On the availability of quandle theory to classifying links up to link-homotopy

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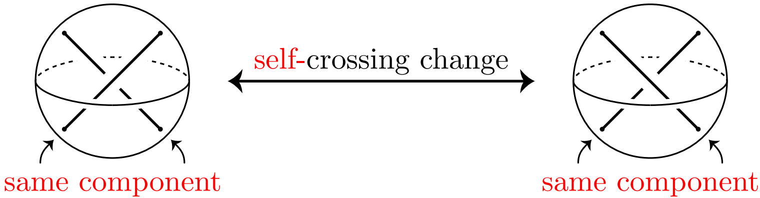
1. Introduction

link-homotopy is ...

ambient isotopy $+$



Rough history

- J. Milnor (1954, 1957)
  - Defined the notion of link-homotopy
  - Defined Milnor invariants ($\overline{\mu}$ invariants)
  - Classified 3-component links up to link-homotopy completely
- J. P. Levine (1988)
  - Enhanced Milnor invariants
  - Classified 4-component links up to link-homotopy completely
- N. Habegger and X. S. Lin (1990)
  - Gave a necessary and sufficient condition for link-homotopic
  - Gave an algorithm judging two links are link-homotopic or not

Motivation

"Classify link-homotopy classes by invariants"

$$\text{numerical invariants} \Rightarrow \begin{cases} \text{easy to compute} \\ \text{easy to compare} \end{cases}$$

This talk

- We have a lot of numerical invariants via quandle theory
- How powerful are them?

Talk plan

1. Introduction
2. Review of quandle theory
3. Numerical link-homotopy invariants
4. Latent ability of the numerical invariants

2. Review of quandle theory

Definition (quandle)

X : set ($\neq \emptyset$)

$* : X \times X \to X$: binary operation

$(X, *)$: quandle

$\overset{\text{def}}{\Leftrightarrow}$ $*$ satisfies the following axioms:

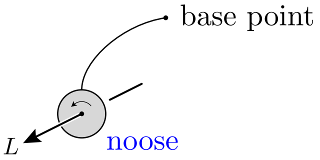
(Q1) ${}^\forall x \in X$, $x * x = x$

(Q2) ${}^\forall x \in X$, $*x : X \to X$ ($\bullet \mapsto \bullet * x$) is bijective

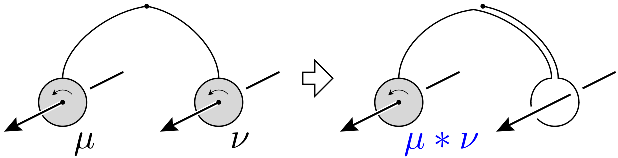
(Q3) ${}^\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$

Definition (knot quandle)

L : link



$Q(L) := \{\text{nooses of } L\}/\text{homotopy}.$



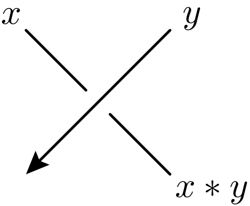
$(Q(L), *)$: knot quandle of L (D. Joyce 1982, S. V. Matveev 1982).

Remark

X : (finite) quandle

$\sharp\{\varphi : Q(L) \to X \ \ \text{homomorphisms}\}$ gives rise to an invariant.

$\varphi : Q(L) \to X$ homo. $\longleftrightarrow$



$(x, y, x * y \in X)$

X-coloring of a diagram of L

X : quandle

$C_n^R(X) := \mathrm{span}_{\mathbb{Z}} X^n$.

Define $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ by

$$\partial_n(x_1, \cdots, x_n) = \sum_{i=2}^{n} (-1)^i \{(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) - (x_1 * x_i, \cdots, x_{i-1} * x_i, x_{i+1}, \cdots, x_n)\}.$$

$\rightsquigarrow (C_n^R(X), \partial_n)$: chain complex.

$C_n^D(X) := \mathrm{span}_{\mathbb{Z}}\{(x_1, \cdots, x_n) \in X^n \mid {}^{\exists} i \text{ s.t. } x_i = x_{i+1}\} \subset C_n^R(X)$.

$\rightsquigarrow \partial_n(C_n^D(X)) \subset C_{n-1}^D(X)$.

$\rightsquigarrow C_n^Q(X) := C_n^R(X)/C_n^D(X)$, $(C_n^Q(X), \partial_n)$: chain complex.

Definition (quandle homology/cohomology)

X : quandle

A : abelian group

$H_n^Q(X;A)$: n-th homology group of
the chain complex $(C_n^Q(X) \otimes A, \partial_n \otimes \mathrm{id})$,

$H_Q^n(X;A)$: n-th cohomology group of
the cochain complex $(\mathrm{Hom}(C_n^Q(X), A), \mathrm{Hom}(\partial_n, \mathrm{id}))$.

