Hasse-Weil zeta functions of SL_2-character varieties of 3-manifolds

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OUTLINE

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- HW zeta of arithmetic two-bridge link
- HW zeta of closed arithmetic 3 manifold
- HW zeta of *A*-polynomials of torus knots

Local (congruence) zeta function 1

$f_1, \cdots, f_r \in \mathbb{Z}[X_1, \cdots, X_m]$
p : prime number
$\mathbb{F}_{p^n}$: finite field with p^n elements
$V(\mathbb{F}_{p^n}) := \{(a_1, \cdots, a_m) \in (\mathbb{F}_{p^n})^m \mid$
$f_1(a_1, \cdots, a_m) = \cdots = f_r(a_1, \cdots, a_m) = 0\}$
Local (congruence) zeta function of V at p

$$Z(V, p, T) := \exp\left(\sum_{n=1}^{\infty} \frac{\#V(\mathbb{F}_{p^n})}{n} T^n\right) \in \mathbb{Q}[\![T]\!]$$

$$\exp(T) := \sum_{n=0}^{\infty} \frac{1}{n!} T^n, \quad \log\left(\frac{1}{1-T}\right) = \sum_{n=1}^{\infty} \frac{1}{n} T^n$$

Example 1

$f = X \in \mathbb{Z}[X], \quad \# V(\mathbb{F}_{p^n}) = 1$

$$Z(V, p, T) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} T^n\right) = \frac{1}{1-T}$$

Example 2

$f = X^2 + 1 \in \mathbb{Z}[X]$

$$Z(V, p, T) = \begin{cases} 1/(1-T), & p = 2 \\ 1/(1-T)^2, & p \neq 2, \left(\frac{-1}{p}\right) = 1 \\ 1/(1-T^2), & p \neq 2, \left(\frac{-1}{p}\right) = -1 \end{cases}$$

Local (congruence) zeta function 2

Theorem (Dwork)

$Z(V, p, T)$ is a rational function.

Therefore

$$Z(V,p,T) = \exp\left(\sum_{n=1}^{\infty} \frac{\#V(\mathbb{F}_{p^n})}{n} T^n\right) = \frac{\prod_i (1-\alpha_i T)}{\prod_j (1-\beta_j T)}$$

$$\#V(\mathbb{F}_{p^n}) = \sum_j \beta_j^n - \sum_i \alpha_i^n \quad (\alpha_i, \beta_j \in \mathbb{C})$$

Computation of $Z(V, p, T) \iff$ computation of each $\#V(\mathbb{F}_{p^n})$

Examples of algebraic sets

- **Set of $\mathrm{SL}_2(\mathbb{C})$-representations of finitely presented group**
- **$\mathrm{SL}_2(\mathbb{C})$-character variety of f.p. group**

Related topics

- **Work of Zink (2000)**
- **Computation of the number of conjugacy classes of $\mathrm{SL}_2(\mathbb{F}_q)$-representations of torus knot groups (Li, Xu 2003,2004)**
- **Computation of the number of conjugacy classes of $\mathrm{SL}_2(\mathbb{F}_p)$ (surjective) representations of knots in the Rolfsen's table (Kitano,Suzuki 2012)**

$f_1, \cdots, f_r \in \mathbb{Z}[X_1, \cdots, X_m]$
$V(\mathbb{F}_{p^n}) := \{(a_1, \cdots, a_m) \in (\mathbb{F}_{p^n})^m |$
$f_1(a_1, \cdots, a_m) = \cdots = f_r(a_1, \cdots, a_m) = 0\}$

$$Z(V, p, T) := \exp\left(\sum_{n=1}^{\infty} \frac{\# V(\mathbb{F}_{p^n})}{n} T^n\right) \in \mathbb{Q}[\![T]\!].$$

$$\zeta(V(f_1, \cdots, f_r), s) := \prod_{p\,:\,\text{prime}} Z(V, p, p^{-s}).$$

$\zeta(V(f_1, \cdots, f_r), s)$ converges absolutely in $\mathrm{Re}(s) > \dim V(f_1, \cdots, f_r)$.

