# Hyperbolic knots with left-orderable, non–$L$–space surgeries

Kimihiko Motegi

*joint with*

Masakazu Teragaito

# Left-orderability of groups

A nontrivial group $G$ is left-orderable (LO) if $\exists$ strict total order $<$ on $G$ which is left invariant:

$$g < h \quad \Rightarrow \quad fg < fh \quad \text{for } \forall f \in G.$$

• $\mathbb{Z}$ is LO

$m < n \quad \Rightarrow \quad k + m < k + n \quad$ for $\forall k \in \mathbb{Z}$.

• $G$ is LO $\quad \Rightarrow \quad G$ has no torsion element

If $g \neq 1$, say $1 < g$, then $g < g^2$ by left-invariance.

So $1 < g^2$ by transitivity. Inductively, $1 < g^n$ for all $n > 0$,

- $G$ is LO and $H \subset G \quad \Rightarrow \quad H$ is LO.

- $Homeo_+(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R};$ orientation preserving homeomorphism$\}$ is LO. (Conrad)

- $G$: a countable group
  $G$ is LO $\quad \Leftrightarrow \quad G \hookrightarrow Homeo_+(\mathbb{R})$

- Burns-Hale criterion:
  $G$ is LO $\quad \Leftrightarrow \quad$ every nontrivial finitely generated subgroup $H \subset G$ has LO quotient.

## Theorem (Boyer-Rolfsen-Wiest)

*$M$: a compact, orientable, irreducible 3–manifold.*
*Then $\pi_1(M)$ is LO $\Leftrightarrow$ $\pi_1(M)$ has an LO quotient.*

Sketch Proof.

$\Rightarrow$ is obvious.

$\Leftarrow$ Apply Burns-Hale criterion.

Take any $H \subset \pi_1(M)$. $H \neq \{1\}$: finitely generated

$H$ infinite index: $M_H \to M$ covering with $\pi_1(M_H) \cong H$
Let $C \subset M_H$ be the Scott core; $C$ compact & $\pi_1(C) \cong \pi_1(M_H)$

Then $\beta_1(C) = \beta_1(M_H) > 0$. Hence $\exists$ epimorphism $H \to \mathbb{Z}$

$H$ finite index: $\exists \varphi : \pi_1(M) \to G'$; epimorphism, $G'$ is LO
$\varphi(H) \subset G'$ finite index, hence $\varphi(H) \neq \{1\}$.

Hence $\beta_1(M) > 0 \quad \Rightarrow \quad \pi_1(M)$ is LO

**Example** For any knot $K \subset S^3$, $\pi_1(E(K))$ is LO.

Restrict our attention to rational homology spheres $M$.

$$H_*(M; \mathbb{Q}) \cong H_*(S^3; \mathbb{Q})$$

**Question**

Which rational homology sphere has the LO fundamental group?

# $L$–spaces

$M$ : rational homology sphere

$\widehat{HF}(M)$ : Heegaard Floer homology with coefficients in $\mathbb{Z}_2$
(Ozsváth-Szabó)

$\widehat{HF}(M) = \widehat{HF}_0(M) \oplus \widehat{HF}_1(M)$ : $\mathbb{Z}_2$–grading

$\chi(\widehat{HF}(M)) = |H_1(M;\mathbb{Z})|$

Hence $\quad \mathrm{rk}\widehat{HF}(M) \geq |H_1(M;\mathbb{Z})|$

$M$ is an $L$–space if equality holds, i.e. $\mathrm{rk}\widehat{HF}(M) = |H_1(M;\mathbb{Z})|$.

**Example** Lens spaces, more generally, spherical 3–manifolds are $L$–spaces.

**Question**

Which rational homology sphere is an $L$–space?

M : lens space → M : $L$-space

spherical 3-manifold → $\pi_1$(M) : *not* LO

**Conjecture (Boyer-Gordon-Watson)**

*$M$ : an irreducible rational homology sphere*

*$M$ is an $L$–space $\Leftrightarrow$ $\pi_1(M)$ is not LO*

## Theorem (Boyer-Gordon-Watson)

*The conjecture is true for Seifert fiber spaces and Sol–manifolds.*

Hence conjecture is open for hyperbolic 3–manifolds and toroidal 3–manifolds.