Notations

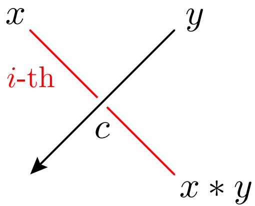
$H_n^Q(X) := H_n^Q(X;\mathbb{Z})$,

$H_Q^n(X) := H_Q^n(X;\mathbb{Z})$.

$L = K_1 \cup \cdots \cup K_n$: n-component link

D : diagram of L

$$C(D;i) := \sum_{c} \mathrm{sign}(c) \cdot (x,y) \in C_2^Q(Q(L));$$



Theorem (J. S. Carter et al. 2003)

- $C(D;i) \in Z_2^Q(Q(L))$
- $[C(D;i)] \in H_2^Q(Q(L))$ does NOT depend on the choice of D

$[K_i] := [C(D;i)] \in H_2^Q(Q(L))$: i-th fundamental class of $Q(L)$

$L = K_1 \cup \cdots \cup K_n$: n-component link (again)
X : quandle

$\varphi : Q(L) \to X$ homo.
$\rightsquigarrow \varphi_\sharp : H_2^Q(Q(L)) \to H_2^Q(X)$ $([K_i] \mapsto \varphi_\sharp([K_i]))$.

A : abelian group

Associated with $\theta \in Z_Q^2(X; A)$, we have the multiset

$$\{\langle \theta, \varphi_\sharp([K_i]) \rangle \in A \mid \varphi : Q(L) \to X \ \text{ homo.}\},$$

called the i-th quandle cocycle invariant.

3. Numerical link-homotopy invariants

Trouble

Knot quandle is NOT invariant under self-crossing change.

$\rightsquigarrow$ Knot quandle is NOT invariant under link-homotopy.

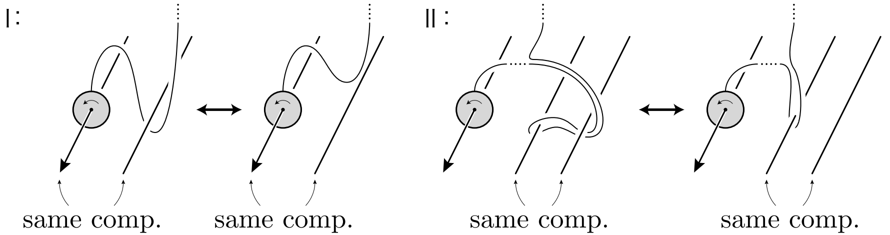
$\rightsquigarrow$ Quandle cocycle invariant is NOT invariant under link-homotopy.

Solution

Take a quotient of knot quandle
to be invariant under self-crossing change.

L : link

$\color{blue}{RQ(L)} := Q(L)/$ the following moves;



$(RQ(L), *)$: reduced knot quandle of L (J. R. Hughes 2011 (, I.)).

Theorem (J. R. Hughes 2011 (, I.))

Reduced knot quandle is invariant under link-homotopy.

X : quandle

$\mathrm{Aut}(X) := \{\varphi : X \to X \ \ \text{auto.}\}$: automorphism group of X

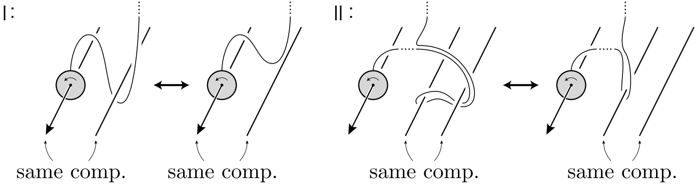
$\mathrm{Inn}(X) := \langle *\, x : X \to X \ \ (x \in X) \rangle \lhd \mathrm{Aut}(X)$

: inner automorphism group of X

$\varphi \in \mathrm{Inn}(X)$: inner automorphism of X

$RQ(L)$: reduced knot quandle

- I-move $\leftrightarrow$ $x * \varphi(x) = x$ $(\varphi \in \mathrm{Inn})$
- II-move $\leftrightarrow$ $x * (y * \varphi(y)) = x * y$ $(\varphi \in \mathrm{Inn})$



Definition (quasi-trivial quandle)

X : quasi-trivial quandle

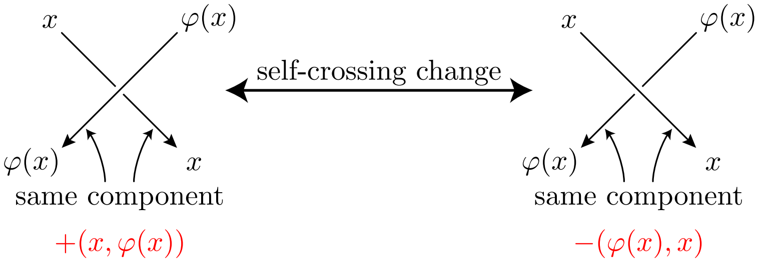
$\overset{\text{def}}{\Leftrightarrow}$ $^{\forall}x \in X$, $^{\forall}\varphi \in \mathrm{Inn}(X)$, $x * \varphi(x) = x$.

