Left-orderable fundamental groups and Dehn surgery on two-bridge knots

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Intelligence of Low-dimensional Topology
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Outline

1. Left-ordering
   - Definition and Examples
2. *L*-space conjecture
   - *L*-space
   - Previous Works
3. Our Result
   - Main Result
   - Basic Ideas for Proof

Left-ordering

A non-trivial group G is said to be left-orderable (LO)
if G admits a strict total ordering "<" which is invariant under left-multiplication.

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Many groups, which appear in Topology, are known to be LO.

- free groups, free abelian groups
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L-space

A rational homology sphere Y is an L-space if Heegaard–Floer homlogy $\widehat{HF}(Y)$ is a free abelian group with rank equal to $|H_1(Y;\mathbb{Z})|$.

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- elliptic manifolds
- double branched covers over non-split alternating knots/links

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L-space conjecture

It is an open problem to find a non-Heegaard–Floer characterization of *L*-spaces.

Conjecture (Boyer-Gordon-Watson 2011)

Let Y be an irreducible rational homology sphere.
Then Y is an L-space if and only if $\pi_1(Y) \neq LO$.

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Dehn surgery

Rational homology spheres are obtained by Dehn surgery on knots in huge quantities.

Let K be a knot in S^3. If K admits Dehn surgery yielding an L-space (L-space surgery), then

- K is fibered (Yi Ni),
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Motivation

Hence we can say that "most" knots have no *L*-space surgery.
If we support *L*-space conjecture, then we can expect

any non-trivial Dehn surgery on "most" knots yields
a 3-manifold whose $\pi_1 = LO$.

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Previous works on Dehn surgery vs LO

Boyer-Gordon-Watson 2011

For the figure-eight knot, any slope r in $(-4, 4)$ is LO.

Idea: Use a continuous family of representations of $\pi_1(S^3 - K)$ to $SL_2(\mathbb{R})$.

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For the figure-eight knot, the slopes ± 4 are also LO.

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Twist knots

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For a hyperbolic twist knot, any slope in $[0, 4]$ is LO.

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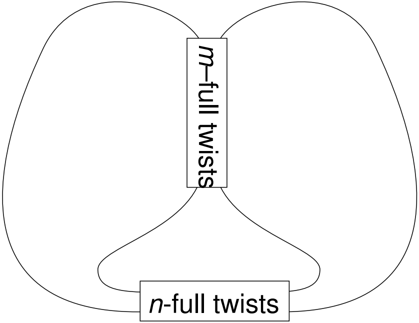
All LO

Theorem (Motegi-Teragaito, 2013)

There exist infinitely many hyperbolic knots such that all non-trivial slopes are LO.

Genus one 2-bridge knots

Let $K = K(m, n)$ be a genus one 2-bridge knot.



- Vertical twists are left-handed if $m > 0$.
- Horizontal twists are right-handed if $n > 0$.
- $K(m, n) = K(-n, -m)$.
- $K(-m, -n)$ is the mirror image of $K(m, n)$.

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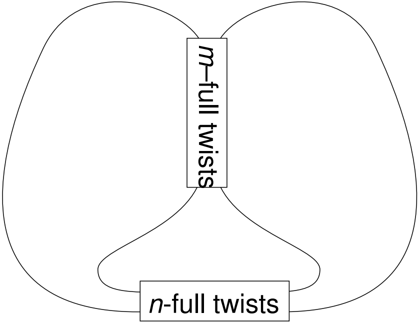

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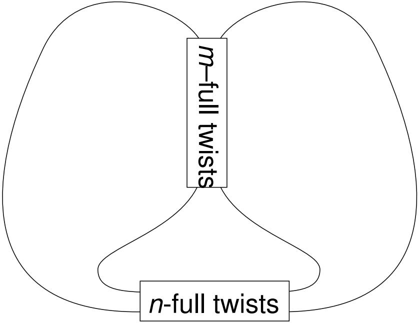

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Main result

Theorem (Hakamata-Teragaito, Tran 2013)

Let $K = K(m, n)$ be a hyperbolic genus one 2-bridge knot. Let I be the interval defined by

$$I = \begin{cases} (-4n, 4m) & \text{if } n > 0, \\ [0, \max\{4m, -4n\}) & \text{if } m > 1 \text{ and } n < -1, \\ [0, 4] & \text{otherwise.} \end{cases}$$

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For twist knots

For positive twist knots, we obtain a wider range of left-orderable slopes.

Corollary

Let $K(1, n)$ be the n-twist knot with $n > 0$. Then any slope in $(-4n, 4]$ is LO.

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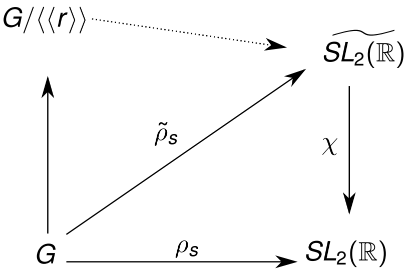
Scheme

Like BRW and BGW, we use a continuous family of representaions of knot group $G = \pi_1(S^3 - K)$ to $SL_2(\mathbb{R})$.