Ex.1 Riemann (Dedekind) zeta function

Example 1

$f = X \in \mathbb{Z}[X], \quad Z(V, p, T) = 1/(1-T)$

$$\zeta(V, s) := \prod_{p:\text{ prime}} (1 - p^{-s})^{-1} = \zeta(s) \text{ : Riemann zeta}$$

Example 2

$K/\mathbb{Q}$: number field, $\quad \mathcal{O}_K$: ring of integers of K
$\mathcal{O}_K \tilde{\to} \mathbb{Z}[X_1, \cdots, X_m]/(f_1, \cdots, f_r)$

$$\zeta(V, s) = \zeta(K, s) \text{ : Dedekind zeta of } K$$
$$= \prod_{\mathfrak{p}:\text{ prime}} (1 - N(\mathfrak{p})^{-s})^{-1}, \quad N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$$

Ex.2 zeta of cyclotomic polynomial

$\Phi_d(X)$: min. poly. of a prim. d-th root ζ_d of 1

$$\Phi_1(X) = X - 1, \quad \Phi_2(X) = X + 1,$$

$$\Phi_3(X) = X^2 + X + 1, \quad \Phi_4(X) = X^2 + 1$$

$$\mathbb{Q}_d := \mathbb{Q}(\zeta_d) \tilde{\to} \mathbb{Q}[X]/(\Phi_d(X))$$

$$\mathcal{O}_d = \mathbb{Z}[\zeta_d] \tilde{\to} \mathbb{Z}[X]/(\Phi_d(X))$$

$$\zeta(\Phi_d(X), s) = \zeta(\mathbb{Q}_d, s)$$

For instance

$$\zeta(X^2 + 1, s) = \zeta(\mathbb{Q}(\sqrt{-1}), s)$$

M **: orientable complete hyperbolic 3 manifold with finite volume**
$X(M) := \{$**characters of** $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})\}$

$X(M)$ **: (affine) algebraic set over** $\mathbb{Q}$ **(the set of the common zeros of** $f_1, \cdots, f_r \in \mathbb{Q}[T_1, \cdots, T_m]$**)**
$X_0(M)$ **: a canonical component of** ***M***
(an irreducible component containing the character corr. to the holonomy rep. of ***M*****)**

Theorem (Thurston)

if ***M*** **complete hyperbolic 3 manifold with cusp** ***n*** **then**

$$\dim X_0(M) = n.$$

Hasse-Weil zeta function of M

M : hyperbolic 3 manifold

Hasse-Weil zeta function **of** M

$$\zeta(M,s) := \prod_{p:prime} Z(X(M),p,p^{-s}).$$

$$Z(X(M),p,T) := \exp\left(\sum_{n=1}^{\infty} \frac{\#X(M)(\mathbb{F}_{p^n})}{n} T^n\right) \in \mathbb{Q}[[T]].$$

Remark

$\zeta(M,s)$ is defined up to rational functions in $\mathbb{Q}(p^{-s})$ for finitely many prime numbers p

$$\zeta((L-1)(1+LM^6),s) =$$
$$\zeta(s-1)^2\zeta(s)^{-1} \times \zeta(\mathbb{Q}_4,s)^{-1}\zeta(\mathbb{Q}_{12},s)^{-1} \times (1-2^{-s})^{-2}(1-3^{-s})^{-1}$$

$$\zeta((L-1)(1+LM^{10}),s) =$$
$$\zeta(s-1)^2\zeta(s)^{-1} \times \zeta(\mathbb{Q}_4,s)^{-1}\zeta(\mathbb{Q}_{20},s)^{-1} \times (1-2^{-4s})^2(1-5^{-s})(1-2^{-s})^{-2}$$

$\mathbb{Q}_n := \mathbb{Q}(\zeta_n)$, ζ_n : primitive n-th root of unity

$\zeta(\mathbb{Q}_n,s)$: Dedekind zeta function

$\zeta(s)$: Riemann zeta function

Hasse-Weil zeta of figure 8 knot

Theorem (H.)