<u>Remarks</u>

- $RQ(L)$ is quasi-trivial.
  $\rightsquigarrow$ $^{\exists}\varphi : RQ(L) \to X$ non-trivial homo. $\Rightarrow$ X : quasi-trivial
- $$\begin{array}{ccc} Q(L) & & \\ \downarrow{\scriptstyle \pi} & \circlearrowleft & \searrow \\ RQ(L) & \xrightarrow{\varphi} & X \end{array} \quad (X : \text{quasi-trivial})$$
- $\sharp\{\varphi \mid Q(L) \xrightarrow{\pi} RQ(L) \xrightarrow{\varphi} X \text{ homo.}\}$
  is invariant under link-homotopy (X : quasi-trivial)

Trouble

Fundamental class of reduced knot quandle is NOT well-defined.



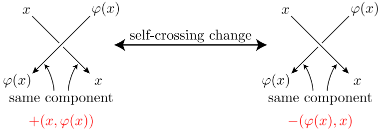
$\rightsquigarrow$ Quandle cocycle invariant is NOT well-defined.

Solution

Modify the definition of quandle homology slightly.

X : quasi-trivial quandle

A : abelian group



$C_n^{D,qt}(X) := C_n^D(X) \cup S \subset C_n^R(X)$;

$$S = \{q(x_1, \varphi(x_1), x_3, \cdots, x_n) + q(\varphi(x_1), x_1, x_3, \cdots, x_n) \mid q \in \mathbb{Z},\ \varphi \in \mathrm{Inn}(X)\}.$$

$\rightsquigarrow \partial_n(C_n^{D,qt}(X)) \subset C_{n-1}^{D,qt}(X)$.

$\rightsquigarrow C_n^{Q,qt}(X) := C_n^R(X)/C_n^{D,qt}(X)$, $(C_n^{Q,qt}(X), \partial_n)$: chain complex.

$\rightsquigarrow H_n^{Q,qt}(X; A)$, $H^n_{Q,qt}(X; A)$.

Proposition

$L = K_1 \cup \cdots \cup K_n$: n-component link

The i-th fundamental class $[K_i]^{qt} \in H_2^{Q,qt}(Q(L))$ is well-defined.

X : quasi-trivial quandle

A : abelian group

$\theta \in Z^2_{Q,qt}(X;A)$

The i-th (modified) quandle cocycle invariant

$$\{\langle \theta, \varphi_\sharp([K_i]^{qt}) \rangle \in A \mid Q(L) \xrightarrow{\pi} RQ(L) \xrightarrow{\varphi} X \ \text{ homo.}\}$$

is invariant under link-homotopy.

4. Latent ability of the numerical invariants

Constituents of the numerical invariant

- reduced knot quandle $RQ(L)$
- fundamental classes $[K_i]^{qt} \in H_2^{Q,qt}(RQ(L))$
- quasi-trivial quandle X
- homomorphism $RQ(L) \to X$
- 2-cocycle $\theta \in H^2_{Q,qt}(X; A)$

Conjecture

$L = K_1 \cup \cdots \cup K_n$, $L' = K'_1 \cup \cdots \cup K'_n$: n-component links

L is link-homotopic to L'

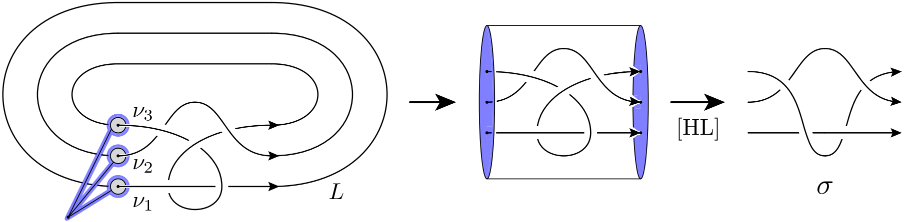
$\Leftrightarrow$ ${}^{\exists}\varphi : RQ(L) \xrightarrow{\cong} RQ(L')$ s.t. $\varphi_\sharp([K_i]^{qt}) = [K'_i]^{qt}$ $(1 \leq i \leq n)$.

