Yokota type invariants derived from Costantino-Murakami's invariants

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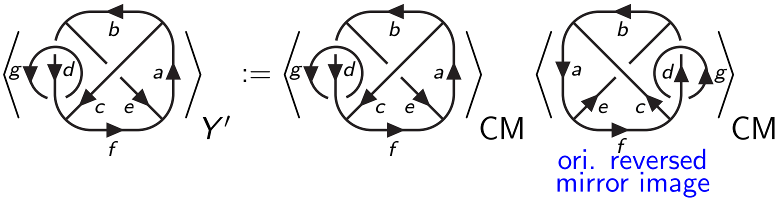
Intelligence of Low-dimensional Topology

Introduction

We define Yokota type invariants for oriented graphs from Costantino-Murakami's invariants (CM invariants).



Let Γ be a plane graph (one component). We conjecture that for appropriate sequence of colors the next equation holds.

$$\frac{\pi}{2} \lim_{n \to \infty} \frac{\log \langle \Gamma \rangle_{Y'}}{n} = \mathrm{Vol}(S_\Gamma),$$

where S_Γ is a hyperbolic polyhedron bounded by Γ whose dihedral angles are corresponds to colors.

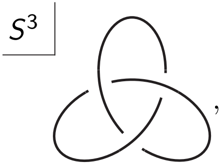
Contents

Knot and spatial graph

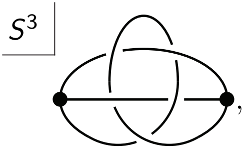
Knots and spatial graphs

Definition 1.1

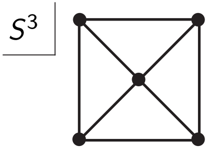
A knot is an embedding of a circle into the three-sphere.
A spatial graph (a knotted graph) is embedding of a graph (V, E) into the three-sphere. Where V is a set of vertices and E is a set of edges.
A plane graph is a spatial graph which can be embedded to the two-sphere.



knot



spatial graph

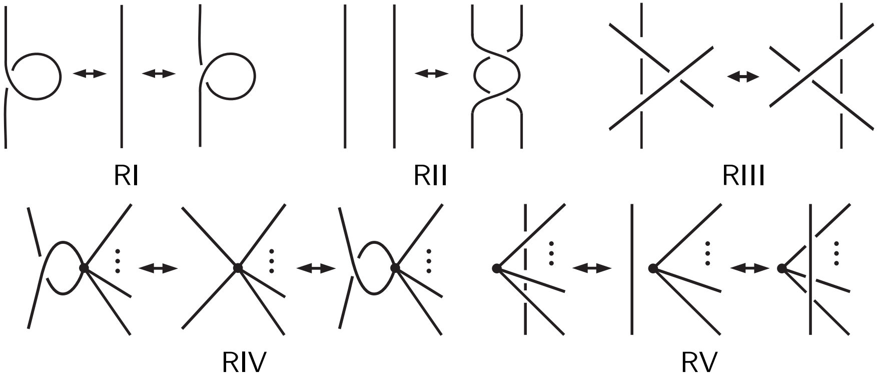


plane graph

We treat them through diagrams derived from regular projections to the two-sphere.

Reidemeister moves

There are 5 local moves called Reidemeister moves for knot and spatial graph diagrams.



Theorem 1.2

Two diagrams are transformed to each other by a sequence of Reidemeister moves. $\Leftrightarrow$ *Two diagrams represent the same knot or spatial graph.*

Volume conjecture

The N-th colored Jones polynomial $J_N(K;q)$ for a knot K is defined by an irreducible N-dimensional representation of the quantum group $\mathcal{U}_q(sl_2)$.

Conjecture 1 ([Kashaev], [H. Mrakami-J. Murakami])

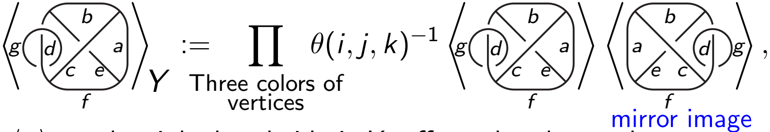
Let K be a hyperbolic knot in the three-sphere. Then

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(K;\exp(2\pi\sqrt{-1}/N))|}{N} = \mathrm{Vol}(S^3 \setminus K)$$

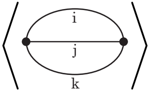
where Vol is the hyperbolic volume.

