The topologies of box complexes and the chromatic numbers of graphs

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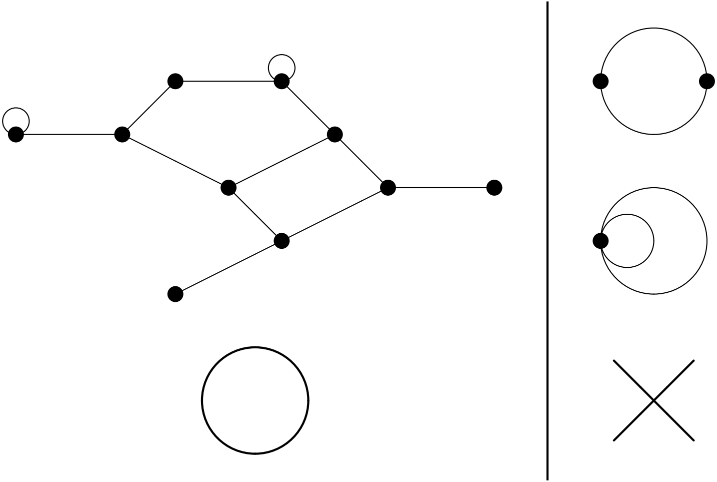
Definition of graphs

Definition 1

A graph is a pair (V, E) s.t.

- V is a set.
- E is a subset of $V \times V$ s.t. $(x, y) \in E$ implies $(y, x) \in E$.

For a graph $G = (V, E)$, V is written by $V(G)$, and E is written by $E(G)$.



Definition of graph homomorphisms

Definition 2

A graph homomorphism from G to H is a map $f : V(G) \to V(H)$ s.t. $(f \times f)(E(G)) \subset E(H)$.

The following is a classical problem in graph theory.

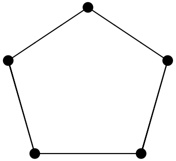
Problem 1 (The existence problem of graph homomorphisms)

Given two graphs G and H. Consider an easy method to determine whether $\exists f : G \to H$ or not.

Odd girth $g_0(G)$

For a positive integer n, the n-cycle graph C_n is defined by

- $V(C_n) = \mathbb{Z}/n\mathbb{Z}$.
- $E(C_n) = \{(x, x \pm 1) \mid x \in \mathbb{Z}/n\mathbb{Z}\}$.



C_5

Odd girth $g_0(G)$

Definition 3

Let G be a graph. The odd girth $g_0(G)$ of G is defined by

$$g_0(G) = \inf\{n \geq 1 \mid n \text{ is odd and } \exists C_n \to G.\}$$

If $g_0(G) = 1$, then G has a loop. (Hence if G is non-looped, then $g_0(G) \geq 3$.)

Lemma 1

If $\exists G \to H$, then $g_0(G) \geq g_0(H)$.

Proof.

Put $n = g_0(G)$. Then $\exists C_n \to G$, hence $\exists C_n \to H$. Therefore $g_0(H) \leq n = g_0(G)$. □

The existence problem of the graph homomorphism is related to the existence problems of the $\mathbb{Z}_2$-equivariant maps, via the box complex $B(G)$.

Definition of simplicial complex

Definition 4

An (abstract) simplicial complex is a pair (V, Δ) satisfying the followings :

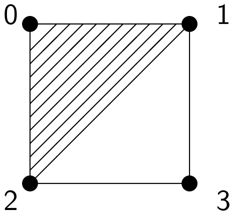
- V is a set.
- Δ is a family of finite subsets of V.
- $\forall v,\ v \in V \Rightarrow \{v\} \in \Delta$.
- $\forall \sigma \in \Delta,\ \forall \tau \in 2^V,\ \tau \subset \sigma \Rightarrow \tau \in \Delta$.