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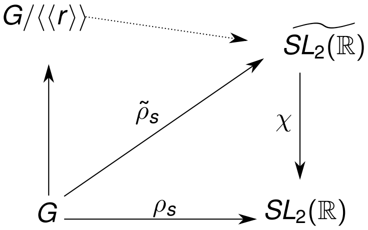

- $\widetilde{SL_2(\mathbb{R})}$ is LO.
- Any non-trivial subgroup of LO group is LO.

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Let M be a prime, connected, compact 3-manifold.
Then $\pi_1 M$ is LO if and only if there is a surjection from $\pi_1 M$ onto a LO group.

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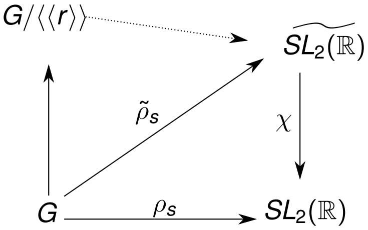

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Presentation of knot group

Let $G = \pi_1(S^3 - K(m,n))$.
It is well known that G has a presentation

$$G = \langle x, y : w^n x = y w^n \rangle,$$

where x and y are meridians, and $w = (xy^{-1})^m (x^{-1}y)^m$.

Representations

For real numbers $s > 0$ and $t > 1$, define

$$\rho_s : G \to SL_2(\mathbb{R})$$

by

$$\rho_s(x) = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}, \quad \rho_s(y) = \begin{pmatrix} \frac{t-s-1}{\sqrt{t}-\frac{1}{\sqrt{t}}} & \frac{s}{(\sqrt{t}-\frac{1}{\sqrt{t}})^2} - 1 \\ -s & \frac{s+1-\frac{1}{t}}{\sqrt{t}-\frac{1}{\sqrt{t}}} \end{pmatrix}$$

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Riley Polynomials

The map $\rho_s : G \to SL_2(\mathbb{R})$ gives an irreducible non-abelian representation if and only if s and t satisfy Riley's equation $\phi_K(s,t) = 0$.

$$\phi_K(s,t) = (\tau_{n+1} - \tau_n) + (s + 2 - t - 1/t) f_{m-1} g_{m-1} \tau_n,$$

where $\tau_k = \frac{\lambda_+^k - \lambda_-^k}{\lambda_+ - \lambda_-}$, $\lambda_\pm$ are eigenvalues of $\rho_s(w)$,
f_i and g_i are certain polynomials of variable s.

* $\mathrm{tr}\rho_s(w) = \lambda_+ + \lambda_- = f_m^2 + f_{m-1}^2 - s(t + 1/t) g_{m-1}^2$.

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Existence of Representaions

In general, we cannot solve the equation $\phi_K(s,t) = 0$. But we can guarantee the existence of real solutions.

Existence of solutions

If $|n| > 1$, then Riley's equation $\phi_K(s,t) = 0$ has a real solution $t > 1$ for any $s > 0$ such that

$$s + 2 + \frac{c}{sg_{m-1}^2} < t + \frac{1}{t} < s + 2 + \frac{d}{sg_{m-1}^2}$$

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Slope

We need to annihilate the image $\rho_s(r)$ of the slope $r = \mu^p \lambda^q$.

- The images of meridian and longitude under ρ_s are diagonal.
- Let A and B be their $(1,1)$-entries.

Hence,

$$\rho_s(r) = I \iff A^p B^q = 1$$
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Range of Slopes

Lemma

If $p/q \in (0, 4m)$, then there exists $s > 0$ such that $\rho_s(r) = I$.

This is established by examining the image of a function

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By definition, $A = \sqrt{t}$. On the other hand,

$$B = \frac{-f_m + tf_{m-1}}{-f_{m-1} + tf_m}.$$

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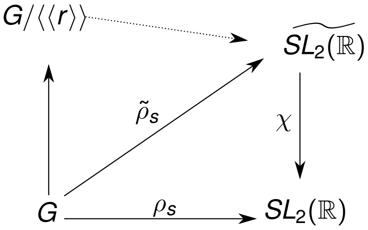
Control of Lift

Thus we have a representation $\rho_s : G \to SL_2(\mathbb{R})$ with $\rho_s(r) = I$. In general, it lifts to $\tilde{\rho}_s : G \to \widetilde{SL_2(\mathbb{R})}$, but $\tilde{\rho}_s(r)$ may not be the identity element.



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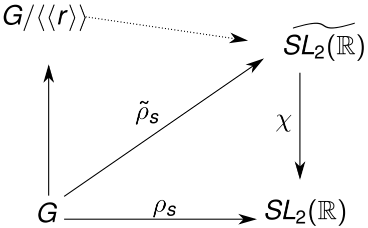

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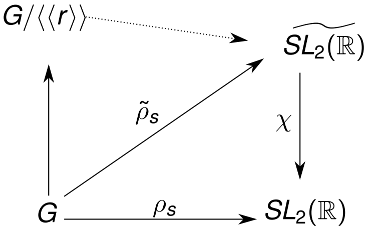

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Proof of Theorem

From the argument so far, any slope in $(0, 4m)$ is LO.

Assume $n > 0$. Apply the argument for $K(n, m)$, which is the mirror of $K(m, n)$. Then we obtain $(0, 4n)$.

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