigate such a construction of fixed-point operators on control categories. Since we have to consider parameters for free names in the $\lambda\mu$-calculus, a control category has three patterns of parameterizations: one is

Object constructors:

$$1 \;\; \bot \;\; A \times B \;\; B^A \;\; A \parr B$$

Morphism constructors:

$$\begin{array}{ll}
\mathrm{id}_A & : A \to A \\
\diamond_A & : A \to 1 \\
\pi_1 & : A \times B \to A \\
\pi_2 & : A \times B \to B \\
\varepsilon_{A,B} & : B^A \times A \to B \\
\mathrm{a}_{A,B,C} & : (A \parr B) \parr C \to A \parr (B \parr C) \\
\mathrm{l}_A & : A \to A \parr \bot \\
\mathrm{c}_{A,B} & : A \parr B \to B \parr A \\
\mathrm{i}_A & : \bot \to A \\
\nabla_A & : A \parr A \to A \\
\mathrm{d}_{A,B,C} & : (A \parr C) \times (B \parr C) \to (A \times B) \parr C \\
\mathrm{s}^{-1}_{A,B,C} & : (B \parr C)^A \to B^A \parr C
\end{array}$$

$$\frac{f : A \to B \quad g : B \to C}{g \circ f : A \to C} \qquad \frac{f : A \to B \quad g : A \to C}{\langle f, g\rangle : A \to B \times C}$$

$$\frac{f : A \times B \to C}{\mathrm{cur}(f) : A \to C^B} \qquad \frac{f : A \to B}{f \parr C : A \parr C \to B \parr C}$$

Fig. 1. Signature of Control Categories

standard *parameterization* and another is a parameterization for names called *co-parameterization*, and the last one is *bi-parameterization* which has both parameterization and co-parameterization. In a control category, a bi-parameterized fixed-point operator can be derived from a co-parameterized one in analogous way of the cartesian closed case. Moreover, a bi-parameterized fixed-point operator can be derived from a non-parameterized one under suitable uniformity principles. An interesting and important observation is that these correspondences are indeed bijective. This result simplifies semantic structure needed in [6].

1.3 Construction of This Paper

Section 2 is a reminder of control categories and the typed call-by-name $\lambda\mu$-calculus. In Section 3, we introduce polynomial categories with respect to control categories. In Section 4, we consider generic parameterized operators on control categories. The rest of this paper gives observations of parameterized fixed-point operators on control categories and their uniformity principles. Uniformity conditions enable us to prove bijective correspondence between uniform bi-parameterized fixed-point operators and uniform non-parameterized ones.

2 Preliminaries

2.1 Control Categories

Control categories introduced by Selinger [11] are sound and complete models of the typed call-by-name $\lambda\mu$-calculus. A control category is a cartesian closed

$$\frac{x\colon A \in \Gamma}{\Gamma \vdash x\colon A \mid \Delta} \qquad \frac{}{\Gamma \vdash *\colon \top \mid \Delta}$$

$$\frac{\Gamma, x\colon A \vdash M\colon B \mid \Delta}{\Gamma \vdash \lambda x^A.\, M\colon A \to B \mid \Delta} \qquad \frac{\Gamma \vdash M\colon A \to B \mid \Delta \quad \Gamma \vdash N\colon A \mid \Delta}{\Gamma \vdash MN\colon B \mid \Delta}$$

$$\frac{\Gamma \vdash M\colon A \mid \Delta \quad \Gamma \vdash N\colon B \mid \Delta}{\Gamma \vdash \langle M, N\rangle\colon A \wedge B \mid \Delta} \qquad \frac{\Gamma \vdash M\colon A_1 \wedge A_2 \mid \Delta}{\Gamma \vdash \pi_i M\colon A_i \mid \Delta}$$

$$\frac{\Gamma \vdash M\colon \bot \mid \alpha\colon A, \Delta}{\Gamma \vdash \mu\alpha^A.\, M\colon A \mid \Delta} \qquad \frac{\Gamma \vdash M\colon A \mid \Delta \quad \alpha\colon A \in \Delta}{\Gamma \vdash [\alpha]\, M\colon \bot \mid \Delta}$$

$$\frac{\Gamma \vdash M\colon \bot \mid \beta\colon B, \alpha\colon A, \Delta}{\Gamma \vdash \mu(\alpha^A, \beta^B).\, M\colon A \vee B \mid \Delta} \qquad \frac{\Gamma \vdash M\colon A \vee B \mid \Delta \quad \alpha\colon A, \beta\colon B \in \Delta}{\Gamma \vdash [\alpha, \beta]\, M\colon \bot \mid \Delta}$$

Fig. 2. Deduction Rules of $\lambda\mu$-Calculus

category together with a premonoidal structure [10]. In this section, we recall some definitions about control categories but may omit the detail, which are found in [11].

Definition 1. *A morphism $f\colon A \to B$ in a premonoidal category $\mathcal{P}$ is* ***central*** *if for every morphism $g \in \mathcal{P}(C, D)$, $(B \parr g) \circ (f \parr C) = (f \parr D) \circ (A \parr g)$ and $(g \parr B) \circ (C \parr f) = (D \parr f) \circ (g \parr A)$.*

Definition 2. *A morphism $f\colon A \to B$ in a symmetric premonoidal category with codiagonals $\mathcal{P}$ is* ***focal*** *if f is central, discardable and copyable [11]. The subcategory formed by the focal morphisms of $\mathcal{P}$ is called the* ***focus*** *of $\mathcal{P}$ and denoted by $\mathcal{P}^\bullet$.*

Remark 1. In general, the focus of a symmetric premonoidal category is not the same as its center. (For example, detailed analysis are found in [2].) However, in a control category, the center and the focus always coincide [11].