Geometric realization

$$V = \{0, 1, 2, 3\}$$

$$\Delta = \{\sigma \mid \sigma \subset \{0, 1, 2\}, \{1, 3\}, \text{ or } \{2, 3\}.\}$$

$$|\Delta| =$$



Definition of order complex

Definition 5

A partially ordered set is often called a poset. A subset $\sigma \subset P$ is called a chain if the restriction of the order of P to σ is totally ordered. The order complex $\Delta(P)$ is the simplicial complex

- $V(\Delta(P)) = P$.
- $\Delta(P) = \{\sigma \subset P \mid \sigma \text{ is a finite chain of } P.\}$.

Definition of box complex

Definition 6

The box complex $B(G)$ of a graph G is a poset

$$B(G) = \{(\sigma, \tau) \mid \sigma, \tau \in 2^{V(G)} \setminus \{\emptyset\}, \sigma \times \tau \subset E(G).\}$$

with the order such that $(\sigma, \tau) \leq (\sigma', \tau') \Leftrightarrow \sigma \subset \sigma'$ and $\tau \subset \tau'$.

Remark that $B(G)$ has the $\mathbb{Z}_2$-action $(\sigma, \tau) \leftrightarrow (\tau, \sigma)$. For a graph homomorphism $f : G \to H$, the map $B(G) \to B(H)$, $(\sigma, \tau) \mapsto (f(\sigma), f(\tau))$ is $\mathbb{Z}_2$-equivariant. Hence if we can show that $\nexists B(G) \xrightarrow{\mathbb{Z}_2} B(H)$, then we have $\nexists G \to H$.

Definition of K_n

Definition 7

Let $n \geq 0$. The graph K_n is defined by

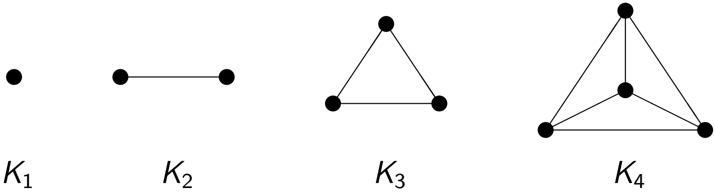
- $V(K_n) = \{1, \cdots, n\}$.
- $E(K_n) = \{(x, y) \mid x \neq y\}$.

A graph homomorphism $G \to K_n$ is called an n-coloring of G. The chromatic number $\chi(G)$ of the graph G is defined by

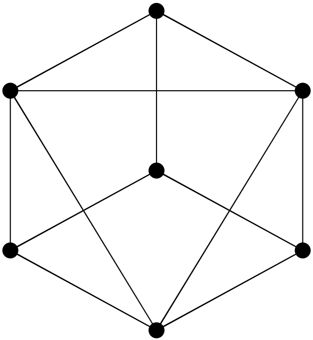
$$\chi(G) = \inf\{n \geq 0 \mid \exists G \to K_n.\}.$$

To compute $\chi(G)$ is called the graph coloring problem. Since $g_0(K_n) = 3$ for $n \geq 3$, the odd girth is not useful to this problem.

K_n

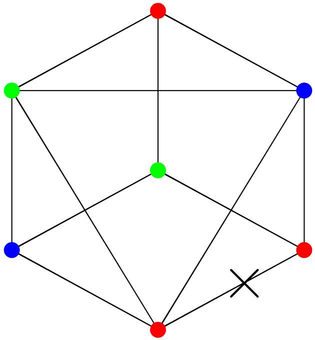


An example of coloring



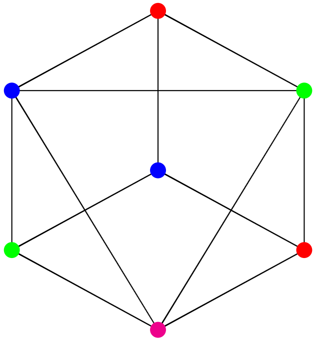
G

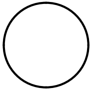
An example of coloring



G

An example of coloring





$$\chi(G) = 4$$

Neighborhood complex

For a graph G and for $v \in V(G)$, the neighborhood $N(v)$ of v is defined by $N(v) = \{w \in V(G) \mid (v, w) \in E(G)\}$.