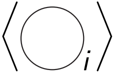
Yokota's invariants

Let Γ be a trivalent spatial graph. For each edge of Γ we add a natural number called color which corresponds to the dimension of the representation of $\mathcal{U}_q(sl_2)$. Yokota's invariants $\langle\cdot\rangle_Y$ are defined through a colored diagram of Γ by the next relation.

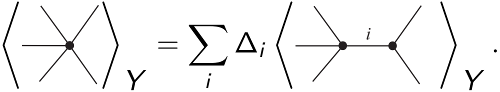


where $\langle\cdot\rangle$ on the right-hand side is Kauffman bracket and

$\theta(i,j,k) :=$ 

. We put $\Delta_i :=$ 

. Yokota's invariants are generalized for more than 3-valent vertex by the next relation.



Yokota's invariants are also generalized for 1 and 2-valent vertex by other relations.

Costantino-Murakami's invariants

Costantino-Murakami's invariants

This section follows the paper [Costantino-Murakami].

F. Costantino and J. Murakami defined invariants for framed oriented trivalent graphs (i.e. invariants for RII, RIII and RV moves) through non-integral representations of $\mathcal{U}_q(sl_2)$ where q is a root of unity.

We prepare notations. Fix a natural number n, $\xi_n := \exp(\frac{\pi\sqrt{-1}}{n})$.

$$\{a\} = \xi_n^a - \xi_n^{-a} \ (a \in \mathbb{C}), \quad [a] = \frac{\{a\}}{\{1\}}, \quad \{k\}! = \prod_{j=1}^{k} \{j\} \ (k \in \mathbb{N})$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \prod_{j=0}^{a-b-1} \frac{\{a-j\}}{\{a-b-j\}} \ (a, b \in \mathbb{C} \text{ s.t. } a-b \in \{0, 1, \ldots, n-1\})$$

$\mathcal{U}_q(sl_2)$ is a Hopf algebra as follows.

Generator: E, F, K, K^{-1}

Relation:

$$[E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \ \ KE = qEK, \ \ KF = q^{-1}FK, \ \ KK^{-1} = K^{-1}K = 1.$$

Structure of a Hopf algebra:

$$\Delta(E) = E \otimes K + K^{-1} \otimes E, \ \ \Delta(F) = F \otimes K + K^{-1} \otimes F, \ \ \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1},$$

$$S(E) = -qE, \ \ S(F) = -q^{-1}F, \ \ S(K) = K^{-1},$$

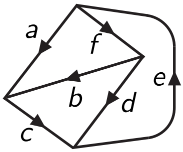
$$\epsilon(E) = \epsilon(F) = 0, \ \ \epsilon(K) = 1.$$

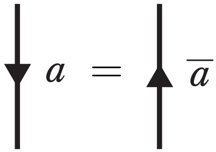
Representation of $\mathcal{U}_q(sl_2)$

For each complex number $a \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$, there is a simple representation of $\mathcal{U}_{\xi_n}(sl_2)$ for n-dimensional vector space V^a whose basis is $\{e_0^a, e_1^a, \cdots, e_{n-1}^a\}$. The actions are given by

$$E(e_j^a) = [j]e_{j-1}^a, \quad F(e_j^a) = [2a-j]e_{j+1}^a, \quad K(e_j^a) = \xi_n^{a-j} e_j^a \quad (e_{-1}^a = e_n^a = 0)$$

For each edge of a framed spatial graph Γ, we add a complex number $a \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$ corresponds to the representation for V^a. Due to the isomorphism $(V^a)^* \cong V^{n-1-a}$, we can consider a colored edge and $n-1-a$ colored opposite direction edge are equal. We put $\overline{a} = n-1-a$.





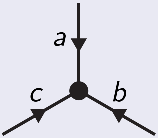
Admissible condition

For non-half-integers a, b, if $a+b$ is nor half-integer, there is a decomposition $V^a \otimes V^b = \bigoplus_c V^c$ here $a+b-c \in \{0, 1, \cdots, n-1\}$.

admissible conditions

If three colors a, b, c of edges at a vertex satisfy the next condition, we call the triple (a, b, c) is admissible.

$$a+b+c \in \{n-1, n, \ldots, 2n-2\},$$

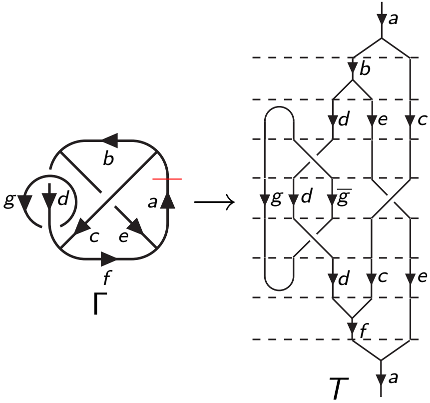


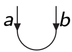
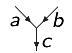
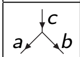
here the orientations of the three edges are all toward the vertex.