Definition 3. *Suppose $\mathcal{P}$ is a symmetric premonoidal category with codiagonals and also suppose $\mathcal{P}$ has finite products. We say that $\mathcal{P}$ is* ***distributive*** *if the projections of products are focal and the functor $(-) \parr C$ preserves finite products for all objects C.*

Definition 4. *Suppose $\mathcal{P}$ is a symmetric premonoidal category with codiagonals and also suppose $\mathcal{P}$ is cartesian closed. $\mathcal{P}$ is a* ***control category*** *if the canonical morphism $\mathrm{s}_{A,B,C} \in \mathcal{P}(B^A \parr C, (B \parr C)^A)$ is a natural isomorphism in A, B and C, satisfying certain coherence conditions.*

Definition 5. *A* ***(strict) functor of control categories*** *is a functor that preserves all the structures of a control category on the nose.*

The structure of control categories is equational in the sense of Lambek and Scott [8]. The object and morphism constructors of control categories are shown in Figure 1. Some other canonical morphisms that are not shown here

$(\beta_\to)$	$(\lambda x^A.\, M)N = M\,[N\,/x]$	$: B$	
$(\eta_\to)$	$\lambda x^A.\, Mx = M$	$: B$	$x \notin \mathrm{FV}(M)$
$(\beta_\wedge)$	$\pi_i\langle M_1, M_2\rangle = M_i$	$: A_i$	
$(\eta_\wedge)$	$\langle \pi_1 M, \pi_2 M\rangle = M$	$: A \wedge B$	
$(\eta_\top)$	$* = M$	$: \top$	
(β_μ)	$[\beta]\mu\alpha^A.\, M = M\,[\beta\,/\alpha]$	$: \bot$	
(η_μ)	$\mu\alpha^A.[\alpha]\,M = M$	$: A$	$\alpha \notin \mathrm{FN}(M)$
$(\beta_\vee)$	$[\gamma,\delta]\mu(\alpha^A, \beta^B).\, M = M\,[\gamma\,/\alpha, \delta\,/\beta]$	$: \bot$	
$(\eta_\vee)$	$\mu(\alpha^A, \beta^B).[\alpha,\beta]\,M = M$	$: A \vee B$	$\alpha, \beta \notin \mathrm{FN}(M)$
$(\beta_\bot)$	$[\beta]\,M = M$	$: \bot$	
$(\zeta_\to)$	$(\mu\alpha^{A\to B}.\, M)N = \mu\beta^B.\, M\,[[\beta](-)N\,/[\alpha](-)]$	$: B$	
$(\zeta_\wedge)$	$\pi_i(\mu\alpha^{A_1\wedge A_2}.\, M) = \mu\beta^{A_i}.\, M\,[[\beta]\pi_i(-)\,/[\alpha](-)]$	$: A_i$	
$(\zeta_\vee)$	$[\gamma,\delta]\mu\alpha^{A\vee B}.\, M = M\,[[\gamma,\delta](-)\,/[\alpha](-)]$	$: \bot$	

Fig. 3. Axioms of $\lambda\mu$-Calculus

but definable from the constructors, $f \times A$, $\mathrm{w_r} = \mathrm{i}_A \invamp B \circ \mathrm{c}_{B,\bot} \circ \mathrm{l}_B : B \to A \invamp B$, $\mathrm{w_l} = \mathrm{c}_{B,A} \circ \mathrm{w_r} : A \to A \invamp B$ and so on are also used in this paper. Since coherence theorems for premonoidal categories have been shown by Power and Robinson in [10], we may elide not only cartesian structural isomorphisms but also premonoidal ones.

2.2 $\lambda\mu$-Calculus

According to Selinger, the typed call-by-name $\lambda\mu$-calculus can be considered as an internal language of control categories [11]. The types and the terms of our $\lambda\mu$-calculus are defined as follows:

$$\begin{aligned} \text{Types } A, B &::= \sigma \mid A \to B \mid \top \mid A \wedge B \mid \bot \mid A \vee B, \\ \text{Terms } M, N &::= x \mid * \mid \lambda x^A.\, M \mid MN \mid \langle M, N\rangle \mid \pi_1 M \mid \pi_2 M \\ &\quad\ \mid \mu\alpha^A.\, M \mid [\alpha]\,M \mid \mu(\alpha^A, \beta^B).\, M \mid [\alpha,\beta]\,M, \end{aligned}$$

where σ, x and α (β) range over base types, variables and names respectively. Every judgment takes the form $\Gamma \vdash M : A \mid \Delta$, where Γ denotes a sequence of pairs $x : A$, and Δ denotes a sequence of pairs $\alpha : A$. The typing rules and the axioms are given by Figure 2 and 3.

3 Parameterizations on Control Categories

3.1 Polynomial Control Categories

Since parameterization is nicely modeled by polynomial categories like the case of cartesian closed categories (see [7]), we introduce polynomial control categories and show their functional completeness a la Lambek. Because we have to deal with not only variables but also names on control categories unlike on cartesian closed categories, an additional parameter for free names is required.