# LO-surgery versus $L$–space surgery

$K$: a knot in $S^3$

$K(p/q)$ : 3–manifold obtained by $p/q$–surgery on $K$.

$H_1(K(p/q)) \cong \mathbb{Z}_p$.

Define the set of left-orderable surgeries for $K$ as

$$\mathcal{S}_{LO}(K) = \{r \in \mathbb{Q} \mid \pi_1(K(r)) \text{ is LO}\}.$$

Similarly define the set of $L$–space surgeries for $K$ as

$$\mathcal{S}_{L}(K) = \{r \in \mathbb{Q} \mid K(r) \text{ is an } L\text{–space}\}.$$

$\mathcal{S}_{LO}(K) \ni 0$ (Gabai), (Boyer-Rolfesn-Wiest)

$\mathcal{S}_{L}(K) \not\ni 0$

BGW-Conjecture, together with the cabling conjecture (González-Acuña-Short), suggests:

**Conjecture**

*Let $K$ be a knot which is not a cable of a nontrivial knot. Then we have:*
$\mathcal{S}_{LO}(K) \cup \mathcal{S}_{L}(K) = \mathbb{Q}$ & $\mathcal{S}_{LO}(K) \cap \mathcal{S}_{L}(K) = \emptyset$.

**Example (trivial knot)**

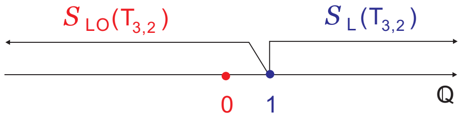
$K$ : trivial knot

$\mathcal{S}_{LO}(K) = \{0\}$ and $\mathcal{S}_{L}(K) = \mathbb{Q} - \{0\}$.

**Example (trefoil knot)** (Clay-Watson)

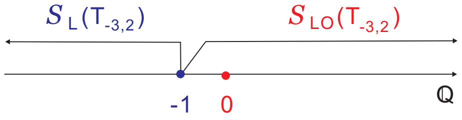
$K$ : trefoil knot $T_{3,2}$

$\mathcal{S}_{LO}(T_{3,2}) = (-\infty, 1) \cap \mathbb{Q}$ and $\mathcal{S}_{L}(T_{3,2}) = [1, \infty) \cap \mathbb{Q}$.




$T_{3,2}(r)$ is orientation reversingly diffeomorphic to $T_{-3,2}(-r)$.

$\mathcal{S}_{LO}(T_{-3,2}) = (-1, \infty) \cap \mathbb{Q}$ and $\mathcal{S}_{L}(T_{-3,2}) = (-\infty, -1] \cap \mathbb{Q}$.



**Example (figure-eight knot)**

$K$ : figure-eight knot

$\mathcal{S}_L(K) = \emptyset$ (Ozsváth-Szabó)

Conjecture $\Rightarrow$ $\mathcal{S}_{LO}(K) = \mathbb{Q}$.

$\mathcal{S}_{LO}(K) \supset (-4, 4) \cap \mathbb{Q}$ (Boyer-Gordon-Watson)

$\mathcal{S}_{LO}(K) \supset \{-4, 4\}$ (Clay-Watson)

$\mathcal{S}_{LO}(K) \supset \mathbb{Z}$ (Fenley)

$\mathcal{S}_L(K) \neq \emptyset$
$\Rightarrow$

- $K$ has the restricted Alexander polynomial. (Ozsváth-Szabó)
- $K$ is a fibered knot. (Ni).

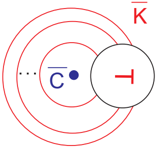
Generically $\mathcal{S}_L(K) = \emptyset$, and hence it is expected that $\mathcal{S}_{LO}(K) = \mathbb{Q}$ for most knots $K$.

### Theorem

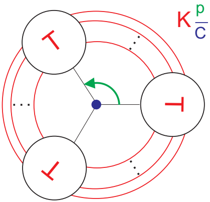
*There exist infinitely many hyperbolic knots $K$ each of which enjoys the following properties.*

1. $K(r)$ *is a hyperbolic 3–manifold for all* $r \in \mathbb{Q}$.
2. $\mathcal{S}_L(K) = \emptyset$.
3. $\mathcal{S}_{LO}(K) = \mathbb{Q}$.