If three colors of a vertex is admissible, we can give a representation canonically at the vertex.
From now on, unless otherwise noted, colors in summations $\sum$ move all admissible colors.

(1, 1)-Tangle

To define Costantino-Murakami's invariants, we cut an edge of the admissibly colored framed spatial graph Γ and make $(1, 1)$-tangle diagram T. Then we slice T so that each piece has only one singular point (maximal, minimal, crossing point or vertex). The slices are regarded as maps from bottom to top as follows.



	$id_a : V^a \to V^a$
	${}^b_a R : V^a \otimes V^b \to V^b \otimes V^a$
	${}^b_a(R^{-1}) : V^a \otimes V^b \to V^b \otimes V^a$
	$\cap_{a,b} : V^a \otimes V^b \to \mathbb{C}$
	$\cup_{a,b} : \mathbb{C} \to V^a \otimes V^b$
	$Y_c^{a,b} : V^c \to V^a \otimes V^b$
	$Y_{a,b}^c : V^a \otimes V^b \to V^c$

Maps

- $${}^{b}_{a}R(e^a_i \otimes e^b_j) = \sum_m \{m\}! \xi_n^{2(a-i)(b-j)-m(a-b-i+j)-\frac{m(m+1)}{2}} \begin{bmatrix} i \\ i-m \end{bmatrix} \begin{bmatrix} 2b-j \\ 2b-j-m \end{bmatrix} e^b_{j+m} \otimes e^a_{i-m}$$

  where $m \in [0, \min(i, n-1-j)] \cap \mathbb{N}$

- $$\cup_{a,b} = \delta_{b,n-1-a} \sum_{i=0}^{n-1} \xi_n^{-(a-i)(n-1)} e^a_i \otimes e^b_{n-1-i}$$

- $$\cap_{a,b}(e^a_i, e^b_j) = \delta_{b,n-1-a} \delta_{i,n-1-j} \xi_n^{-(a-i)(n-1)}$$

- $$Y^{a,b}_c(e^c_k) = \sum_{\substack{i+j-k \\ =a+b-c}} C^{a,b,c}_{i,j,k} e^a_i \otimes e^b_j$$

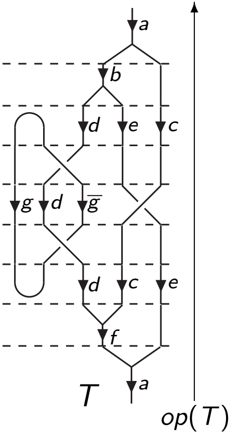
- $$Y^c_{a,b}(e^a_i \otimes e^b_j) = \sum_{\substack{i+j-k \\ =a+b-c}} C^{n-1-b,n-1-a,n-1-c}_{n-1-j,n-1-i,n-1-k} e^c_k$$

  where

$$C^{a,b,c}_{i,j,k} = \sqrt{-1}^{c-a-b} (-1)^{j-k} \xi_n^{\frac{j(2b-j+1)-i(2a-i+1)}{2}} \begin{bmatrix} 2c \\ 2c-k \end{bmatrix}^{-1} \begin{bmatrix} 2c \\ a+b+c-(n-1) \end{bmatrix}$$

$$\sum_{z+w=k} (-1)^z \xi_n^{\frac{(2z-k)(2c-k+1)}{2}} \begin{bmatrix} a+b-c \\ i-z \end{bmatrix} \begin{bmatrix} 2a-i+z \\ a2-i \end{bmatrix} \begin{bmatrix} 2b-j+w \\ 2b-j \end{bmatrix}$$

Definition of Costantino-Murakami's invariants



We have a morphism $op(T) : V^a \to V^a$ by composing the maps derived from slices of T. By Schur's lemma, $op(T)$ is a scalar multiplied identity $\lambda(T) id_a$. Then Costantino-Murakami's invariant $\langle \cdot \rangle_{CM}$ is defined as

$$\langle \Gamma \rangle_{CM} := \lambda(T) \begin{bmatrix} 2a+n \\ 2a+1 \end{bmatrix}^{-1}.$$

The value is independent of the choice of the edge that was cut.