We construct a new category $\mathcal{P}_Y^X$ from a control category $\mathcal{P}$ by

$$\mathcal{P}_Y^X(A,B) = \mathcal{P}(X \times A, B \parr Y).$$

The identity arrow of $\mathcal{P}_Y^X$ is the projection from $X \times A$ to A followed by the weakening arrow to $A \parr Y$ in $\mathcal{P}$. We define $g \circ_Y^X f$, the composite of $f \in \mathcal{P}_Y^X(A,B)$ and $g \in \mathcal{P}_Y^X(B,C)$, by

$$\begin{array}{l} X \times A \xrightarrow{\ \Delta \times A\ } X \times X \times A \xrightarrow{\ X \times f\ } X \times (B \parr Y) \\ \xrightarrow{\ \mathrm{w_l} \times (B \parr Y)\ } (X \parr Y) \times (B \parr Y) \xrightarrow{\ \mathrm{d}_{X,B,Y}\ } (X \times B) \parr Y \\ \xrightarrow{\ g \parr Y\ } C \parr Y \parr Y \xrightarrow{\ C \parr \nabla\ } C \parr Y. \end{array}$$

Hereafter, we write $marhrmd'_{A,B,C}$ for $\mathrm{d}_{A,B,C} \circ \mathrm{w_l} \times (B \parr C) \in \mathcal{P}(A \times (B \parr C), (A \times B) \parr C)$.

Proposition 1. $\mathcal{P}_Y^X$ *is a control category.*

We can regard $\mathcal{P}_Y^X$ as the polynomial control category obtained from $\mathcal{P}$ by adjoining an indeterminate of X and a name of Y. We call $\mathrm{w_l}: X \to X \parr Y$ in $\mathcal{P}$ the indeterminate variable of $\mathcal{P}_Y^X$ and $\pi_2: X \times Y \to Y$ in $\mathcal{P}$ the indeterminate name of $\mathcal{P}_Y^X$. $\mathcal{P}$ can be embedded into $\mathcal{P}_Y^X$ through the weakening functor $\mathrm{I}_Y^X: \mathcal{P} \to \mathcal{P}_Y^X$ defined by

$$\mathrm{I}_Y^X(f) = X \times A \xrightarrow{\pi_2} A \xrightarrow{f} B \xrightarrow{\mathrm{w_l}} B \parr Y.$$

The following theorem justifies to regard $\mathcal{P}_Y^X$ as a polynomial category with respect to control categories.

Theorem 1. *Let $\mathcal{P}$ be a control category. Given a control category $\mathcal{O}$ and a functor of control categories $F: \mathcal{P} \to \mathcal{O}$ with morphisms $a \in \mathcal{O}(1, FX)$ and $k \in \mathcal{O}^\bullet(FY, \bot)$, there exists a unique functor of control categories $F': \mathcal{P}_Y^X \to \mathcal{O}$ such that it sends the indeterminate variable and the indeterminate name of $\mathcal{P}_Y^X$ to a and k respectively and the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\ \mathrm{I}_Y^X\ } & \mathcal{P}_Y^X \\ & {\scriptstyle F} \searrow & \downarrow {\scriptstyle F'} \\ & & \mathcal{O} \end{array}$$

Proof. (Outline) F' is constructed by

$$F'f = $$
$$FA \cong 1 \times FA \xrightarrow{\ a \times FA\ } FX \times FA \xrightarrow{\ Ff\ } FB \parr FY \xrightarrow{\ FB \parr k\ } FB \parr \bot \cong FB$$

for $f \in \mathcal{P}(X \times A, B \parr Y)$. □

Remark 2. If we introduce a polynomial control category "$\mathcal{P}[x:X \mid \alpha:Y]$" syntactically (as in [8]), it is characterized by the universal property above. Hence we have $\mathcal{P}_Y^X \cong \mathcal{P}[x:X \mid \alpha:Y]$ and the functional completeness of control categories: for any "polynomial" $\phi(x,\alpha) \in \mathcal{P}[x:X \mid \alpha:Y](A,B)$, there exists a unique morphism $f \in \mathcal{P}(X \times A, B \parr Y)$ such that $\phi(x,\alpha) = (B \parr \alpha) \circ f \circ (x \times A)$.

By trivializing the parameter Y, we obtain a control category $\mathcal{P}^X$ with $\mathcal{P}^X(A,B) = \mathcal{P}(X \times A, B)$, and by trivializing the parameter X, we obtain a control category $\mathcal{P}_Y$ with $\mathcal{P}_Y(A,B) = \mathcal{P}(A, B \parr Y)$.

It can be seen easily that $\mathcal{P}_Y^X$, $(\mathcal{P}_Y)^X$ and $(\mathcal{P}^X)_Y$ are the same control category.

It is also useful to see that the mapping $(X,Y) \mapsto \mathcal{P}_Y^X$ gives rise to an indexed category $\mathcal{P}^{\mathrm{op}} \times \mathcal{P}^\bullet \to \mathbf{ContCat}$, where **ContCat** is the category of small control categories and functors of control categories. Indeed, $g \in \mathcal{P}(X, X')$ and $h \in \mathcal{P}^\bullet(Y', Y)$ determine the re-indexing functor from $\mathcal{P}_{Y'}^{X'}$ to $\mathcal{P}_Y^X$ which sends $f : X' \times A \to B \parr Y'$ to $(B \parr h) \circ f \circ (g \times A) : X \times A \to B \parr Y$.

Lemma 1. *The focus of $\mathcal{P}_Y$ agrees with the focus of $\mathcal{P}$,* i.e.,

$$(\mathcal{P}_Y)^\bullet(A,B) = \mathcal{P}^\bullet(A, B \parr Y).$$

For the focus of $\mathcal{P}^X$, see the next subsection.