$M_8 := S^3 \setminus \mathcal{K}$: fig. 8 knot complement in S^3

$$\zeta(M_8, s) = \frac{\zeta(E, s)}{\zeta(s)\zeta(\mathbb{Q}(\sqrt{5}), s)},$$

$\zeta(E, s)$: Hasse-Weil zeta of the elliptic curve $/\mathbb{Q}$

$$E : Y^2 = X^3 - 2X + 1$$

Hasse-Weil zeta of figure 8 knot 2

- functional equation

$$\xi(M_8, s) := \frac{4\pi^{(3s/2)+1}}{(10\sqrt{2})^s \Gamma(s/2)^3} \times \zeta(M_8, s)$$

$\xi(M_8, 2-s)\xi(2-s)\xi(\mathbb{Q}(\sqrt{5}), 2-s) = -\xi(M_8, s)\xi(s)\xi(\mathbb{Q}(\sqrt{5}), s).$

- special value

$$\lim_{s\to 1} \frac{\xi(M_8, s)}{s-1} = -\frac{\mathrm{AGM}(\varphi, \varphi - 1)}{\sqrt{10}\log(\varphi)},$$

$\varphi = (\sqrt{5}+1)/2$
$\mathrm{AGM}(\varphi, \varphi - 1)$: arith. geom. mean

Remark

$\log(\varphi^2) = m(\Delta_{\mathcal{K}}(T))$, $\Delta_{\mathcal{K}}(T)$: Alexander poly. of $\mathcal{K}$

$$\mathrm{AGM}(\varphi, \varphi - 1) = \frac{1}{2}\left(\frac{d}{dk} m(P_k)(\sqrt{5})\right)^{-1}$$

$P_k := x + \frac{1}{x} + y + \frac{1}{y} - 4k$: elliptic curve for $4k \neq 0, 1$. $P = P(t_1, \cdots, t_n) \in \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$, Mahler measure of P

$$m(P) := \int_0^1 \cdots \int_0^1 \log|P(e^{2\pi t_1 \sqrt{-1}}, \cdots, e^{2\pi t_n \sqrt{-1}})| dt_1 \cdots dt_n$$

remark

Value at $s = 2$

$$\lim_{s\to 2}(s-2)\xi(M_8, s) = \frac{75}{2\sqrt{5}\pi^2\mathcal{L}(E,2)}$$

$$\mathcal{L}(E,2) = m(P_{\sqrt{-4}/4}) \text{ (Mellit)}, E \tilde{\to} E(P_{\sqrt{-4}/4})$$

$\mathcal{L}(E,s)$: L-series of $E/\mathbb{Q}$
$L(E,s)\zeta(E,s) = \zeta(s)\zeta(s-1)$,
$\mathcal{L}(E,s) := (40)^{s/2}(2\pi)^{-s}\Gamma(s)L(E,s)$

The genus of $\mathrm{SL}_2(\mathbb{C})$-character curves of certain family of hyperbolic two-bridge knots (containing twist knots) are studied (Macasieb, Petersen, Van Luijk, 2011)
Two of them have genus 1, elliptic curves

Example

Up to (a, b) $(a, b < 50)$ $(5, 2) = 4_1, (15, 4) = 7_4, (21, 8) = 7_7, (27, 8) = 8_{11}, (45, 14) = 10_{21}$ are elliptic curves

Question

Are there finitely many elliptic curves appeared as canonical comp. of $\mathrm{SL}_2(\mathbb{C})$-char. varieties of hyperbolic two-bridge knots?