Remark

1. For a half-integer $a \in \frac{1}{2}\mathbb{Z}$,

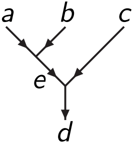
$$\begin{bmatrix} 2a+n \\ 2a+1 \end{bmatrix} = 0.$$

Hence for half-integer colors, Costantino-Murakami's invariants may become infinity.

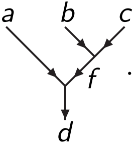
2. If graphs are restricted to links, Costantino-Murakami's invariants correspond to Akutsu-Deguchi-Ohtsuki (colored Alexander) invariants. Akutsu-Deguchi-Ohtsuki invariants have a property of volume conjecture for cone manifolds whose singular sets are the links [J. Murakami].

6j-symbols

The 6j-symbols are the coefficients of the next relation.



$$= \sum_f \left\{ \begin{matrix} a & b & e \\ c & d & f \end{matrix} \right\}$$



.

They satisfy the following relations.

[Orthogonal relation]

$$\sum_f \left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\} \left\{ \begin{matrix} d & b & f \\ a & c & g \end{matrix} \right\} = \delta_{eg}.$$

[Pentagon relation]

$$\sum_h \left\{ \begin{matrix} a & b & f \\ g & c & h \end{matrix} \right\} \left\{ \begin{matrix} a & h & g \\ e & d & i \end{matrix} \right\} \left\{ \begin{matrix} b & c & h \\ d & i & j \end{matrix} \right\} = \left\{ \begin{matrix} f & c & g \\ d & e & j \end{matrix} \right\} \left\{ \begin{matrix} a & b & f \\ j & e & i \end{matrix} \right\}.$$

6j-symbols

For $a, b, c, d, e, f \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$ and $a + b - c, a + f - e, b + d - f, d + c - e \in \mathbb{Z}$, the 6$j$-symbols are calculated by the next equation.

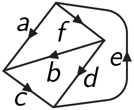
$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\} =$$

$$(-1)^{n-1+B_{afe}} \left[\begin{array}{c} 2f+n \\ 2f+1 \end{array} \right]^{-1} \frac{\{B_{dce}\}!\{B_{abc}\}!}{\{B_{bdf}\}!\{B_{afe}\}!} \left[\begin{array}{c} 2c \\ A_{abc}+1-n \end{array} \right] \left[\begin{array}{c} 2c \\ B_{ced} \end{array} \right]^{-1}$$

$$\times \sum_{z=s}^{S} (-1)^z \left[\begin{array}{c} A_{afe}+1 \\ 2e+z+1 \end{array} \right] \left[\begin{array}{c} B_{aef}+z \\ B_{aef} \end{array} \right] \left[\begin{array}{c} B_{bfd}+B_{dce}-z \\ B_{bfd} \end{array} \right] \left[\begin{array}{c} B_{dec}+z \\ B_{dfb} \end{array} \right]$$

where $s = \max(0, -B_{bdf} + B_{dce})$, $S = \min(B_{dce}, B_{afe})$, $A_{xyz} = x + y + z$, $B_{xyz} = x + y - z$.

Value of tetrahedron



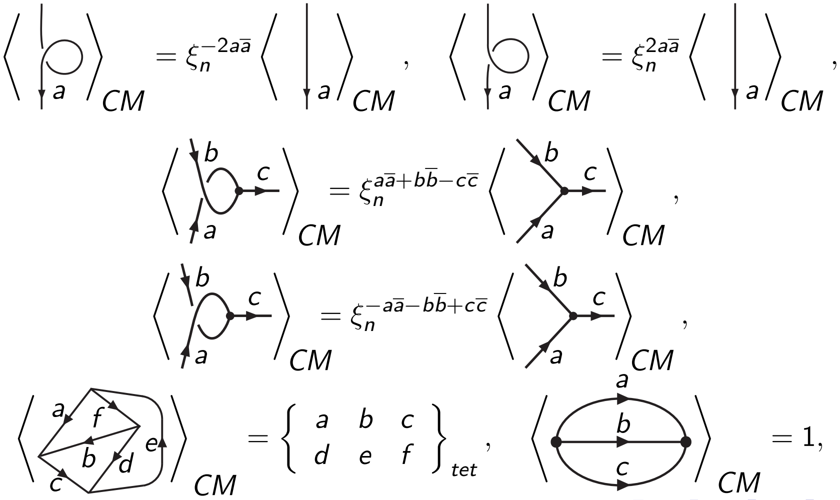
$$\Bigg\langle \quad \Bigg\rangle_{CM} = \begin{bmatrix} 2f+n \\ 2f+1 \end{bmatrix} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} =: \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_{tet}.$$