Definition 8

The neighborhood complex $N(G)$ of a graph G is the simplicial complex

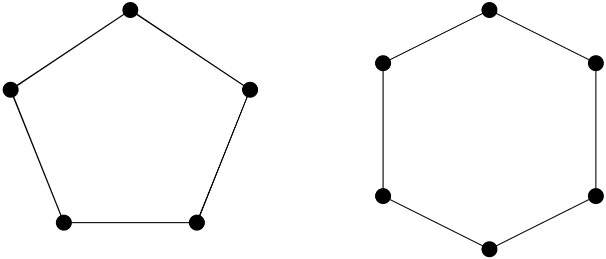
- $V(N(G)) = \{v \in V(G) \mid N(v) \neq \emptyset\}$.
- $N(G) = \{\sigma \subset V(G) \mid \#\sigma < \infty$, and $\exists v \in V(G)$ s.t. $\sigma \subset N(v).\}$.

Theorem 2 (Babson-Kozlov '06)

$$N(G) \simeq B(G).$$

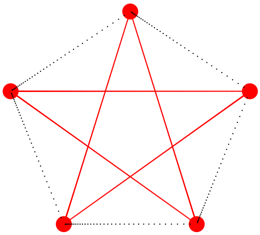
C_5 and C_6

C_5 C_6



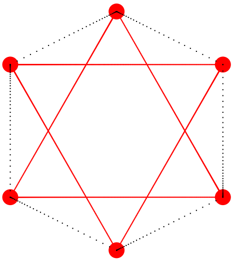
$N(C_5)$ and $N(C_6)$

C_5



$N(C_5) \simeq S^1$

C_6



$N(C_6) \simeq S^1 \sqcup S^1$

Lovász's theorem

Theorem 3 (Lovász)

Let $n \geq -1$. If $N(G)$ is n-connected, then $\chi(G) \geq n+3$.

Proof.

Since $B(G) \simeq N(G)$, $B(G)$ is n-connected. By the Gysin sequence, we have $w_1(B(G))^{n+1} \neq 0$. On the other hand, suppose $\exists G \to K_m$. Then $B(G) \xrightarrow{\mathbb{Z}_2} B(K_m) \simeq S^{m-2}$, we have $w_1(B(G))^{m-1} = 0$. Hence we have

$$n + 1 < m - 1.$$

□

Kneser graphs

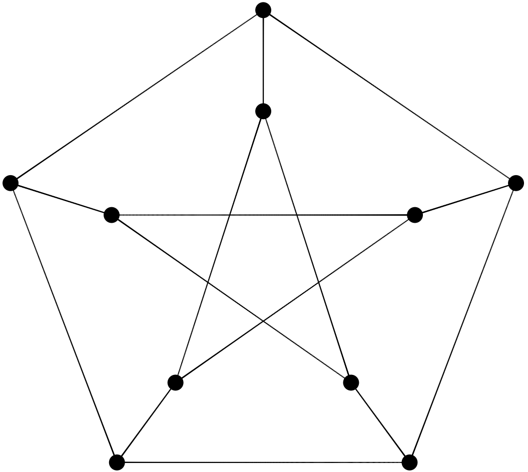
Definition 9

Let $k \geq 0$ and $n \geq 2k$. The Kneser graph $KG_{n,k}$ is defined by

- $V(KG_{n,k}) = \{\sigma \subset \{1, \cdots, n\} \mid \# \sigma = k\}$.
- $E(KG_{n,k}) = \{(\sigma, \tau) \mid \sigma \cap \tau = \emptyset.\}$.