Expectation

$$
\begin{array}{ccc}
S^3 \setminus \mathcal{K} & \xleftarrow{\text{Dehn filling}} & S^3 \setminus \mathcal{K} \cup \mathcal{K}' \\
\downarrow & & \downarrow \\
X(S^3 \setminus \mathcal{K}) & \xrightarrow{\subset} & X(S^3 \setminus \mathcal{K} \cup \mathcal{K}') \\
\downarrow & & \downarrow \\
\zeta(S^3 \setminus \mathcal{K}, s) & \xrightarrow{?} & \zeta(S^3 \setminus \mathcal{K} \cup \mathcal{K}', s)
\end{array}
$$

Describe the relation between $\zeta(S^3 \setminus \mathcal{K}, s)$ and $\zeta(S^3 \setminus \mathcal{K} \cup \mathcal{K}', s)$
$\longrightarrow$ relation between $\zeta(S^3 \setminus \mathcal{K}, s)$ and $\zeta(S^3 \setminus \mathcal{K}', s)$

In what follows, we use the notation 'arithmetic' for
M : arithmetic 3 manifold $\Longleftrightarrow$ up to isom. the image of the holonomy representation $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ is in $\mathrm{SL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of some $K/\mathbb{Q}$

Theorem (Gehring, Maclachlan, Martin)

Arithmetic two-bridge links in the 3-sphere are figure 8 knot, Whitehead link $5_1^2 = (8,3)$, $6_2^2 = (10,3)$, $6_3^2 = (12,5)$

Hasse-Weil zeta function of arithmetic two-bridge links

Theorem (H.)

X_0, X_1, X_2 : canonical components of Whitehead link $5^2_1 = (8, 3)$, $6^2_2 = (10, 3)$, $6^2_3 = (12, 5)$

$$\zeta(X_0, s) = \zeta(\mathbb{Q}(\sqrt{2}), s-1)\zeta(s)^2\zeta(s-1)^2\zeta(s-2).$$

$$\zeta(X_1, s) = \zeta(\mathbb{Q}(\sqrt{5}), s-1)^2\zeta(s)^3\zeta(s-2).$$

$$\zeta(X_2, s) = \zeta(s)^2\zeta(s-2).$$

$$\zeta(X_0, s) = \zeta(\mathbb{Q}(\sqrt{2}), s-1)\zeta(s)^2\zeta(s-1)^2\zeta(s-2).$$
$$\zeta(X_1, s) = \zeta(\mathbb{Q}(\sqrt{5}), s-1)^2\zeta(s)^3\zeta(s-2).$$
$$\zeta(X_2, s) = \zeta(s)^2\zeta(s-2).$$

remark

$\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}$ for $\zeta(X_i, s)$

$$= \mathbb{Q}\left(\begin{array}{c}\text{all the } \mathbb{P}^1\text{-coordinates} \\ \text{of the degenerate fibers of } X_i\end{array}\right)$$

Remark

trace fields (= invariant trace fields) of 5_1^2, 6_2^2 and 6_3^2 = $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-7})$

Character varieties of arith. two-bridge links

Defining polys of Whitehead link, 6^2_2, 6^2_3 in $\mathbb{C}^3$

$$f_0 := z^3 - xyz^2 + (x^2 + y^2 - 2)z - xy,$$
$$f_1 := z^4 - xyz^3 + (x^2 + y^2 - 3)z^2 - xyz + 1,$$
$$f_2 := z^3 - xyz^2 + (x^2 + y^2 - 1)z - xy$$