It is proved that this value is well-defined for half-integer colors.
From the symmetry of the tetrahedron for rotations, we have

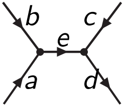
$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_{tet} = \begin{Bmatrix} \overline{d} & \overline{b} & \overline{f} \\ \overline{a} & \overline{e} & \overline{c} \end{Bmatrix}_{tet}.$$

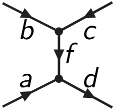
Relations

We can calculate Costantino-Murakami's invariants axiomatically by using following relations.

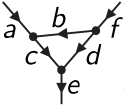


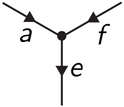
Relations

$\Big\langle$ 

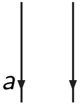
$\Big\rangle_{CM} = \sum_f \left\{ \begin{matrix} a & b & e \\ c & d & f \end{matrix} \right\} \Big\langle$ 

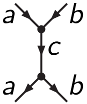
$\Big\rangle_{CM}$,

$\Big\langle$ 

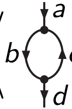
$\Big\rangle_{CM} = \left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}_{tet} \Big\langle$ 

$\Big\rangle_{CM}$,

$\Big\langle$ 

$\Big\rangle_{CM} = \sum_c \begin{bmatrix} 2c+n \\ 2c+1 \end{bmatrix}^{-1} \Big\langle$ 

$\Big\rangle_{CM}$,

$\Big\langle$ 

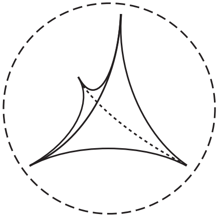
$\Big\rangle_{CM} = \delta_{ad} \begin{bmatrix} 2a+n \\ 2a+1 \end{bmatrix} \Big\langle$ 

$\Big\rangle_{CM}$.

Ideal tetrahedra

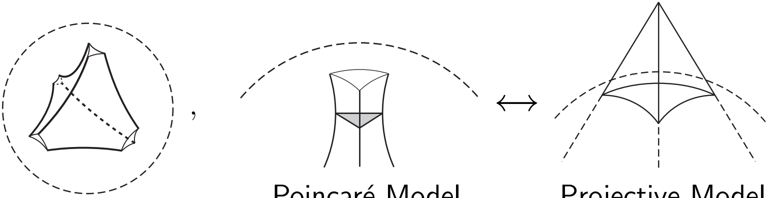
Shapes of hyperbolic tetrahedra are determined by their 6 dihedral angles.

A vertex at an infinity point of hyperbolic space is called ideal vertex. The tetrahedra whose 4 vertices are all ideal are called ideal tetrahedra. Two dihedral angles of the opposite edges of ideal tetrahedra are equal. Hence a shape of ideal tetrahedron is determined by three dihedral angle α, β, γ $(\alpha + \beta + \gamma = \pi)$.



Truncated tetrahedra

We can consider a vertex outside the infinity points of hyperbolic space. This vertex appears in the projective model of hyperbolic space.
For the three faces around the vertex, there is a geodesic surface which is perpendicular to them. Cutting the tetrahedron at each vertex by the surface, we have a finite polyhedron. This polyhedron is called truncated tetrahedron.