# Periodic Construction

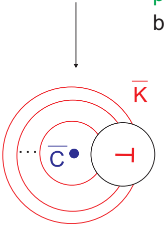


# Periodic Construction

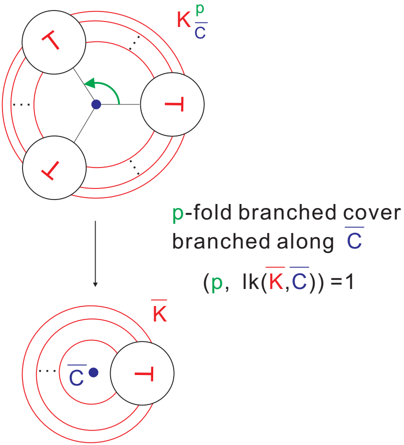



p-fold branched cover branched along $\overline{C}$

$(p, \ lk(\overline{K}, \overline{C})) = 1$



# Theorem

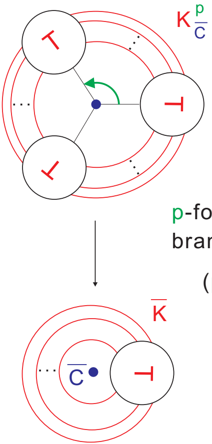



If $\overline{K}$ is a fibered knot, $\overline{C}$ satisfies the inequality

$$|\overline{C} \cap \overline{S}| > \langle \overline{C}, \overline{S} \rangle = lk(\overline{C}, \overline{K})$$

for any fiber surface $\overline{S}$.

# Theorem




p-fold branched cover branched along $\overline{C}$

$(p, \ lk(\overline{K},\overline{C}))=1$

$\mathcal{S}_L(K_{\frac{p}{C}}) = \emptyset$

$\mathcal{S}_{LO}(K_{\frac{p}{C}}) \supset p\,\mathcal{S}_{LO}(\overline{K})$

$p\,\mathcal{S} = \{pr \mid r \in \mathcal{S}\}$

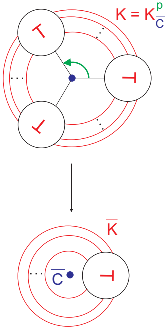
If $\overline{K}$ is a fibered knot, $\overline{C}$ satisfies the inequality

$$|\overline{C} \cap \overline{S}| > \langle \overline{C}, \overline{S} \rangle = lk(\overline{C}, \overline{K})$$

for any fiber surface $\overline{S}$.

# Left-orderable surgeries on periodic knots

$K^p_{\overline{C}}$ is a periodic knot with period $p$, and $\overline{K}$ is its factor knot.




**Theorem**

*$K$ : knot in $S^3$ with cyclic period $p$*

*$\overline{K}$ : factor knot*

$\mathcal{S}_{LO}(K) \supset p\mathcal{S}_{LO}(\overline{K})$

Sketch Proof. $r = \frac{m}{pn} \in \mathcal{S}_{LO}(\overline{K})$

$$\begin{array}{ccc} E(K) & \xrightarrow{\pi} & E(\overline{K}) \\ \text{Dehn filling} \downarrow & & \downarrow \text{Dehn filling} \\ K(\frac{m}{n}) & \xrightarrow[\pi']{} & \overline{K}(\frac{m}{pn}) \end{array}$$

$$\begin{array}{ccc} \pi_1(E(K)) & \xrightarrow{\pi_*} & \pi_1(E(\overline{K})) \\ \downarrow & & \downarrow \\ \pi_1(K(\frac{m}{n})) & \xrightarrow[\pi'_*]{} & \pi_1(\overline{K}(\frac{m}{pn})) \end{array}$$

## Lemma

$\pi_* : \pi_1(K(\frac{m}{n})) \to \pi_1(\overline{K}(\frac{m}{pn}))$ *is surjective.*