It is easy to see $\chi(KG_{n,k}) \leq n - 2k + 2$, and Kneser conjectured $\chi(KG_{n,k}) = n - 2k + 2$ in 1955 (Kneser's conjecture). Lovász proved that $N(KG_{n,k})$ is $(n - 2k - 1)$-connected, and show that $\chi(KG_{n,k}) = n - 2k + 2$ in 1978.

$KG_{5,2}$



$KG_{5,2}$

($\mathbb{Z}_2$)-topologies of $B(G)$ and $N(G)$ and $\chi(G)$

- (Lovász) For a connected graph G, $N(G)$ (or $B(G)$) is connected iff $\chi(G) \geq 3$.

Lovász expected that there is a topological invariant of $N(G)$ which is equivalent to $\chi(G)$.

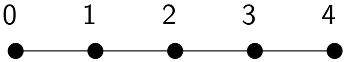
- (Walker '83) There is no homotopy invariant of $N(G)$ (hence of $B(G)$) which is equivalent to $\chi(G)$.
- (M) There is no topological invariant of $N(G)$ and $B(G)$ which is equivalent to $\chi(G)$.
- (M) There is no $\mathbb{Z}_2$-homotopy invariant of $B(G)$ which is equivalent to $\chi(G)$.
- Whether there is a $\mathbb{Z}_2$-topological invariant of $B(G)$ which is equivalent to $\chi(G)$ is still open.

r-fundamental groups

From now on, we fix a positive integer r. A based graph is a pair (G, v) where G is a graph and $v \in V(G)$. The r-fundamental group $\pi_1^r(G, v)$ is a group whose definition is similar to the fundamental group of topological spaces. Especially, the 2-fundamental group is similar to the fundamental group of $N(G)$. But $\pi_1^r(G, v)$ can be directly used to the existence problem of the graph homomorphisms.

r-fundamental groups

Let L_n denote the graph defined by $V(L_n) = \{0, 1, \cdots, n\}$ and $E(L_n) = \{(x, y) \mid |x - y| = 1\}$.



L_4

Let (G, v) be a based graph. A graph homomorphism $L_n \to G$ s.t. $0, n \mapsto v$ is called a loop of (G, v) with length n.

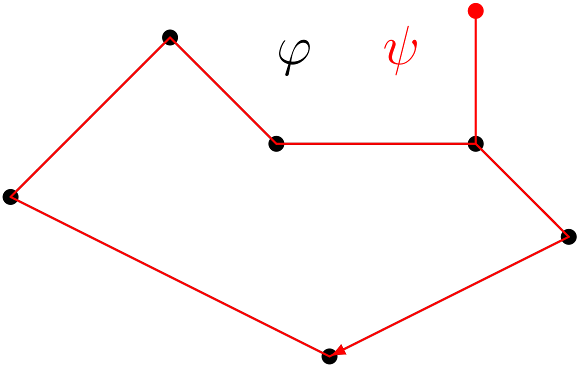
r-fundamental groups

Let $L(G, v)$ denote the set of loops of (G, v). For $\varphi \in L(G, v)$, we write $l(\varphi)$ for the length of φ.
Fix a positive integer r, consider the following two conditions (I) and $(\mathrm{II})_r$ for loops φ, ψ.

(I) $l(\psi) = l(\varphi) + 2$ and $\exists x \in \{0, 1, \cdots, n\}$ s.t. $\varphi(i) = \psi(i)$ for $i \leq x$ and $\varphi(i) = \psi(i+2)$ for $i \geq x$.

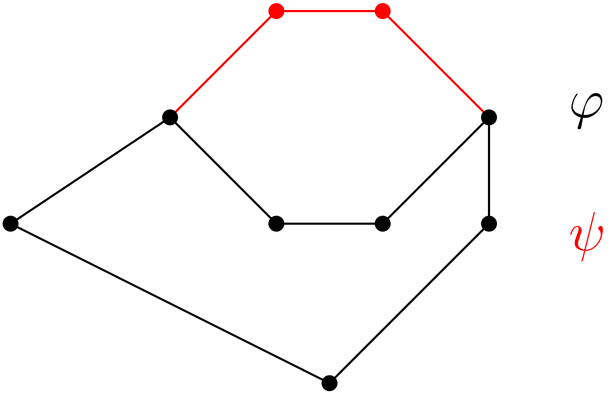
$(\mathrm{II})_r$ $l(\varphi) = l(\psi)$ and $\#\{i \in \{0, 1, \cdots, l(\varphi) \mid \varphi(i) = \psi(i)\}\} < r$.

