Edge state integrals on shaped triangulations

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Motivation: quantum Chern–Simons theory with a non-compact gauge group

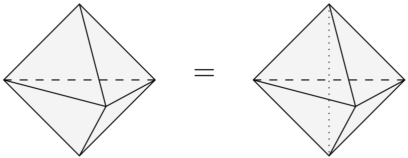
Given a Lie group G, a 3-manifold M.
Gauge fields: G-connections $A \in \mathcal{A} = \Omega^1(M, Lie\, G)$.
Chern–Simons action functional
$CS_M(A) = \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$.
Group of gauge transformations $\mathcal{G} = \mathcal{C}^\infty(M, G)$,

$$\mathcal{A} \times \mathcal{G} \to \mathcal{A}, \quad (A, g) \mapsto A^g := g^{-1} A g + g^{-1} dg$$

Phase space = space of flat connections = $\hom(\pi_1(M), G)/G$.
Partition function: $Z_\hbar(M) = \int_{\mathcal{A}/\mathcal{G}} e^{\frac{i}{\hbar} CS_M(A)} \mathcal{D}A$.
Problem: give a mathematically rigorous definition of $Z_\hbar(M)$.
Applications: quantum $2+1$ gravity, invariants of 3-manifolds.
Previous works: Witten, Hikami, Dijkgraaf, Fuji, Manabe, Dimofte, Gukov, Lenells, Zagier, Gaiotto, Andersen, K.

Combinatorics of triangulated 3-manifolds

Topological invariance and the $2-3$ Pachner move



Ponzano–Regge model of $2+1$-dimensional quantum gravity: states on edges (finite-dimensional representations of $sl(2)$) and weights on tetrahedra ($6j$-symbols).
Turaev–Viro model: $sl(2)$ replaced by $U_q(sl(2))$ with $q = \sqrt[N]{1}$.
Next steps: ∞-dimensional representations with generic q.
Need for special functions.

Faddeev's quantum dilogarithm

For $\hbar \in \mathbb{R}_{>0}$, Faddeev's quantum dilogarithm defined by

$$\Phi_\hbar(z) = \exp\left(\int_{\mathbb{R}+i\epsilon} \frac{e^{-i2xz}}{4\sinh(xb)\sinh(xb^{-1})x} dx \right)$$

in the strip $|\Im z| < \frac{1}{2\sqrt{\hbar}}$, where $\hbar = (b + b^{-1})^{-2}$, and extended to the whole $\mathbb{C}$ through the functional equations

$$\Phi_\hbar(z - ib^{\pm 1}/2) = (1 + e^{2\pi b^{\pm 1} z})\Phi_\hbar(z + ib^{\pm 1}/2)$$

Choose $\Re b > 0$ and $\Im b \geq 0$. If $\Im b > 0$ (i.e. $\hbar > 1/4$), then

$$\Phi_\hbar(z) = \frac{(-qe^{2\pi bz}; q^2)_\infty}{(-\bar{q}e^{2\pi b^{-1}z}; \bar{q}^2)_\infty}, \quad q := e^{i\pi b^2}, \ \bar{q} := e^{-i\pi b^{-2}},$$

with the notation $(x; y)_\infty := (1 - x)(1 - xy)(1 - xy^2)\dots$

Analytical properties

Zeros and poles:

$$(\Phi_\hbar(z))^{\pm 1} = 0 \;\Leftrightarrow\; z = \mp\left(\frac{i}{2\sqrt{\hbar}} + mib + nib^{-1}\right), \;\; m, n \in \mathbb{Z}_{\geq 0}$$

Behavior at infinity:

$$\Phi_\hbar(z)\Big|_{|z|\to\infty} \approx \begin{cases} 1 & |\arg z| > \frac{\pi}{2} + \arg b \\ \zeta_{inv}^{-1} e^{i\pi z^2} & |\arg z| < \frac{\pi}{2} - \arg b \\ \frac{(\bar{q}^2;\bar{q}^2)_\infty}{\Theta(ib^{-1}z;-b^{-2})} & |\arg z - \frac{\pi}{2}| < \arg b \\ \frac{\Theta(ibz;b^2)}{(q^2;q^2)_\infty} & |\arg z + \frac{\pi}{2}| < \arg b \end{cases}$$

where $\zeta_{inv} := e^{\pi i(2+\hbar^{-1})/12}$, $\Theta(z;\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i \tau n^2 + 2\pi i z n}$, $\Im\tau > 0$.
Inversion relation: $\Phi_\hbar(z)\Phi_\hbar(-z) = \zeta_{inv}^{-1} e^{i\pi z^2}$.
Complex conjugation: $\overline{\Phi_\hbar(z)}\Phi_\hbar(\bar{z}) = 1$.