## Lemma

*$K(\frac{m}{n})$ is irreducible.*

Sketch Proof. Suppose : $K(\frac{m}{n})$ is reducible

$\Rightarrow$

$K$ is cabled – Cabling conjecture for periodic knots (Hayashi-Shimokawa)

$\Rightarrow$

$\overline{K}(\frac{m}{pn})$ has nontrivial torsion – invariant Seifert fibration theorem & invariant torus decomposition theorem (Meeks-Scott)

Contradiction. □

$\exists$ epimorphism $\pi_1(K(\frac{m}{n})) \to \pi_1(\overline{K}(\frac{m}{pn}))$ : LO

Boyer-Rolfsen-Wiest criterion shows $\pi_1(K(\frac{m}{n}))$ is also LO.

$pr = \frac{m}{n} \in \mathcal{S}_{LO}(K)$.

□

# $L$–space surgeries on periodic knots

**Theorem**

$K$ : *periodic knot in* $S^3$ *with the axis* $C$

$\overline{K}$ : *factor knot with the branch circle* $\overline{C}$

$K$ *has an* $L$*–space surgery.*
$\Rightarrow$
$E(\overline{K})$ *has a fibering over the circle with a fiber surface* $\overline{S}$
*such that* $|\overline{C} \cap \overline{S}| = \langle \overline{C}, \overline{S} \rangle$*: the algebraic intersection number between* $\overline{C}$ *and* $\overline{S}$*, i.e. the linking number* $lk(\overline{C}, \overline{K})$.

Uses Ni's result and the invariant fibration theorem (Edmonds-Livingston).

In particular, we have:

**Corollary**

*$K$ : periodic knot with the factor knot $\overline{K}$*

*$\overline{K}$ is not fibered,*

$\Rightarrow$

$\mathcal{S}_L(K) = \emptyset$.

# Periodic Construction

$S^3 \supset K^{p}_{C}$

$\downarrow$ $p$-fold branched cover branched along $\overline{C}$

$(p,\ lk(\overline{K}, \overline{C})) = 1$

$S^3 \supset \overline{K}$ (knot) $\quad \overline{C}$ (unknotted circle)

$\mathcal{S}_L(K^{p}_{C}) = \emptyset$

$\mathcal{S}_{LO}(K^{p}_{C}) \supset p\,\mathcal{S}_{LO}(\overline{K})$

If $\overline{K}$ is a fibered knot, $\overline{C}$ satisfies the inequality

$$|\overline{C} \cap \overline{S}| > \langle \overline{C}, \overline{S} \rangle = lk(\overline{C}, \overline{K})$$

for any fiber surface $\overline{S}$.

$g(K^{p}_{C}) \geq pg(\overline{K})$ (Naik)

## Theorem

$\overline{K}$ : *nontrivial knot in* $S^3$,

1. $\exists$ *infinitely many unknotted circles* $\overline{C}$ *such that* $\overline{K} \cup \overline{C}$ *is a hyperbolic link.*

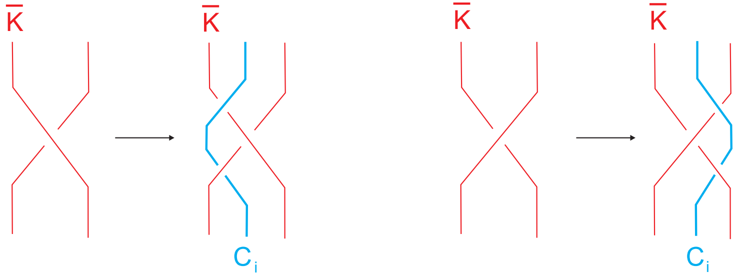
2. $\overline{K} \cup \overline{C}$ *is a hyperbolic link and* $p > 2$,
$\Rightarrow$
$K_{\frac{p}{C}}$ *is a hyperbolic knot* & $K_{\frac{p}{C}}(r)$ *is a hyperbolic 3–manifold for all* $r \in \mathbb{Q}$.