$L = K_1 \cup \cdots \cup K_n$: n-component link

$x_i \in RQ(L)$: classes of nooses intersecting with i-th comp.

ν_i : nooses representing x_i

We have a pure braid σ:



Remark

σ is NOT unique, because the choices of $\nu_1, \cdots, \nu_n$ are not unique.

Notation

$\sigma^{(i)} := \sigma - (i\text{-th comp.})$.

Theorem

$L = K_1 \cup \cdots \cup K_n$, $L' = K'_1 \cup \cdots \cup K'_n$: n-component links

Assume that

$^{\exists}\varphi : RQ(L) \xrightarrow{\cong} RQ(L')$ s.t. $\varphi_{\sharp}([K_i]^{qt}) = [K'_i]^{qt} \quad (1 \leq i \leq n)$.

Choose and fix $x_1, \cdots, x_n \in RQ(L)$.

$^{\exists}\sigma$, σ' : pure braids for $(L, \{x_i\})$ and $(L', \{\varphi(x_i)\})$ respectively
s.t. $\sigma^{(i)}$ is link-homotopic to $\sigma'^{(i)}$ for some i

$\Rightarrow$ L is link-homotopic to L'.

Corollary

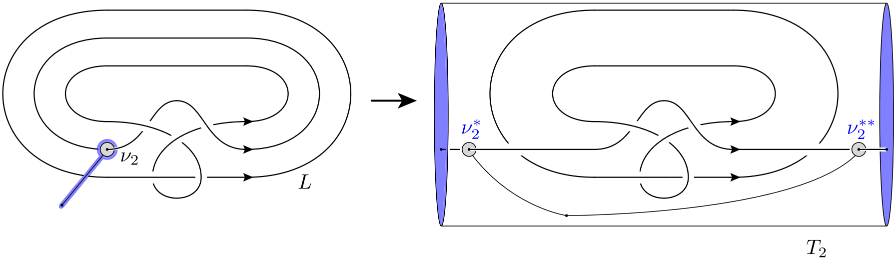
The conjecture is true for links with ≤ 3 components.

$L = K_1 \cup \cdots \cup K_n$: n-component link

$x_i \in RQ(L)$: class of nooses intersecting with i-th comp.

ν_i : noose representing x_i

We have a $(1,1)$-tangle T_i:



Remark

T_i is NOT unique, because the choice of ν_i is not unique.

Proposition

$L = K_1 \cup \cdots \cup K_n$, $L' = K'_1 \cup \cdots \cup K'_n$: n-component links

Assume that

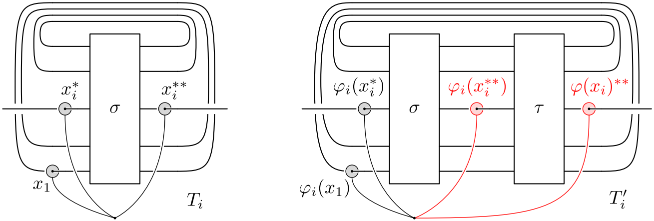
$$^{\exists}\varphi : RQ(L) \xrightarrow{\cong} RQ(L') \text{ s.t. } \varphi_\sharp([K_i]^{qt}) = [K'_i]^{qt} \quad (1 \leq i \leq n).$$

Choose and fix $x_1, \cdots, x_n \in RQ(L)$.

$^{\forall}T_i$, T'_i : $(1,1)$-tangles for (L, x_i) and $(L, \varphi(x_i))$ respectively,

$^{\exists}\varphi_i : RQ(T_i) \xrightarrow{\cong} RQ(T'_i)$ s.t.

$$\varphi_i(x_j) = \varphi(x_j) \ (i \neq j), \ \varphi_i(x_i^*) = \varphi(x_i)^*, \ \varphi_i(x_i^{**}) = \varphi(x_i)^{**}.$$



Corollary

$L = K_1 \cup \cdots \cup K_n$, $L' = K'_1 \cup \cdots \cup K'_n$: n-component links

Assume that

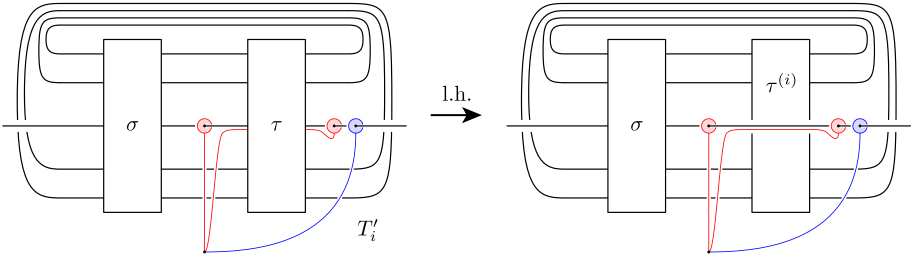
$$^{\exists}\varphi : RQ(L) \xrightarrow{\cong} RQ(L') \text{ s.t. } \varphi_\sharp([K_i]^{qt}) = [K'_i]^{qt} \quad (1 \leq i \leq n).$$

Choose and fix $x_1, \cdots, x_n \in RQ(L)$.