Poincaré Model

Projective Model

Property of volume conjecture

Theorem 2.1

Let S be a hyperbolic tetrahedron and Γ be a graph made of edges of S. $\theta_a, \cdots, \theta_f$ are dihedral angles of S. Let $a_n, \cdots, f_n$ be sequences of integral colors such that $\lim_{n\to\infty} \frac{2\pi a_n}{n} = \pi - \theta_a, \cdots, \lim_{n\to\infty} \frac{2\pi f_n}{n} = \pi - \theta_f$.
If S is ideal (i.e. dihedral angles of opposite edges are equal),

$$\begin{aligned} \mathrm{Vol}(S) &= \lim_{n\to\infty} \frac{\pi}{n} \log\left((-1)^{n-1} \left\{ \begin{array}{ccc} a_n & b_n & c_n \\ a_n & b_n & c_n \end{array} \right\}_{tet} \right) \\ &= \lim_{n\to\infty} \frac{\pi}{n} \log\left((-1)^{n-1} \left\{ \begin{array}{ccc} \overline{a_n} & \overline{b_n} & \overline{c_n} \\ \overline{a_n} & \overline{b_n} & \overline{c_n} \end{array} \right\}_{tet} \right). \end{aligned}$$

If S is a truncated tetrahedron,

$$\mathrm{Vol}(S) = \lim_{n\to\infty} \frac{\pi}{2n} \log\left(\left\{ \begin{array}{ccc} a_n & b_n & c_n \\ d_n & e_n & f_n \end{array} \right\}_{tet} \left\{ \begin{array}{ccc} \overline{a_n} & \overline{b_n} & \overline{c_n} \\ \overline{d_n} & \overline{e_n} & \overline{f_n} \end{array} \right\}_{tet} \right). \quad (1)$$

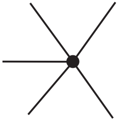
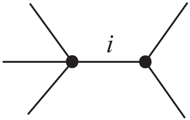
Yokota type invariants

Definition

Let Γ be admissibly colored oriented trivalent graph and D be its diagram. Yokota type invariant $\langle \cdot \rangle_{Y'}$ is defined from Costantino-Murakami's invariants by the next relation.

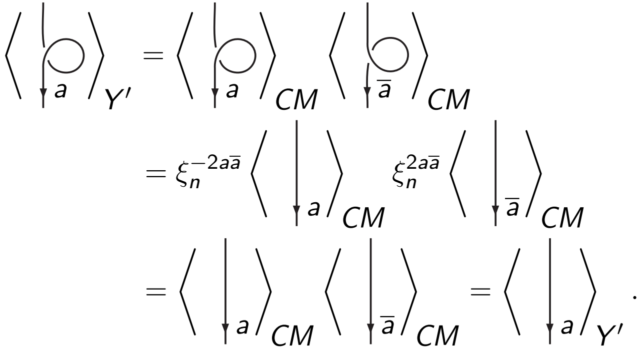
$$\langle \Gamma \rangle_{Y'} = \langle D \rangle_{CM} \left\langle \overline{D}^{\,r} \right\rangle_{CM},$$

where $\overline{\cdot}$ means a mirror image, $\cdot^{\,r}$ means reversing orientations. For more than 3-valent vertices, we reduce the valence to three by the next relation.

$\Big\langle$  $\Big\rangle_{Y'} = \sum_i \begin{bmatrix} 2i+n \\ 2i+1 \end{bmatrix}^{-1} \Big\langle$  $\Big\rangle_{Y'}$,

where we omit colors and orientations of surrounding edges. We assume they have the same colors and orientations in the both sides. The orientation of the i colored edge is arbitrary.

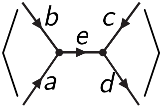
The invariance of Yokota type invariants for RII, RIII and RV are from that of Costantino-Murakami's invariants.
Invariance for RI:



The invariance for RIV is shown in a similar way.

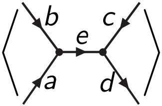
We show that the values of Yokota type invariants are independent of the way to expand an edge at the more than 3-valent vertices. It is enough to see the next equations. (cf. [Yetter])

$$\sum_e \begin{bmatrix} 2e+n \\ 2e+1 \end{bmatrix}^{-1}$$

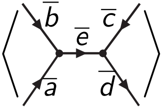


Y'

$$= \sum_e \begin{bmatrix} 2e+n \\ 2e+1 \end{bmatrix}^{-1}$$

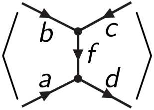


CM



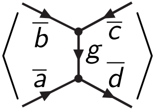
CM

$$= \sum_e \begin{bmatrix} 2e+n \\ 2e+1 \end{bmatrix}^{-1} \sum_f \begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}$$

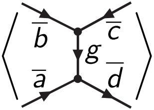


CM

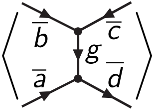
$$\sum_g \begin{Bmatrix} \bar{a} & \bar{b} & \bar{e} \\ \bar{c} & \bar{d} & g \end{Bmatrix}$$