3.2 Currying

The theorem below tells us how an indeterminate variable can be eliminated with a name, hence gives us a way to reduce the parameterized constructs on control categories to a simpler form: from the bi-parameterized form to the co-parameterized form.

Theorem 2. *The isomorphisms*

$$\mathcal{P}(X \times A, B \parr Y) \underset{\lfloor - \rfloor}{\overset{\lceil - \rceil}{\rightleftarrows}} \mathcal{P}(A, B \parr Y^X)$$

give rise to isomorphisms of control categories between $\mathcal{P}_Y^X$ and $\mathcal{P}_{Y^X}$.

Since a direct proof is very lengthy, we find it much easier to use the $\lambda\mu$-calculus as an internal language of control categories.

$$\frac{f = x:X, a:A \vdash M : B \mid \gamma : Y}{\lceil f \rceil = a:A \vdash \mu\beta^B.[\delta](\lambda x^X.\mu\gamma^Y.[\beta]M) : B \mid \delta : Y^X}$$

$$\frac{g = a:A \vdash N : B \mid \delta : Y^X}{\lfloor g \rfloor = x:X, a:A \vdash \mu\beta^B.[\gamma]((\mu\delta^{Y^X}.[\beta]N)x) : B \mid \gamma : Y}$$

It is routine to verify $\lceil - \rceil$ and $\lfloor - \rfloor$ preserve all the structures of control categories.

Remark 3. The above theorem suggests that any reasonable parameterized construct on a control category $\mathcal{P}$ must be compatible with $\lceil - \rceil$. Furthermore, the indexed categorical view mentioned before requires such a construct must be natural in parameter X in $\mathcal{P}$ and parameter Y in $\mathcal{P}^\bullet$. This consideration leads us to introduce the axiomatization of bi-parameterized fixed-point operators shortly.

The following results are part of this theorem, but we shall state them separately for future reference.

Lemma 2. $\lceil - \rceil$ *and* $\lfloor - \rfloor$ *preserve composites:*

$$\lceil g \circ_Y^X f \rceil = \lceil g \rceil \circ_{Y^X} \lceil f \rceil$$
$$\lfloor g \circ_{Y^X} f \rfloor = \lfloor g \rfloor \circ_Y^X \lfloor f \rfloor$$

Lemma 3. $\lceil - \rceil$ *and* $\lfloor - \rfloor$ *preserve focuses:*

$$f \in (\mathcal{P}_Y^X)^\bullet(A, B) \quad \textit{iff} \quad \lceil f \rceil \in (\mathcal{P}_{Y^X})^\bullet(A, B)$$
$$f \in (\mathcal{P}_{Y^X})^\bullet(A, B) \quad \textit{iff} \quad \lfloor f \rfloor \in (\mathcal{P}_Y^X)^\bullet(A, B)$$

In particular, from Lemma 1 and 3, we have $\mathcal{P}^{X^\bullet}(A, B) \cong \mathcal{P}^\bullet(A, B^X)$.

4 Parameterized Operators on Control Categories

In this section, we introduce three parameterization patterns based on our observation of polynomial control categories. One is standard parameterization in cartesian categories, and another parameterization is co-parameterization, which has a parameter for free names. The last one is bi-parameterization, which combines both parameterization and co-parameterization. Interaction between co-(bi-)parameterization and focuses of control categories is crucial.

Definition 6. *A* ***parameterized operator*** *of type* $(A_1, B_1) \times \cdots \times (A_n, B_n) \to (A, B)$ *on a control category* $\mathcal{P}$ *is a family of functions of the form*

$$\alpha^X : \mathcal{P}^X(A_1, B_1) \times \ldots \times \mathcal{P}^X(A_n, B_n) \to \mathcal{P}^X(A, B)$$

indexed by X*, such that natural in* X *in* $\mathcal{P}$*.*

Since a control category is a cartesian closed category, the following proposition holds.

Proposition 2. *Parameterized operators of type* $(A_1, B_1) \times \cdots \times (A_n, B_n) \to (A, B)$ *on a control category* $\mathcal{P}$ *are in bijective correspondence with arrows of* $\mathcal{P}(B_1^{A_1} \times \ldots \times B_n^{A_n}, B^A)$*.*

Definition 7. *A* ***co-parameterized operator*** *of type* $(A_1, B_1) \times \cdots \times (A_n, B_n) \to (A, B)$ *on a control category* $\mathcal{P}$ *is a family of functions of the form*

$$\alpha_Y : \mathcal{P}_Y(A_1, B_1) \times \ldots \times \mathcal{P}_Y(A_n, B_n) \to \mathcal{P}_Y(A, B)$$

indexed by Y*, such that natural in* Y *in* $\mathcal{P}^\bullet$*.*

Proposition 3. *Co-parameterized operators of type* $(A_1, B_1) \times \cdots \times (A_n, B_n) \to (A, B)$ *on a control category* $\mathcal{P}$ *are in bijective correspondence with arrows of* $\mathcal{P}(B_1^{A_1} \times \ldots \times B_n^{A_n}, B^A)$*.*

Proof. It follows from the isomorphisms $\mathcal{P}_Y(A, B) \cong \mathcal{P}^\bullet(\bot^{B^A}, Y)$ and $\mathcal{P}^\bullet(\bot^A, \bot^{A'}) \cong \mathcal{P}(A', A)$. □

Corollary 1. *Parameterized operators and co-parameterized operators of the same type are in bijective correspondence.*

The following bi-parameterization is important for control categories as semantic models of the $\lambda\mu$-calculus.