Condition (I)



Condition $(\mathrm{II})_r$

The case $r = 3$.



r-fundamental group $\pi_1^r(G, v)$

Definition 10

Let $\simeq_r$ denote the equivalence relation generated by the conditions (I) and (II)$_r$. Put

$$\pi_1^r(G, v) = L(G, v)/\simeq_r$$

and call this the r-fundamental group of the based graph (G, v).

$\pi_1^r(G, v)$ become a group with compositions of loops.

Parities

The map

$$\pi_1^r(G, v) \to \mathbb{Z}_2, [\varphi]_r \mapsto (l(\varphi) \bmod. 2)$$

is a well-defined group homomorphism, and the kernel is written by $\pi_1^r(G, v)_{ev}$, and is called the even part of $\pi_1^r(G, v)$. Let G_0 denote the connected component of G containing v. Then $\pi_1^r(G, v) = \pi_1^r(G, v)_{ev}$ iff $\chi(G_0) \leq 2$.

r-neighborhood complex

Let G be a graph and $v \in V(G)$. The s-neighborhood $N_s(v)$ is defined as follows.

- $N_1(v) = N(v)$.
- $N_{s+1}(v) = \bigcup_{w \in N_s(v)} N(w)$.

Definition 11

The r-neighborhood complex $N_r(G)$ is the simplicial complex

- $V(N_r(G)) = \{v \in V(G) \mid N(v) \neq \emptyset\}$.
- $N_r(G) = \{\sigma \subset V(G) \mid \#\sigma < \infty, \exists v \in V(G) \text{ s.t. } \sigma \subset N_r(v).\}$.

In particular, $N_1(G) = N(G)$.

$\pi_1(N_r(G))$

Theorem 4 (M)

Let (G, v) be a based graph s.t. $N(v) \neq \emptyset$. Then

$$\pi_1(N_r(G), v) \cong \pi_1^{2r}(G, v)_{ev}.$$

Especially $\pi_1(N(G), v) \cong \pi_1^2(G, v)_{ev}$.

Length and stable length

Let $\alpha \in \pi_1^r(G, v)$. Put

$$l(\alpha) = \inf\{l(\varphi) \mid \varphi \in \alpha\}$$

and call this the length of α.

Length and stable length

Proposition 5

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of non-negative real numbers s.t.

$$a_{n+m} \leq a_n + a_m \ (\forall n, m \in \mathbb{N}).$$

Then $\lim_{n \to \infty} a_n / n$ exists and

$$\lim_{n \to \infty} \frac{a_n}{n} = \inf_{n \in \mathbb{N}} \frac{a_n}{n}.$$

For $\alpha \in \pi_1^r(G, v)$, the sequence $(l(\alpha^n))_{n \in \mathbb{N}}$ satisfies the above hypothesis, and we define the stable length of α by

$$l_s(\alpha) := \lim_{n \to \infty} \frac{l(\alpha^n)}{n}.$$

Application of π_1^r

r-fundamental groups can be applied to the existence problem of graph homomorphisms.
Let $f : (G, v) \to (H, w)$ be a based graph homomorphism. Then the map $f_* : \pi_1^r(G, v) \to \pi_1^r(H, w)$, $[\varphi]_r \mapsto [f \circ \varphi]_r$ is well-defined, and satisfies the followings:

(0) f_* is a group homomorphism.

(1) f_* preserves parities.

(2) $l(f_*(\alpha)) \leq l(\alpha)$.