3. *Assume* $p > 2$ *and* $\overline{C_i}$ $(i = 1, 2)$ *is an unknotted circle such that* $lk(\overline{C_i}, \overline{K})$ *and* $p$ *are relatively prime, and* $\overline{K} \cup \overline{C_i}$ *is a hyperbolic link.*

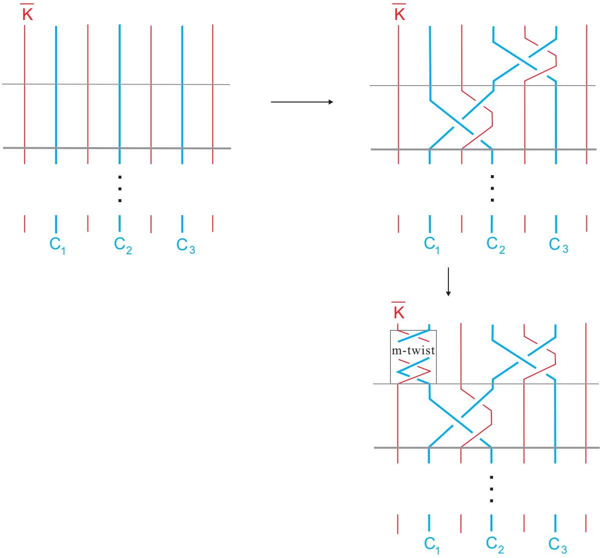
   $\overline{K} \cup \overline{C_1}$ *and* $\overline{K} \cup \overline{C_2}$ *are not isotopic.*
   $\Rightarrow$
   $K_{\frac{p}{C_1}}$ *and* $K_{\frac{p}{C_2}}$ *are not isotopic.*

Sketch Proof.
(1) (Aitchison) Arrange $\overline{K}$ as a closed $n$–braid for some integer $n$.

Introduce $(n-1)$–strands $C_i$ $(i = 1, \dots, n-1)$ between the $n$–strands of the original braid so that the crossings introduced, together with the original crossings, are alternatively positive and negative.



Arrange $C_i$ so that the closed braid is a 2–component link consisting of $\overline{K}$ and an unknotted circle $\overline{C} = C_1 \cup \cdots \cup C_{n-1}$ and $\overline{K} \cup \overline{C}$ is a non-split prime alternating link.



Following Menasco, $\overline{K} \cup \overline{C}$ is a torus link or a hyperbolic link.

Since $\overline{K}$ is knotted, but $\overline{C}$ is unknotted,

$\overline{K} \cup \overline{C}$ is not a torus link, and hence a hyperbolic link.

$\square$

(2) Uses invariant torus decomposition and invariant Seifert fibration of $E(K_{\frac{p}{C}})$ (Meeks-Scott) to show that $K_{\frac{p}{C}}$ is hyperbolic.

Classification of non-hyperbolic surgeries on periodic knots with period $> 2$ (Miyazaki-M) shows that $K_{\frac{p}{C}}(r)$ is a hyperbolic 3–manifold for all $r \in \mathbb{Q}$.

$\square$

(3) Assume : $K^p_{C_1}$ and $K^p_{C_2}$ are *isotopic*

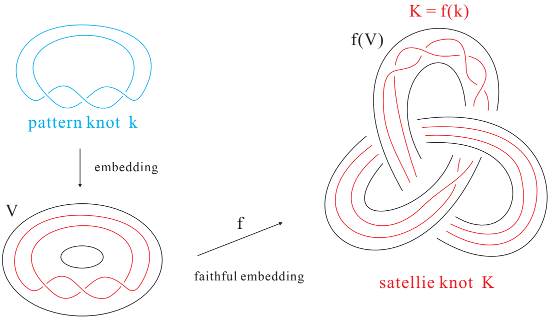
$\Rightarrow$
$K^p_{C_1}$ has two cyclic period of period $p$ with axes $C_1$ and $C_2$.

Since the symmetry group $\mathrm{Sym}(S^3, K^p_{C_1})$ is a cyclic group or a dihedral group,

$\overline{K} \cup \overline{C_1}$ and $\overline{K} \cup \overline{C_2}$ are isotopic.