Definition 8. *A* ***bi-parameterized operator*** *of type* $(A_1, B_1) \times \cdots \times (A_n, B_n) \to (A, B)$ *on a control category* $\mathcal{P}$ *is a family of functions of the form*

$$\alpha_Y^X : \mathcal{P}_Y^X(A_1, B_1) \times \ldots \times \mathcal{P}_Y^X(A_n, B_n) \to \mathcal{P}_Y^X(A, B)$$

indexed by X *and* Y*, such that natural in* X *in* $\mathcal{P}$ *and natural in* Y *in* $\mathcal{P}^\bullet$*. A bi-parameterized operator is* ***strongly bi-parameterized*** *if it is compatible with currying:*

$$\ulcorner \alpha_Y^X(f_1, \ldots, f_n) \urcorner = \alpha_{Y^X}^1(\ulcorner f_1 \urcorner, \ldots, \ulcorner f_n \urcorner).$$

The following lemma is immediate from the compatibility.

Lemma 4. *Co-parameterized operators and strongly bi-parameterized operators of the same type are in bijective correspondence.*

5 Parameterized Fixed-Point Operators on Control Categories

5.1 Uniform Co-parameterized Fixed-Point Operators

In this section, general approach to parameterizations in the previous section is specialized to fixed-point operators on control categories.

First, we define uniform non-parameterized fixed-point operators and uniform co-parameterized ones, and investigate their bijective correspondence.

Definition 9. *A* ***fixed-point operator*** *on a control category* $\mathcal{P}$ *is a family of functions* $(-)^* : \mathcal{P}(A, A) \to \mathcal{P}(1, A)$ *such that* $f^* = f \circ f^*$ *hold. A fixed-point operator on* $\mathcal{P}$ *is* ***uniform*** *if* $h \circ f = g \circ h$ *implies* $h \circ f^* = g^*$ *for any morphisms* $f \in \mathcal{P}(A, A)$*,* $g \in \mathcal{P}(B, B)$ *and* $h \in \mathcal{P}^\bullet(A, B)$*.*

Definition 10. *A* ***co-parameterized fixed-point operator*** *on a control category* $\mathcal{P}$ *is a family of functions* $(-)^\dagger : \mathcal{P}(A, A \parr Y) \to \mathcal{P}(1, A \parr Y)$ *such that the following conditions hold:*

1. *(naturality)*
   $(A \parr h) \circ f^\dagger = ((A \parr h) \circ f)^\dagger$ *for any* $f \in \mathcal{P}(A, A \parr Y')$ *and* $h \in \mathcal{P}^\bullet(Y', Y)$

2. *(fixed-point property)*
   $f^\dagger = f \circ_Y f^\dagger$ *for any* $f \in \mathcal{P}(A, A \parr Y)$*, that is,*

$$f^\dagger = 1 \xrightarrow{f^\dagger} A \parr Y \xrightarrow{f \parr Y} A \parr Y \parr Y \xrightarrow{A \parr \nabla} A \parr Y$$

It is ***uniform*** *if* $h \circ_Y f = g \circ_Y h$ *implies* $h \circ_Y f^\dagger = g^\dagger$ *for any morphisms* $f \in \mathcal{P}_Y(A, A)$*,* $g \in \mathcal{P}_Y(B, B)$ *and* $h \in (\mathcal{P}_Y)^\bullet(A, B)$*.*

$$\begin{array}{ccccc} A & \xrightarrow{f} & A \parr Y & \xrightarrow{h \parr Y} & B \parr Y \parr Y \\ \downarrow{\scriptstyle h} & & & & \downarrow{\scriptstyle B \parr \nabla} \\ B \parr Y & \xrightarrow{g \parr Y} & B \parr Y \parr Y & \xrightarrow{B \parr \nabla} & B \parr Y \end{array}$$

$$\Rightarrow (B \parr \nabla) \circ (h \parr Y) \circ f^\dagger = g^\dagger$$

Remark 4. In other words, a (uniform) co-parameterized fixed-point operator on $\mathcal{P}$ is a family of (uniform) fixed-point operators on $\mathcal{P}_Y$ that are preserved by re-indexing functors.

Remark 5. The word 'operator' in this section has not the same meaning as that of the previous section. In Section 4, a co-parameterized operator is a family of functions $(-)^\dagger : \mathcal{P}(A, A \parr Y) \to \mathcal{P}(1, A \parr Y)$ with fixed A, only indexed by Y. In this section, however, a co-parameterized fixed-point operator is a family of functions $(-)^\dagger : \mathcal{P}(A, A \parr Y) \to \mathcal{P}(1, A \parr Y)$ indexed by both A and Y.

Proposition 4. *On a control category, uniform co-parameterized fixed-point operators are in bijective correspondence with uniform fixed-point operators.*

Proof. Given a uniform fixed-point operator $(-)^*$, we define a uniform co-parameterized operator $(-)^\dagger$ by $f^\dagger = ((A \parr \nabla) \circ (f \parr Y))^*$ for $f \in \mathcal{P}(A, A \parr Y)$. Though we can directly check the naturality, the fixed-point property and the uniformity of $(-)^\dagger$ through chasing many diagrams, we will give a simpler proof via adjunctions later.