(3) $l_s(f_*(\alpha)) \leq l_s(\alpha)$.

For example, let us consider the existence of graph homomorphisms to odd cycles.

$\pi_1^r(C_n)$

Proposition 6

The followings hold.

(1) *For odd $n \geq 3$, we have*

$$\pi_1^r(C_n) = \begin{cases} \mathbb{Z}\alpha & (r < n) \\ \mathbb{Z}/2 & (r \geq n), \end{cases}$$

and the generator α is odd and $l_s(\alpha) = n$ if $r < n$.

(2) *For even $n \geq 4$, we have*

$$\pi_1^r(C_n) = \begin{cases} \mathbb{Z}\alpha & (r < n/2) \\ 1 & (r \geq n/2). \end{cases}$$

and the generator α is even and $l_s(\alpha) = n$ if $r < n/2$.

Theorem 7 (M)

Let n be an odd integer s.t. $n \geq 3$, and G a connected graph. If $\exists G \to C_n$, then $l_s(\beta) \geq n$ for any $r < n$ and any odd element β of $\pi_1^r(G, v)$.

Proof.

Suppose there is a graph homomorphism $f : G \to C_n$. Since $f_*(\beta)$ is odd, $\exists k \in \mathbb{Z}$ s.t. $f_*(\beta) = \alpha^{2k+1}$. Hence

$$l_s(\beta) \geq l_s(f_*(\beta)) = l_s(\alpha^{2k+1}) = |2k+1| l_s(\alpha) \geq l_s(\alpha) = n.$$

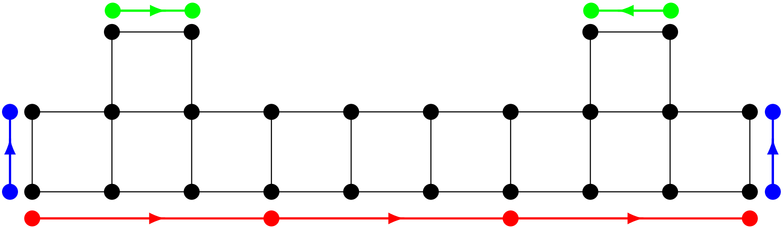
□

Examples

Recall that $\chi(KG_{n,k}) = n - 2k + 2$. Hence $\exists KG_{2k+1,k} \to K_3 \cong C_3$. For a positive integer $k \geq 1$, $\pi_1^3(KG_{2k+1,k}) \cong \mathbb{Z}/2$. Hence by the previous theorem, we have $\nexists KG_{2k+1,k} \to C_5$. On the other hand, it is known that the odd girth $g_0(KG_{2k+1,k})$ is equal to $2k + 1$.

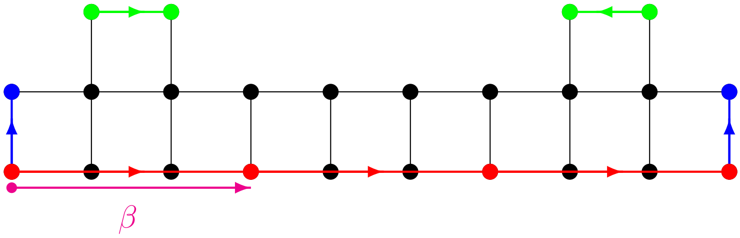
The graph X

$$\pi_1^2(X) \cong \mathbb{Z}$$



Let β denote the generator of $\pi_1^2(X) \cong \mathbb{Z}$.

Then we have $l_s(\beta) = \dfrac{7}{3}$.