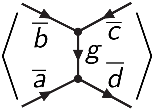
CM

- $\sum_g \begin{Bmatrix} \bar{a} & \bar{b} & \bar{e} \\ \bar{c} & \bar{d} & g \end{Bmatrix}$ 

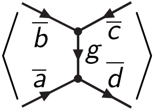
$_{CM}$

$= \sum_g \begin{bmatrix} 2g+n \\ 2g+1 \end{bmatrix}^{-1} \begin{Bmatrix} \bar{a} & \bar{b} & \bar{e} \\ \bar{c} & \bar{d} & g \end{Bmatrix}_{tet}$ 

$_{CM}$

$= \sum_g \begin{bmatrix} 2g+n \\ 2g+1 \end{bmatrix}^{-1} \begin{Bmatrix} c & b & \bar{g} \\ a & d & e \end{Bmatrix}_{tet}$ 

$_{CM}$

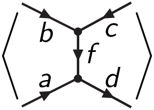
$= \sum_g \begin{bmatrix} 2g+n \\ 2g+1 \end{bmatrix}^{-1} \begin{bmatrix} 2e+n \\ 2e+1 \end{bmatrix} \begin{Bmatrix} c & b & \bar{g} \\ a & d & e \end{Bmatrix}$ 

$_{CM}$.

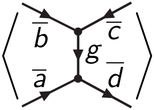
- $\begin{bmatrix} 2\bar{f}+n \\ 2\bar{f}+1 \end{bmatrix} = \cdots = \begin{bmatrix} 2f+n \\ 2f+1 \end{bmatrix}.$

Extension to more than 3-valent vertices

$$(2\text{ prev. slide}) = \sum_f \sum_g \begin{bmatrix} 2g+n \\ 2g+1 \end{bmatrix}^{-1} \sum_e \begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} \begin{Bmatrix} c & b & \overline{g} \\ a & d & e \end{Bmatrix}$$

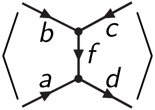


CM

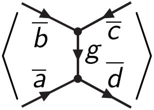


CM

$$= \sum_f \sum_g \begin{bmatrix} 2g+n \\ 2g+1 \end{bmatrix}^{-1} \delta_{f\overline{g}}$$

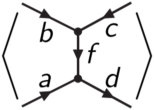


CM

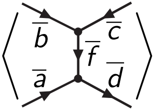


CM

$$= \sum_f \begin{bmatrix} 2\overline{f}+n \\ 2\overline{f}+1 \end{bmatrix}^{-1}$$

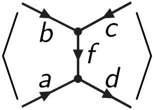


CM



CM

$$= \sum_f \begin{bmatrix} 2f+n \\ 2f+1 \end{bmatrix}^{-1}$$



Y'

Volume conjecture for polyhedra

In Theorem 2.1, the value inside $\log(\cdot)$ of Equation (1) is the value of Yokota type invariants for tetrahedron graphs. Using the Yokota type invariants, we conjecture the extension of Theorem 2.1.

Conjecture 2

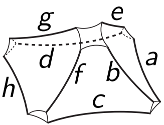
Let Γ be a plane graph and S_Γ be a hyperbolic convex polyhedron which is bounded by Γ. If sequences of integral colors of Γ are taken as in Theorem 2.1 for corresponding dihedral angles of S_Γ,

$$\mathrm{Vol}(S_\Gamma) = \lim_{n\to\infty} \frac{\pi}{2n} \log\left(\langle\Gamma\rangle_{Y'}\right).$$

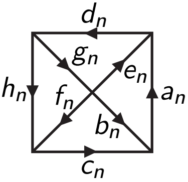
Examples

We did algebraic and numerical calculations for following two cases.