Conversely, from a uniform co-parameterized fixed-point operator $(-)^\dagger$, we obtain an operator $(-)^*$ just by trivializing the parameter. In this case, it is obvious that $(-)^*$ is a uniform fixed-point operator.

It is sufficient for a bijective correspondence to show that

$$((A \parr \nabla) \circ (f \parr Y))^\dagger = f^\dagger : 1 \to A \parr Y$$

holds for a uniform co-parameterized fixed-point operator $(-)^\dagger$. Since $(f \parr Y) \circ_Y^X (\mathrm{w_l} \circ \mathrm{w_l}) = (\mathrm{w_l} \circ \mathrm{w_l}) \circ_Y^X f$ holds, by the uniformity we have

$$(f \parr Y)^\dagger = (\mathrm{w_l} \circ \mathrm{w_l}) \circ_Y^X f^\dagger : 1 \to A \parr Y \parr Y.$$

Applying $A \parr \nabla$ to the both sides of the equation, we get $((A \parr \nabla) \circ (f \parr Y))^\dagger = f^\dagger$. □

For the rest of the proof, we consider the weakening functor and the following results.

Proposition 5. *The weakening functor* $\mathrm{I}_Y : \mathcal{P} \to \mathcal{P}_Y$ *has a right adjoint* U_Y *given by* $\mathrm{U}_Y(A) = A \parr Y$ *and*

$$\mathrm{U}_Y(f) = A \parr Y \xrightarrow{f \parr Y} B \parr Y \parr Y \xrightarrow{B \parr \nabla} B \parr Y.$$

Moreover U_Y *preserves the focus.*

Corollary 2. *The weakening functor* $\mathrm{I}_Y^X : \mathcal{P} \to \mathcal{P}_Y^X$ *has a right adjoint* U_Y^X *given by* $\mathrm{U}_Y^X(A) = A \parr Y^X$ *and* $\mathrm{U}_Y^X(f) = \mathrm{U}_{Y^X}(\lfloor f \rfloor)$*. Moreover* U_Y^X *preserves the focus.*

This adjunction gives us a simpler proof of the construction and bijectivity between uniform fixed-point operators and co-parameterized ones.

As before, we define a uniform co-parameterized operator $(-)^\dagger$ from a uniform fixed-point operator $(-)^*$ by $f^\dagger = ((A \parr \nabla) \circ (f \parr Y))^*$. This $(-)^\dagger$ is just the same as $(\mathrm{U}_Y(f))^*$. We show that $(-)^\dagger$ is indeed a uniform co-parameterized fixed-point operator. Now we note that $\mathrm{U}_Y(g) \circ f = g \circ_Y f$.

- Naturality:
  For $f \in \mathcal{P}_Y(A, A')$ and $h \in \mathcal{P}^\bullet(Y', Y)$, $(A \parr h) \circ \mathrm{U}_Y(f) = \mathrm{U}_Y((A \parr Y) \circ f) \circ (A \parr h)$ holds. So, the uniformity of $(-)^*$ gives us the equation $(A \parr h) \circ f^\dagger = ((A \parr h) \circ f)^\dagger$.
- Fixed-point property:
  The co-parameterized fixed-point property trivially follows from the fixed-point property of $(-)^*$:

$$f^\dagger = (\mathrm{U}_Y(f))^* = \mathrm{U}_Y(f) \circ (\mathrm{U}_Y(f))^* = f \circ_Y f^\dagger$$

- Uniformity:
  We assume $h \circ_Y f = g \circ_Y h$ holds and h is focal. It follows that $\mathrm{U}_Y(h) \circ \mathrm{U}_Y(f) = \mathrm{U}_Y(g) \circ \mathrm{U}_Y(h)$ holds and $\mathrm{U}_Y(h)$ is focal. Therefore the uniformity of $(-)^*$ induces $\mathrm{U}_Y(h) \circ f^\dagger = g^\dagger$.

Similar result about the weakening functors and their adjunctions also help us to understand the relation between uniform fixed-point operators and uniform parameterized ones such as sketched in the introduction.

Definition 11. *A* ***parameterized fixed-point operator*** *on a control category* $\mathcal{P}$ *is a family of functions* $(-)^\# : \mathcal{P}(X \times A, A) \to \mathcal{P}(X, A)$ *such that the following conditions hold:*

1. *(naturality)*
   $f^\# \circ g = (f \circ (g \times A))^\#$ *for any* $f \in \mathcal{P}(X' \times A, A)$ *and* $g \in \mathcal{P}^\bullet(X, X')$
2. *(fixed-point property)*
   $f^\# = f \circ^X f^\#$ *for any* $f \in \mathcal{P}(X \times A, A)$.

It is ***uniform*** *if* $h \circ^X f = g \circ^X h$ *implies* $h \circ^X f^\# = g^\#$ *for any morphisms* $f \in \mathcal{P}^X(A, A)$*,* $g \in \mathcal{P}^X(B, B)$ *and* $h \in (\mathcal{P}^X)^\bullet(A, B)$*.*

Proposition 6. *The weakening functor* $\mathrm{I}^X : \mathcal{P} \to \mathcal{P}^X$ *has a right adjoint* U^X *given by* $\mathrm{U}^X(A) = A^X$ *and*

$$\mathrm{U}^X(f) = A^X \xrightarrow{\mathrm{cur}(f)^X} (B^X)^X \cong B^{X \times X} \xrightarrow{B^{\Delta}} B^X.$$

Moreover U^X *preserves the focus.*

We construct a uniform parameterized fixed-point operator $(-)^{\#}$ from a uniform fixed-point operator $(-)^*$ by $f^{\#} = \varepsilon \circ^X \mathrm{I}^X(\mathrm{U}^X(f)^*)$, where $\varepsilon \in \mathcal{P}(X \times A^X, A)$ is the counit of the adjunction. Bijectivity follows from the equation $f^{\#} = \varepsilon \circ^X \mathrm{I}^X((\mathrm{U}^X(f))^{\#})$, which is derived from the uniformity of $(-)^{\#}$ from the focality of ε and $\varepsilon \circ^X \mathrm{I}^X(\mathrm{U}^X(f)) = f \circ^X \varepsilon$.