$X(f_i) \subset \mathbb{C}^3$ are smooth affine surfaces
Compactification of these surfaces in $\mathbb{P}^2 \times \mathbb{P}^1$

$$F_0 := u^2z^3 - xyz^2w + (x^2 + y^2 - 2u^2)zw^2 - xyw^3,$$
$$F_1 := u^2z^4 - xyz^3w + (x^2 + y^2 - 3u^2)z^2w^2 - xyzw^3$$
$$F_2 := u^2z^3 - xyz^2w + (x^2 + y^2 - u^2)zw^2 - xyw^3.$$

$$F_0 := u^2z^3 - xyz^2w + (x^2 + y^2 - 2u^2)zw^2 - xyw^3,$$
$$F_1 := u^2z^4 - xyz^3w + (x^2 + y^2 - 3u^2)z^2w^2 - xyzw^3$$
$$F_2 := u^2z^3 - xyz^2w + (x^2 + y^2 - u^2)zw^2 - xyw^3.$$

$$\mathbb{P}^2 \times \mathbb{P}^1 := \left\{ (x : y : u,\ z : w) \,\middle|\, \begin{matrix} (x : y : u) \in \mathbb{P}^2 \\ (z : w) \in \mathbb{P}^1 \end{matrix} \right\}$$
$$X(F_i) := \{(x : y : u,\ z : w) \mid F_i(x, y, u, z, w) = 0\}$$

$X(F_i)$: (singular) conic bundles over $\mathbb{P}^1$ by $(x : y : u,\ z : w) \mapsto (z : w)$ (Landes, H. 2011)
possible to compute the number of $\mathbb{F}_q$-rational points of F_i

Theorem (Tran, 2014)

- **The char. var. of the two-bridge link $(2m, 3)$ $(m \neq 3)$ have 2 irr. components. Canonical components are defined in terms of Chebyshev polynomials. Their compactification have conic bundle structure.**
- **Determination of the number of irreducible components of the character varieties of $(-2, 2m+1, 2n)$-pretzel links and k-twisted Whitehead links.**

remark

In general, canonical components of the SL_2 char. var. of a hyperbolic two-bridge link does not have a conic bundle structure over $\mathbb{P}^1$.

question

Does the canonical component of the SL_2 character variety of a hyperbolic two-bridge link have a fibered structure over $\mathbb{P}^1$?

Examples in arith. closed 3 mfd.

M	def.poly. of $X(M)_{\mathrm{Irr}}(\mathbb{C})$	$\zeta(M,s)$
Weeks	$T^3 - T - 1$	$\zeta(K_M, s)$
Meyerhoff	$T^4 - 3T^3 + T^2 + 3T - 1$	$\zeta(K_M, s)$
m010 (-1,2)	$T^4 - 2T^2 + 4$	$\zeta(K_M, s)$
m003 (-4,3)	$T^4 - T^3 - 2T^2 + 2T + 1$	$\zeta(K_M, s)$
m004 (6,1)	$T^6 - 7T^4 + 14T^2 - 4$	$\zeta(K_M, s)$
m003 (-3,4)	$T^6 + T^4 - 1$	$\zeta(K_M, s)$

ρ_M : holonomy representation of M
K_M : trace field of M ($K_M = \mathbb{Q}(\mathrm{Tr}\,\rho_M(\pi_1(M))$)
$X(M)_{\mathrm{Irr}}(\mathbb{C})$: open subset of $X(M)$ consisting of irreducible characters

Questions

M : (arith.) closed 3 mfd, K_M : trace field of M

$$\zeta(M, s) = \zeta(K_M, s)?$$

$$\zeta(M, s) = \zeta(M', s) \iff K_M \tilde{\to} K_{M'}?$$

Theorem (Chinburg, Hamilton, Long, Reid)

K, K' : number field having only one complex place

$$K \tilde{\to} K' \iff \zeta(K, s) = \zeta(K', s)$$

$\mathcal{K}$: knot in S^3
$\lambda, \mu \in \pi_1(S^3 \setminus \mathcal{K})$: canonical longitude, meridian of $\mathcal{K}$

$$R(\mathcal{K}) = \left\{\rho : \pi_1(S^3 \setminus \mathcal{K}) \to \mathrm{SL}_2(\mathbb{C})\right\}$$

$$R_U = \{\rho(\lambda), \rho(\mu) \in R(\mathcal{K}) : \text{ upper triangular } \}$$