Quantum five term identity

Heisenberg's (normalized) selfadjoint operators in $L^2(\mathbb{R})$

$$\mathbf{p}f(x) := \frac{1}{2\pi i}f'(x), \quad \mathbf{q}f(x) := xf(x)$$

Quantum five term identity for unitary operators

$$\Phi_\hbar(\mathbf{p})\Phi_\hbar(\mathbf{q}) = \Phi_\hbar(\mathbf{q})\Phi_\hbar(\mathbf{p}+\mathbf{q})\Phi_\hbar(\mathbf{p})$$

Equivalent integral formula

$$\int_{\mathbb{R}} \frac{\Phi_\hbar(x+u)}{\Phi_\hbar\left(x-\frac{i}{2\sqrt{\hbar}}+i0\right)} e^{-2\pi i w x}\, dx = \zeta_o \frac{\Phi_\hbar(u)\,\Phi_\hbar\left(\frac{i}{2\sqrt{\hbar}}-w\right)}{\Phi_\hbar(u-w)}$$

where $\zeta_o := \exp\left(\frac{\pi i}{12}\left(1+\frac{1}{\hbar}\right)\right)$, and $0 < \Im w < \Im u < \frac{1}{2\sqrt{\hbar}}$.
In particular,

$$\int_{\mathbb{R}+i\epsilon} \Phi_\hbar(x) e^{-2\pi i w x}\, dx = \zeta_o e^{-\pi i w^2} \Phi_\hbar\left(\frac{i}{2\sqrt{\hbar}}-w\right)$$

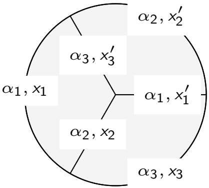
Labeled tetrahedra

Notation for CW-complexes:

- $\Delta_i(X)$ = the set of i-dimensional simplices of X
- $\Delta_{i,j}(X) = \{(a,b)|\ a \in \Delta_i(X),\ \ b \in \Delta_j(a)\}$

Two types of edge labelings:

- State variables $x\colon \Delta_1(X) \to \mathbb{R}$;
- Shape variables $\alpha\colon \Delta_{3,1}(X) \to]0,\pi[,\ \alpha(t,e) = \alpha(t,e^{\mathrm{op}})$, $\sum_e \alpha(t,e) = 2\pi$.



$$\alpha_1 + \alpha_2 + \alpha_3 = \pi$$

Neumann–Zagier symplectic structure: $\omega_{NZ} = d\alpha_1 \wedge d\alpha_2$

Weight functions

A tetrahedron T in state x and with shape α:

$$W_\hbar(T, x, \alpha) = \prod_{j=1}^{3} \Psi_\hbar \left(x_{j+1} + x'_{j+1} - x_{j-1} - x'_{j-1} + \frac{i}{\sqrt{\hbar}} \left(\frac{1}{2} - \frac{\alpha_j}{\pi} \right) \right)$$

where

$$\Psi_\hbar(x) = \frac{\Phi_\hbar(x)}{\Phi_\hbar(0)} e^{-i\pi x^2/2}, \quad \Psi_\hbar(x)\Psi_\hbar(-x) = 1$$

A triangulation X in state x and with shape α:

$$W_\hbar(X, x, \alpha) = \prod_{T \in \Delta_3(X)} W_\hbar(T, x, \alpha)$$

Partition function

Denote

$$\mathbb{R}^{\Delta_j(X)} = \{f : \Delta_j(X) \to \mathbb{R}\}, \quad j \in \{0,1\}.$$