This correspondence is generalized to a relation between uniform co-parameterized fixed-point operators and uniform bi-parameterized ones in the next subsection.

5.2 Uniform Bi-parameterized Fixed-Point Operators

Our goal is to show a bijective correspondence between uniform non-parameterized fixed-point operators and uniform bi-parameterized fixed-point operators introduced below.

Definition 12. *A* ***(strongly) bi-parameterized fixed-point operator*** *on a control category* $\mathcal{P}$ *is a family of functions* $(-)^{\ddagger} : \mathcal{P}(X \times A, A \parr Y) \to \mathcal{P}(X, A \parr Y)$ *such that the following conditions hold:*

1. *(naturality)*
   $(A \parr h) \circ f^{\ddagger} \circ g = ((A \parr h) \circ f \circ (g \times A))^{\ddagger}$ *for any* $f \in \mathcal{P}(X' \times A, A \parr Y')$, $g \in \mathcal{P}(X, X')$ *and* $h \in \mathcal{P}^{\bullet}(Y', Y)$.
2. *(compatibility with currying)*
   $\lceil f^{\ddagger} \rceil = \lceil f \rceil^{\ddagger}$ *for any* $f \in \mathcal{P}(X \times A, A \parr Y)$.
3. *(fixed-point property)*
   $f^{\ddagger} = f \circ^X_Y f^{\ddagger}$ *for any* $f \in \mathcal{P}(X \times A, A \parr Y)$*, that is,*

$$f^{\ddagger} = X \xrightarrow{\Delta; X \times f^{\ddagger}} X \times (A \parr Y) \xrightarrow{\mathrm{d}'} (X \times A) \parr Y \xrightarrow{f \parr Y; A \parr \nabla} A \parr Y$$

It is ***uniform*** *if* $h \circ^X_Y f = g \circ^X_Y h$ *implies* $h \circ^X_Y f^{\ddagger} = g^{\ddagger}$ *for any morphisms* $f \in \mathcal{P}^X_Y(A, A)$, $g \in \mathcal{P}^X_Y(B, B)$ *and* $h \in (\mathcal{P}^X_Y)^{\bullet}(A, B)$.

$$\begin{array}{ccccc}
X \times A & \xrightarrow{\langle \pi_1, f \rangle} & X \times (A \parr Y) & \xrightarrow{\mathrm{d}'_{X,A,Y}} & (X \times A) \parr Y \\
\downarrow{\scriptstyle \langle \pi_1, h \rangle} & & & & \downarrow{\scriptstyle (B \parr \nabla) \circ (h \parr Y)} \\
X \times (B \parr Y) & \xrightarrow{\mathrm{d}'_{X,B,Y}} & (X \times B) \parr Y & \xrightarrow{(B \parr \nabla) \circ (g \parr Y)} & B \parr Y
\end{array}$$

$$\Rightarrow (B \parr \nabla) \circ (h \parr Y) \circ \mathrm{d}'_{X,B,Y} \circ (X \times f^{\ddagger}) \circ \Delta = g^{\ddagger}$$

Remark 6. In other words, a (uniform) bi-parameterized fixed-point operator on $\mathcal{P}$ is a family of (uniform) fixed-point operators on $\mathcal{P}_Y^X$ that are preserved by re-indexing functors and compatible with currying.

Proposition 7. *On a control category, uniform bi-parameterized fixed-point operators are in bijective correspondence with uniform co-parameterized fixed-point operators.*

Proof. Given a uniform co-parameterized fixed-point operator$(-)^\dagger$, we define a uniform bi-parameterized fixed-point operator $(-)^\ddagger$ by $f^\ddagger = \lfloor \lceil f \rceil^\dagger \rfloor$.

$(-)^\ddagger$ satisfies the naturality, the compatibility with $\lfloor - \rfloor$, the bi-parameterized fixed-point property and the uniformity.

– Naturality:

$$\begin{array}{ll} ((B \parr h) \circ f \circ (g \times A))^\ddagger & \\ = \lfloor \lceil (B \parr h) \circ f \circ (g \times A) \rceil^\dagger \rfloor & \text{definition of } (-)^\ddagger \\ = \lfloor ((B \parr h^g) \circ \lceil f \rceil)^\dagger \rfloor & \text{naturality of } \lceil - \rceil \\ = \lfloor (B \parr h^g) \circ \lceil f \rceil^\dagger \rfloor & \text{naturality of } (-)^\dagger \\ = (B \parr h) \circ \lfloor \lceil f \rceil^\dagger \rfloor \circ g & \text{naturality of } \lfloor - \rfloor \\ = (B \parr h) \circ f^\ddagger \circ g & \text{definition of } (-)^\ddagger \end{array}$$

– Compatibility with $\lceil - \rceil$:

$$\lceil f^\ddagger \rceil = \lceil \lfloor \lceil f \rceil^\dagger \rfloor \rceil = \lceil f \rceil^\dagger = \lceil f \rceil^\ddagger.$$

– Fixed-point property:

$$f^\ddagger = \lfloor \lceil f \rceil^\dagger \rfloor = \lfloor \lceil f \rceil \circ_{Y^X} \lceil f \rceil^\dagger \rfloor = \lfloor \lceil f \rceil \rfloor \circ_Y^X \lfloor \lceil f \rceil^\dagger \rfloor = f \circ_Y^X f^\ddagger$$

follows from Lemma 2.