$$\rho(\lambda) = \begin{pmatrix} a_\rho(\lambda) & b_\rho(\lambda) \\ 0 & a_\rho(\lambda)^{-1} \end{pmatrix}, \quad \rho(\mu) = \begin{pmatrix} a_\rho(\mu) & b_\rho(\mu) \\ 0 & a_\rho(\mu)^{-1} \end{pmatrix}$$

$$\xi : R_U \to \mathbb{C}^\times \times \mathbb{C}^\times$$

$$\rho \mapsto (a_\rho(\lambda), a_\rho(\mu))$$

$$\rho(\lambda) = \begin{pmatrix} a_\rho(\lambda) & b_\rho(\lambda) \\ 0 & a_\rho(\lambda)^{-1} \end{pmatrix}, \quad \rho(\mu) = \begin{pmatrix} a_\rho(\mu) & b_\rho(\mu) \\ 0 & a_\rho(\mu)^{-1} \end{pmatrix}$$

$$\xi : R_U \to \mathbb{C}^\times \times \mathbb{C}^\times, \quad \rho \mapsto (a_\rho(\lambda), a_\rho(\mu))$$

$$\mathbb{C}^2 \supset \bigcup_C \overline{\xi(C)} = V(A_{\mathcal{K}}(L, M)),$$

C : irr. component of R_U s.t. $\overline{\xi(C)}$: curve in $\mathbb{C}^2$

$$A_{\mathcal{K}}(L, M) \in \mathbb{C}[L, M]$$

$$V(A_{\mathcal{K}}(L, M)) = \left\{(x, y) \in \mathbb{C}^2 \mid A_{\mathcal{K}}(x, y) = 0\right\}$$

$A_{\mathcal{K}}(L, M)$: A-polynomial of $\mathcal{K}$ (defined up to constant, multiplicity)

Properties

- $A_{\mathcal{K}}(L, M) \in \mathbb{Z}[L, M]$
- $A_{\bigcirc}(L, M) = L - 1$
- $L - 1 \mid A_{\mathcal{K}}(L, M)$
- $\mathcal{K}$: non-trivial $\Longrightarrow A_{\mathcal{K}}(L, M) \neq L - 1$

A-polynomial detects boundary slopes of incompressible surfaces in $S^3 \setminus \mathcal{K}$ attached to ideal points of $V(A_{\mathcal{K}}(L, M))$

a, b : positive integers, $(a, b) = 1$
A-polynomials of (a, b)-torus knots $T(a, b)$
$A_{a,b}(L, M) := A_{T(a,b)}(L, M) =$

$$\begin{cases} (L-1)\left(-1+(LM^{ab})^2\right), & \text{if } a, b > 2, \\ (L-1)(1+LM^{2(2m+1)}), & \text{if } (a, b) = (2, 2m+1). \end{cases}$$

Hasse-Weil zeta functions of A-polynomials of torus knots

Theorem(H.-Terashima)

Up to rational functions in $\mathbb{Q}(\{p^{-s}\}_{p|2ab})$ the zeta function $\zeta(A_{a,b}(L,M),s)$ is equal to

$$\begin{cases} \left(\prod_{d|2ab} \frac{1}{\zeta(\mathbb{Q}_d,s)}\right)\frac{\zeta(s-1)^3}{\zeta(s)^2}, & \text{if } a,\ b>2, \\ \left(\prod_{d|2ab,\ d\nmid ab} \frac{1}{\zeta(\mathbb{Q}_d,s)}\right)\frac{\zeta(s-1)^2}{\zeta(s)}, & \text{if } \begin{array}{c}(a,b)= \\ (2,2m+1)\end{array}. \end{cases}$$

$$\zeta(A_{\bigcirc}(L, M)) = \zeta(s-1)$$

$$\zeta(A_{2,3}(L, M), s) = \frac{\zeta(s-1)^2}{\zeta(s)} \times \zeta(\mathbb{Q}_4, s)^{-1}\zeta(\mathbb{Q}_{12}, s)^{-1}$$

$$\zeta(A_{2,5}(L, M), s) = \frac{\zeta(s-1)^2}{\zeta(s)} \times \zeta(\mathbb{Q}_4, s)^{-1}\zeta(\mathbb{Q}_{20}, s)^{-1}$$