State gauge transformations

$$\mathbb{R}^{\Delta_1(X)} \times \mathbb{R}^{\Delta_0(X)} \to \mathbb{R}^{\Delta_1(X)}, \quad (x, g) \mapsto x^g,$$

$$x^g(e) = x(e) + g(v_1) + g(v_2), \quad \partial e = \{v_1, v_2\}.$$

State gauge invariance of the weight function:

$$W_\hbar(X, x, \alpha) = W_\hbar(X, x^g, \alpha), \quad \forall g \in \mathbb{R}^{\Delta_0(X)}.$$

The partition function (the case $\partial X = \emptyset$):

$$Z_\hbar(X, \alpha) = \int_{\mathbb{R}^{\Delta_1(X)}/\mathbb{R}^{\Delta_0(X)}} W_\hbar(X, x, \alpha) dx$$

Shape gauge invariance

Let X be a closed ($\partial X = \emptyset$) shaped triangulated oriented pseudo 3-manifold (all tetrahedra are oriented, all gluings respect the orientations).

Shape gauge growp action in the space of shapes is generated by total dihedral angles around edges acting through the Neumann–Zagier Poisson bracket.

A gauge reduced shape is the Hamiltonian reduction of a shape over fixed values of the total dihedral angles.

An edge is balanced if the total dihedral angle around it is 2π. A shape with all edges balanced is known as an angle structure (Casson, Lackenby, Rivin).

Invariants of 3-manifolds

Theorem

For a triangulated closed oriented pseudo 3-manifold X with shape α, the partition function $Z_\hbar(X,\alpha)$ is well defined (the integral is absolutely convergent), and it

- *depends on only the gauge reduced class of α;*
- *is invariant under shaped $3-2$ Pachner moves along balanced edges.*

Remark

This construction can be extended to manifolds with boundary eventually giving rize to a TQFT.

One vertex H-triangulations of knots in 3-manifolds

Let $K \subset M$ be a knot in a closed oriented compact 3-manifold. Let X be a one vertex H-triangulation of the pair (M, K), i.e. a one vertex triangulation of M where K is represented by an edge e_0 of X.

Fix another edge e_1, and for any small $\epsilon > 0$, consider a shape structure α_ϵ such that the total dihedral angle is ϵ around e_0, $2\pi - \epsilon$ around e_1, and 2π around any other edge.

Theorem

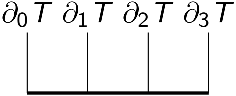
The limit

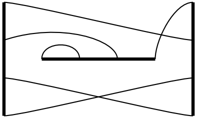
$$\tilde{Z}_\hbar(X) := \lim_{\epsilon \to 0} Z_\hbar(X, \alpha_\epsilon) \left| \Phi_\hbar \left(\frac{\pi - \epsilon}{2\pi i \sqrt{\hbar}} \right) \right|^2$$

is finite and is invariant under shaped $3 - 2$ *Pachner moves of triangulated pairs* (M, K).

An H-triangulation of the pair $(S^3, 4_1)$ (figure-eight knot)

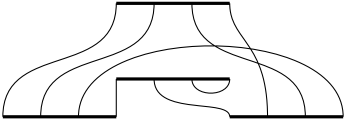
Graphical notation: $T \quad = $





$$\tilde{Z}_\hbar(S^3, 4_1) = 2\left|\int_{\mathbb{R}-i\epsilon} \frac{e^{i\pi z^2}}{\Phi_\hbar(z)^2} dz\right|^2$$

An H-triangulation of the pair $(S^3, 5_2)$



$$\tilde{Z}_\hbar(S^3, 5_2) = 2\left|\int_{\mathbb{R}-i\epsilon} \frac{e^{i\pi z^2}}{\Phi_\hbar(z)^3} dz\right|^2$$

A conjectural relation to the Teichmüller TQFT

The Teichmüller TQFT (constructed in: J.E. Andersen–RK, arXiv:1109.6295, arXiv:1305.4291)

Conjecture

For any closed 1-vertex triangulation of a closed 3-manifold X with shape α, one has

$$Z_\hbar(X,\alpha) = 2\left|Z_\hbar^{(\mathrm{Teichm.})}(X,\alpha)\right|^2$$