– Uniformity:
$g \circ_Y^X h = h \circ_Y^X f$ implies $\lceil g \rceil \circ_{Y^X} \lceil h \rceil = \lceil h \rceil \circ_{Y^X} \lceil f \rceil$ by Lemma 2. Lemma 3 means that if h is focal, so is $\lceil h \rceil$. Hence by the uniformity of $(-)^\dagger$, $\lceil g \rceil^\dagger = \lceil h \rceil \circ_{Y^X} \lceil f \rceil^\dagger$ for $f \in \mathcal{P}_Y^X(A, A)$, $g \in \mathcal{P}_Y^X(B, B)$ and $h \in (\mathcal{P}_Y^X)^\bullet(A, B)$ such that $g \circ_Y^X h = h \circ_Y^X f$. $\lceil g \rceil^\dagger = \lceil h \rceil \circ_{Y^X} \lceil f \rceil^\dagger$ implies $g^\ddagger = h \circ_Y^X f^\ddagger$.

For bijectivity, it is sufficient to show

$$\lfloor \lceil f \rceil^\ddagger \rfloor = f^\ddagger : X \to A \parr Y$$

for a uniform bi-parameterized fixed-point operator $(-)^\ddagger$, but that is equivalent to $(-)^\ddagger$'s compatibility with $\lceil - \rceil$. □

The following theorem is deduced from Proposition 4 and Proposition 7.

Theorem 3. *On a control category, uniform bi-parameterized fixed-point operators are in bijective correspondence with uniform fixed-point operators.*

6 Fixed-Point Operator in $\lambda\mu$-Calculus

In the previous work [6], we have extended the call-by-name $\lambda\mu$-calculus with a uniform fixed-point operator. In this section, we recall its definition (including the syntactic notion of focality which is used for determining the uniformity) and refine the completeness theorem of [6] using the results from Section 5.

Definition 13. *In a call-by-name $\lambda\mu$-theory [11], $\Gamma \vdash H : A \to B \mid \Delta$ is* ***focal*** *if*

$$\Gamma, k : \neg\neg A \vdash H(\mu\alpha^A.\, k(\lambda x^A.[\alpha]\, x)) = \mu\beta^B.\, k(\lambda x^A.[\beta]\, Hx) : B \mid \Delta$$

holds.

This syntactic notion of focus precisely corresponds to the semantic focality.

Proposition 8. *Given a control category $\mathcal{P}$, $h \in \mathcal{P}^X_Y(A, B)$ is focal if and only if the term $x : X \vdash \lambda a^A.\mu\beta^B.[\beta, \gamma]h\langle x, a\rangle : A \to B \mid \gamma : Y$ in the internal language is focal in the sense of Definition 13.*

Definition 14. *A type-indexed family of closed terms $\{\mathtt{fix}^A : (A \to A) \to A\}$ in a $\lambda\mu$-theory is called a* ***uniform fixed-point operator*** *if the following conditions hold:*

1. *(fixed-point property)*
   $\mathtt{fix}^A\, F = F(\mathtt{fix}^A\, F)$ *holds for any term $F : A \to A$.*
2. *(uniformity)*
   For any terms $F : A \to A$, $G : B \to B$ and focal $H : A \to B$, $H \circ F = G \circ H$ implies $H(\mathtt{fix}^A\, F) = \mathtt{fix}^B\, G$.

Indeed, this axiomatization is sound and complete for control categories with uniform bi-parameterized fixed-point operators [6]. However, since we know that uniform bi-parameterized fixed-point operators are reducible to non-parameterized ones, we have the following theorem, which strengthens and simplifies the completeness result under the uniformity conditions.

Theorem 4. *Control categories with uniform fixed-point operators provide a sound and complete class of models of the $\lambda\mu$-calculus extended with a uniform fixed-point operator.*

7 Conclusion

In this paper, we have introduced polynomial categories for control categories, which are required to deal with not only free variables but also free names, and shown their functional completeness a la Lambek [7]. Based on those consideration, we defined strongly bi-parameterized operators. Bi-parameterized operators have more complicated forms than standard parameterized ones since they have both parameterization and co-parameterization. Co-parameterization is for free names while usual parameterization is for free variables. Our strongly bi-parameterized operators can be reduced to co-parameterized operators by the compatibility with currying.

General approach to parameterizations is specialized to parameterizations on fixed-point operators. In this paper, we introduced uniform co-parameterized fixed-point operators and bi-parameterized ones. Our bi-parameterized fixed-point operators are in bijective correspondence with co-parameterized ones. As we have shown in the paper, the uniformity conditions imply the bijective correspondence between bi-parameterized fixed-point operators and non-parameterized ones.

The technical novelty of this approach is that we closely look at co-parameterization and its interaction with the focus of a control category. We believe our observations are useful not merely for fixed-point operators but also for other parameterized constructs on control categories and the $\lambda\mu$-calculus, in particular, for modeling parameterized data-type constructions as in [5].

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