Remark

$$\begin{cases} (L-1)\left(-1+(LM^{ab})^2\right), & \text{if } a,b>2,\\ (L-1)(1+LM^{2(2m+1)}), & \text{if } (a,b)=(2,2m+1). \end{cases}$$

$$\zeta(L-1,s)=\zeta(s-1),\quad \zeta(-1+(LM^{ab})^2,s)=\frac{\zeta(s-1)^2}{\zeta(s)^2}$$

$$\zeta(1+LM^{2(2m+1)},s)=\zeta(s-1)/\zeta(s)$$

We have to consider the intersection of two components of A-polynomial to retrieve essential information!

A min. model $\mathcal{M}^{a,b}_{n,m}$ = an irr. highest wt. rep. of the Virasoro alg. with some conformal wt. and central charge ($1 \leq n \leq a-1, 1 \leq m \leq b-1$)

$$\Phi^{a,b}_{n,m}(\tau) = \sum_{k=0}^{\infty} \chi^{(n,m)}_{2ab}(k) q^{\frac{k^2}{4ab}}.$$

The 'character' $\chi^{(n,m)}_{2ab} : \mathbb{Z} \to \mathbb{C}$ is defined by

$$\chi^{(n,m)}_{2ab}(k) = \begin{cases} 1, & k \equiv \pm(nb - ma) \pmod{2ab}, \\ -1, & k \equiv \pm(nb + ma) \pmod{2ab}, \\ 0, & \text{otherwise.} \end{cases}$$

Note : $\chi^{(n,m)}_{2ab}(0) = 0$.

Remark

Consider the normalized Alexander polynomial of the torus knot $T(a, b)$

$$\Delta_{a,b}(T^2) = \frac{(T^{ab} - T^{-ab})(T - T^{-1})}{(T^a - T^{-a})(T^b - T^{-b})}.$$

$$\frac{T - T^{-1}}{\Delta_{a,b}(T^2)} = \frac{(T^a - T^{-a})(T^b - T^{-b})}{(T^{ab} - T^{-ab})} = \sum_{k\geq 0} \chi_{2ab}^{(a-1,1)}(k) T^{-k}.$$

$$\tilde{\Phi}^{a,b}_{n,m}(\tau) = -\frac{1}{2}\sum_{k=0}^{\infty} k\chi_{2ab}^{(n,m)}(k)q^{\frac{k^2}{4ab}}.$$

When $(n, m) = (a - 1, 1)$,

Theorem (Hikami-Kirillov)

$\langle T(a, b)\rangle_N =$

$$\tilde{\Phi}^{a,b}_{a-1,1}(1/N) \times \exp\left(\frac{(ab - a - b)^2}{2abN}\pi i\right)$$

$\langle T(a, b)\rangle_N$: Kashaev invariant of $T(a, b)$

Asymptotic expansion for $N \to \infty$:

$$\tilde{\Phi}^{a,b}_{n,m}(1/N) + (-iN)^{\frac{3}{2}}\text{-term} \sim \sum_{k=0}^{\infty} \frac{T^{n,m}(k)}{k!} \left(\frac{\pi}{2abiN}\right)^k,$$

$$T^{n,m}(k) = \frac{1}{2}(-1)^{k+1} L(-2k-1, \chi_{2ab}^{(n,m)}).$$

$$L(s, \chi_{2ab}^{(n,m)}) = \sum_{k \geq 1} \frac{\chi_{2ab}^{(n,m)}(k)}{k^s}$$

$$\sum_{k\geq 1} \chi_{2ab}^{(a-1,1)}(k)T^{-k} = \frac{T - T^{-1}}{\Delta_{a,b}(T^2)}.$$