$\Gamma_{1,n}$:



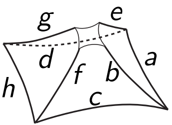
$\leftrightarrow$



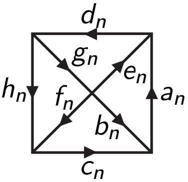
$a, c, d, h : \pi/4$
$b, e, f, g : \pi/3$

$$\begin{cases} a_n = 3n/8\,(+\varepsilon) & b_n = n/3\ \ (+2\varepsilon) \\ c_n = 3n/8\,(+3\varepsilon) & d_n = 3n/8\,(+4\varepsilon) \\ e_n = n/3\ \ (+3\varepsilon) & f_n = n/3\ \ (-6\varepsilon) \\ g_n = n/3\ \ (+5\varepsilon) & h_n = 3n/8\,(+9\varepsilon) \end{cases}$$

$\Gamma_{2,n}$:



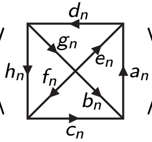
$\leftrightarrow$

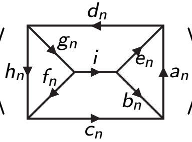


a, c, d, h
b, e, f, g $: \pi/3$

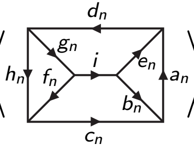
$$\begin{cases} a_n = n/3\,(+\varepsilon) & b_n = n/3\,(+2\varepsilon) \\ c_n = n/3\,(+3\varepsilon) & d_n = n/3\,(+4\varepsilon) \\ e_n = n/3\,(+3\varepsilon) & f_n = n/3\,(-6\varepsilon) \\ g_n = n/3\,(+5\varepsilon) & h_n = n/3\,(+9\varepsilon) \end{cases}$$

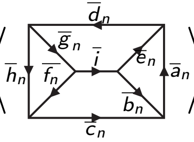
Square pyramids

$\Bigg\langle$ 

$\Bigg\rangle_{Y'} = \sum_i \begin{bmatrix} 2i+n \\ 2i+1 \end{bmatrix}^{-1} \Bigg\langle$ 

$\Bigg\rangle_{Y'}$

$= \sum_i \begin{bmatrix} 2i+n \\ 2i+1 \end{bmatrix}^{-1} \Bigg\langle$ 

$\Bigg\rangle_{CM} \Bigg\langle$ 

$\Bigg\rangle_{CM}$

$$= \sum_i \begin{bmatrix} 2i+n \\ 2i+1 \end{bmatrix}^{-1} \begin{Bmatrix} a_n & e_n & d_n \\ i & c_n & b_n \end{Bmatrix}_{tet} \begin{Bmatrix} d_n & g_n & h_n \\ f_n & c_n & i \end{Bmatrix}_{tet}$$

$$\times \begin{Bmatrix} \overline{a_n} & \overline{e_n} & \overline{d_n} \\ \overline{i} & \overline{c_n} & \overline{b_n} \end{Bmatrix}_{tet} \begin{Bmatrix} \overline{d_n} & \overline{g_n} & \overline{h_n} \\ \overline{f_n} & \overline{c_n} & \overline{i} \end{Bmatrix}_{tet} .$$

Regularity of formula for square pyramids

We calculated the above formula as a rational function of q, reduced the numerator and the denominator by common factors then substituted $q = \xi_n$.

$\Gamma_1 : n = 24, \{a, b, c, d, e, f, g, h\} = \{9, 8, 9, 9, 8, 8, 8, 9\}$

$$\frac{27025539214627761048737732625739438682 88}{414445402563377 5}$$

$\Gamma_2 : n = 12, \{a, b, c, d, e, f, g, h\} = \{4, 4, 4, 4, 4, 4, 4, 4\}$

$$\frac{94785522391588664840 0}{20660630690 7}$$

$\Gamma_2 : n = 24, \{a, b, c, d, e, f, g, h\} = \{8, 8, 8, 8, 8, 8, 8, 8\}$

$$\frac{18417276716781939060567652343662582870272 00}{19743796020815679008287}$$

Numerical calculations

Table: Numerical calculations at $\varepsilon = 0.0000001$

n	$\pi/2n * \log(\lvert\langle \Gamma_{1,n}\rangle_{Y'}\rvert)$
24	3.440464669
48	3.653713460
72	3.741391100
120	3.824413802
240	3.900859202
600	3.959111190
900	3.986845579
1200	3.983212953
Vol.	4.01536

n	$\pi/2n * \log(\lvert\langle \Gamma_{2,n}\rangle_{Y'}\rvert)$
24	2.597872961
48	2.603015626
72	2.594719877
120	2.581962148
240	2.566523650
600	2.552634909
900	2.548604997
1200	2.546357950
Vol.	2.53735

Problem

Problem

Prove Conjecture 2 for some polyhedra which have more than 3-valent vertices.