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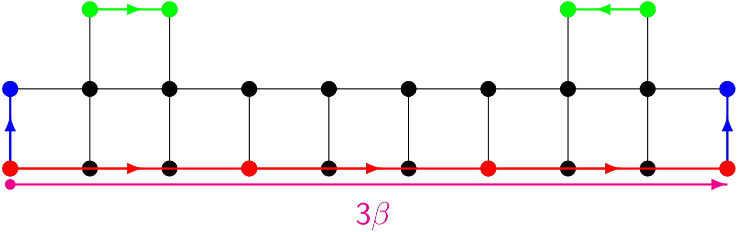

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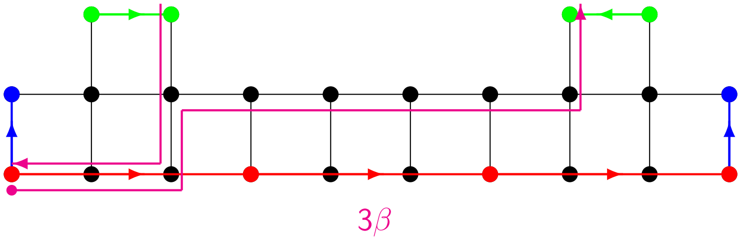

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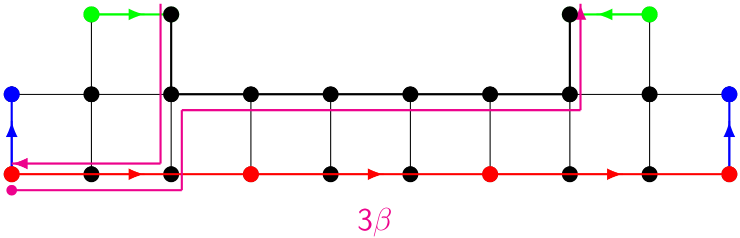
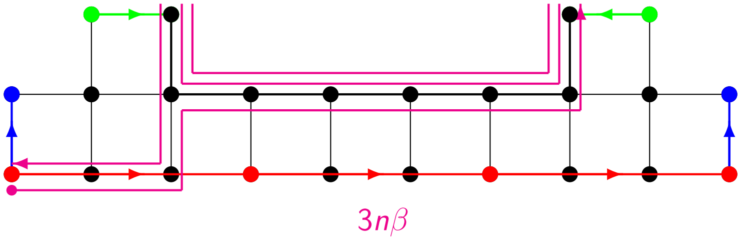


From the following, we have $l(\beta^{3n}) \approx 7n$.

Hence we have $l_s(\beta) = \lim_{n\to\infty} \dfrac{l(\beta^{3n})}{3n} = \lim_{n\to\infty} \dfrac{7n}{3n} = \dfrac{7}{3}$



Hence the stable length of the generator β of $\pi_1^2(X) \cong \mathbb{Z}$ is smaller than 3. Since β is odd, we have $\nexists X \to C_3 \cong K_3$. This implies that $\chi(X) > 3$.
Since $\pi_1^2(G)_{ev} \cong \pi_1(N(G))$, $\pi_1(N(K_3)) \to \pi_1(N(X))$ is an isomorphism. Indeed, this $N(G) \hookrightarrow N(X)$ is homotopy equivalence (hence $B(G) \hookrightarrow B(X)$ is $\mathbb{Z}_2$-homotopy equivalence) s.t. $\chi(G) \neq \chi(X)$.

r-covering maps

Recall that the s-neighborhood $N_s(v)$ is defined by

$$N_1(v) = N(v) \text{ and } N_{s+1}(v) = \bigcup_{w \in N_s(v)} N(w).$$

Definition 12

A graph homomorphism $p : G \to H$ is said to be an r-covering map if for any $v \in V(G)$,

$$p|_{N_s(v)} : N_s(v) \to N_s(p(v))$$

is bijective for $1 \leq s \leq r$.

r-covering maps

There is similar relations between $\pi_1^r(G, v)$ and r-covering maps, as is the case of covering space theory.