$\square$

# Hyperbolic knots with $\mathcal{S}_L = \emptyset$ and $\mathcal{S}_{LO} = \mathbb{Q}$




## Theorem (Clay-Watson)

$K$ : *satellite knot*

$k$ : *pattern knot*

$\pi_1(k(r))$ *is LO, and* $K(r)$ *is irreducible*

$\Rightarrow$

$\pi_1(K(r))$ *is LO*

## Proposition

*$K_1, K_2$ : nontrivial knots*
$\mathcal{S}_{LO}(K_1 \sharp K_2) \supset \mathcal{S}_{LO}(K_1) \cup \mathcal{S}_{LO}(K_2)$.

Proof:
Recall: $K_1 \sharp K_2(r)$ is irreducible for $\forall r \in \mathbb{Q}$

(1) Regard $K_1 \sharp K_2$ as a satellite knot with a pattern knot $K_1$ and a companion knot $K_2$.

$\Rightarrow \mathcal{S}_{LO}(K_1 \sharp K_2) \supset \mathcal{S}_{LO}(K_1)$ (Clay-Watson)

(2) Regard $K_1 \sharp K_2$ as a satellite knot with a pattern knot $K_2$ and a companion knot $K_1$.

$\Rightarrow \mathcal{S}_{LO}(K_1 \sharp K_2) \supset \mathcal{S}_{LO}(K_2)$ (Clay-Watson)

Hence $\mathcal{S}_{LO}(K_1 \sharp K_2) \supset \mathcal{S}_{LO}(K_1) \cup \mathcal{S}_{LO}(K_2)$.

□

Take $T_{-3,2} \sharp T_{3,2}$.

$\mathcal{S}_{LO}(T_{-3,2} \sharp T_{3,2}) \supset \mathcal{S}_{LO}(T_{-3,2}) \cup \mathcal{S}_{LO}(T_{3,2}) \supset ((-1, \infty) \cup (-\infty, 1)) \cap \mathbb{Q}$.

Hence $\mathcal{S}_{LO}(T_{-3,2} \sharp T_{3,2}) = \mathbb{Q}$

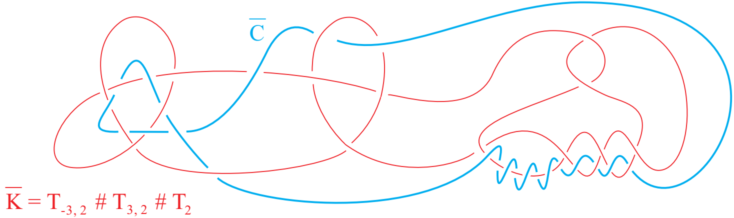
Note that $T_{-3,2} \sharp T_{3,2}$ is fibered.

For ease of handling, take the connected sum $(T_{-3,2} \sharp T_{3,2}) \sharp T_2$, where $T_2$ is a twist knot.

Its Alexander polynomial is $(t^2 - t + 1)^2 (2t^2 - 5t + 2)$, which is not monic, hence $\overline{K} = (T_{-3,2} \sharp T_{3,2}) \sharp T_2$ is not fibered.

$\mathcal{S}_{LO}(\overline{K}) = \mathcal{S}_{LO}(T_{-3,2} \sharp T_{3,2} \sharp T_2) \supset \mathcal{S}_{LO}(T_{-3,2} \sharp T_{3,2}) = \mathbb{Q}$, i.e. $\mathcal{S}_{LO}(\overline{K}) = \mathbb{Q}$

Take an unknotted circle $\overline{C}$ as follows.




$lk(\overline{C}, \overline{K}) = 1$ & $\overline{K} \cup \overline{C}$ is alternating (hence hyperbolic).

For any $p > 2$,
apply the periodic construction to $(\overline{K}, \overline{C})$, we have:

1. $K_{\frac{p}{C}}$ is a hyperbolic knot in $S^3$.
2. $K_{\frac{p}{C}}(r)$ is a hyperbolic 3–manifold for all $r \in \mathbb{Q}$.
3. $\mathcal{S}_L(K_{\frac{p}{C}}) = \emptyset$.
4. $\mathcal{S}_{LO}(K_{\frac{p}{C}}) \supset p\mathcal{S}_{LO}(\overline{K}) = p\mathbb{Q} = \mathbb{Q}$, i.e. $\mathcal{S}_{LO}(K_{\frac{p}{C}}) = \mathbb{Q}$.