Consider $s = \log T$ for $L\left(s, \chi_{2ab}^{(n,m)}\right)$

$$\sum_{k\geq 1} \chi_{2ab}^{(n,m)}(k)T^{-\log k} = L\left(\log T, \chi_{2ab}^{(n,m)}\right)$$

When $(n, m) = (a - 1, 1)$, essentially
$L\left(s, \chi_{2ab}^{(n,m)}\right)$ corresponds to Alexander poly.

Relation between $\chi_{2ab}^{(n,m)}$ and Dirichlet char.

$$\chi_{2ab}^{(n,m)} = \frac{1}{\phi(2ab)} \sum_{\chi\,:\,\text{even}} c_\chi(a,b,n,m)\chi,$$

χ runs through all the even Dirichlet characters modulo $2ab$ (that is, $\chi(-1) = 1$)

$$c_\chi(a,b,n,m) = 2\overline{(\chi(nb-ma) - \chi(nb+ma))}.$$

$$L\left(s, \chi_{2ab}^{(n,m)}\right) = \frac{1}{\phi(2ab)} \sum_{\chi\,:\,\text{even}} c_\chi(a,b,n,m)L(s,\chi).$$

$$\phi(2ab) = \#\left(\mathbb{Z}/2ab\mathbb{Z}\right)^\times$$

$$\zeta(A_{a,b}(L,M),s) =$$

$$\begin{cases} \left(\prod_{d|2ab} \frac{1}{\zeta(\mathbb{Q}_d,s)}\right) \frac{\zeta(s-1)^3}{\zeta(s)^2}, & \text{if } a,\ b > 2, \\ \left(\prod_{d|2ab,\ d\nmid ab} \frac{1}{\zeta(\mathbb{Q}_d,s)}\right) \frac{\zeta(s-1)^2}{\zeta(s)}, & \text{if } \begin{matrix}(a,b) = \\ (2,2m+1)\end{matrix}. \end{cases}$$

$$\zeta(\mathbb{Q}_d,s) = \prod_{\chi} L(s,\chi)$$

$\chi : (\mathbb{Z}/d\mathbb{Z}) \to \mathbb{C}$: Dirichlet character

$$L\left(s,\chi_{2ab}^{(n,m)}\right) = \frac{1}{\phi(2ab)} \sum_{\chi\,:\,\text{even}} c_\chi(a,b,n,m) L(s,\chi).$$

In particular, for $(2,3)$-torus knot, there is only one character

$$\chi_{12}^{(1,1)} = \chi_{12}^{(1,2)} = \text{ Dirichlet character modulo 12}$$

which is the unique even Dirichlet character modulo 12

$$\zeta(A_{2,3}(L,M),s) \simeq \frac{\zeta(s-1)^2}{\zeta(s)} \times \zeta(\mathbb{Q}_4,s)^{-1}\zeta(\mathbb{Q}_{12},s)^{-1}$$

$$\zeta(\mathbb{Q}_{12},s) = \zeta(s)L(s,\chi_{12}^{(1,1)})L(s,\chi_2)L(s,\chi_3)$$

χ_2, χ_3 : the other (odd) Dirichlet char. of $(\mathbb{Z}/12\mathbb{Z})^{\times}$

question

Is there a topological interpretation for $\tilde{\Phi}^{a,b}_{n,m}(\tau)$ and $L\left(s, \chi^{(n,m)}_{2ab}\right)$ for $(n,m) \neq (a-1, 1)$?

question

Is there a direct relation between $\langle T(a,b)\rangle_N$ (or colored Jones polynomial of $T(a,b)$) and $L\left(s, \chi^{(n,m)}_{2ab}\right)$ (or othere components of $\zeta(A_{a,b}(L,M), s)$)?

END