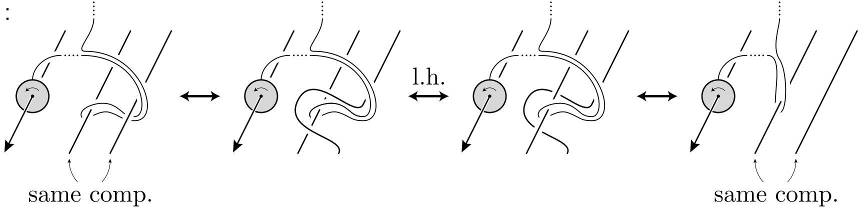
σ, $\sigma' = \sigma \cdot \tau$: pure braids for $(L, \{x_i\})$ and $(L', \{\varphi(x_i)\})$ resp.

T_i, T'_i : $(1,1)$-tangles obtained from σ and σ' by closing the ends of each string other than i-th

T'_i is link-homotopic to the $(1,1)$-tangle obtained from $\sigma \cdot (\tau^{(i)} \cup (\text{trivial } i\text{-th string}))$ by closing the ends of each string other than i-th.



II :



Theorem (again)

$L = K_1 \cup \cdots \cup K_n$, $L' = K'_1 \cup \cdots \cup K'_n$: n-component links

Assume that

$^{\exists}\varphi : RQ(L) \xrightarrow{\cong} RQ(L')$ s.t. $\varphi_\sharp([K_i]^{qt}) = [K'_i]^{qt} \quad (1 \leq i \leq n)$.

Choose and fix $x_1, \cdots, x_n \in RQ(L)$.

$^{\exists}\sigma$, σ' : pure braids for $(L, \{x_i\})$ and $(L', \{\varphi(x_i)\})$ respectively s.t. $\sigma^{(i)}$ is link-homotopic to ${\sigma'}^{(i)}$ for some i

$\Rightarrow L$ is link-homotopic to L'.

X : quasi-trivial quandle

$\widetilde{C}_n^{D,qt}(X) := C_n^D(X) \cup \widetilde{S} \subset C_n^R(X)$;

$$\widetilde{S} = \mathrm{span}_{\mathbb{Z}}\{(x_1, \varphi(x_1), x_3, \cdots, x_n) \in X^n \mid \varphi \in \mathrm{Inn}(X)\}.$$

$\rightsquigarrow \partial_n(\widetilde{C}_n^{D,qt}(X)) \subset \widetilde{C}_{n-1}^{D,qt}(X)$.

$\rightsquigarrow \widetilde{C}_n^{Q,qt}(X) := C_n^R(X)/\widetilde{C}_n^{D,qt}(X)$, $(\widetilde{C}_n^{Q,qt}(X), \partial_n)$: chain complex.

$\rightsquigarrow \widetilde{H}_n^{Q,qt}(X; A)$, $\widetilde{H}^n_{Q,qt}(X; A)$.

Remarks

- $C_n^{D,qt}(X) \subset \widetilde{C}_n^{D,qt}(X)$.
  $\rightsquigarrow \widetilde{H}_n^{Q,qt}(X; A) \triangleleft H_n^{Q,qt}(X; A)$
- $[K_i]^{qt} \in \widetilde{H}_n^{Q,qt}(X; A)$

Theorem

$L = K_1 \cup \cdots \cup K_n$: n-component link

$K_1, \ldots, K_m$: non-trivial up to link-homotopy

$K_{m+1}, \ldots, K_n$: trivial up to link-homotopy

$\widetilde{H}_2^{Q,qt}(RQ(L)) = \langle [K_1]^{qt} \rangle \oplus \cdots \oplus \langle [K_m]^{qt} \rangle$.

$\rightsquigarrow$ The numerical invariant knows

which components are trivial up to link-homotopy.

Remark

$|\langle [K_i]^{qt} \rangle| = (\text{order of } i\text{-th longitude}) \leq \infty \quad (1 \leq i \leq m)$.