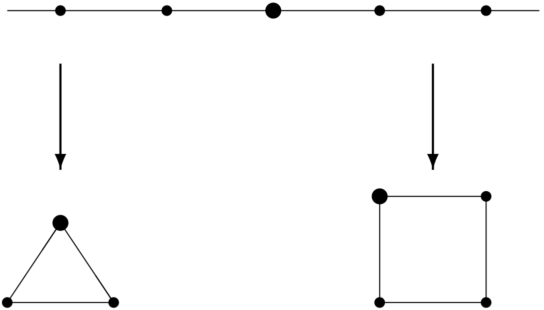
(1) If $p : (G, v) \to (H, w)$ is an r-covering map, then $p_* : \pi_1^r(G, v) \to \pi_1^r(H, w)$ is injective.

(2) For each $\Gamma \leq \pi_1^r(G, v)$, there is an r-covering map $(G_\Gamma, v_\Gamma) \to (G, v)$ s.t. G_Γ is connected, and $p_*(\pi_1^r(G_\Gamma, v_\Gamma)) = \Gamma$, and this is unique up to isomorphisms.

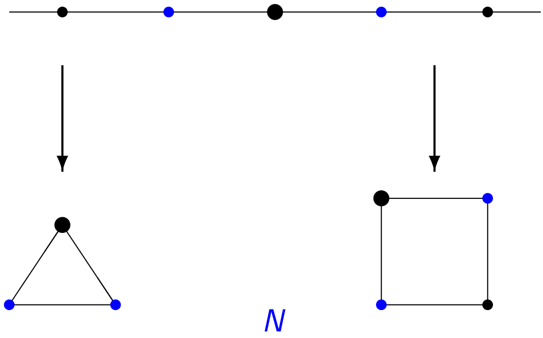
(3) Suppose $f : (T, x) \to (H, w)$ is a graph homomorphism and $p : (G, v) \to (H, w)$ an r-covering map. If T is connected and $f_* \pi_1^r(T, x) \subset p_* \pi_1^r(G, v)$, then $\exists g : (T, x) \to (G, v)$ s.t. $p \circ g = f$.

r-covering maps

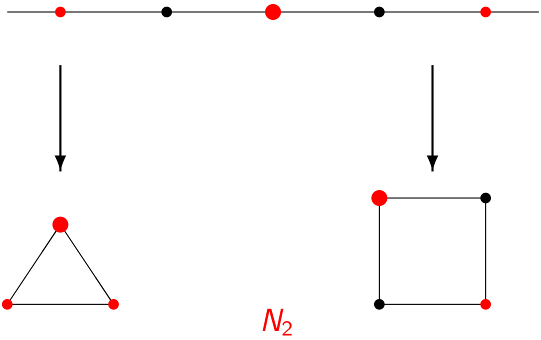
The 2nd projection $K_2 \times G \to G$ is an r-covering map for any $r \geq 1$. If G is connected and $\chi(G) \geq 3$, then $K_2 \times G$ is connected, and the associated subgroup of $\pi_1^r(G)$ is the even part $\pi_1^r(G)_{ev}$.



Examples



Examples



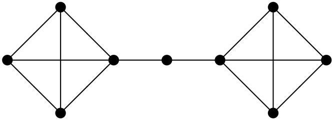
Examples

- $\pi_1^2(K_n) \cong \mathbb{Z}/2$ for $n \geq 4$, connected 2-covering over K_n $(n \geq 4)$ is G or $K_2 \times G$.
- Since $\pi_1^3(KG_{2k+1,k}) \cong \mathbb{Z}/2$, connected 3-covering over $KG_{2k+1,k}$ is $KG_{2k+1,k}$ or $K_2 \times KG_{2k+1,k}$. But since $\pi_1^2(KG_{2k+1,k})$ is a free group, and hence there are many connected 2-covering maps over $KG_{2k+1,k}$.

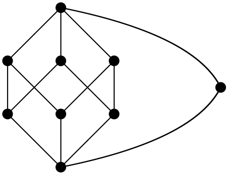
Hence if $K_2 \times G \cong K_2 \times H$, we have

$$\pi_1^r(G)_{ev} \cong \pi_1^r(K_2 \times G) \cong \pi_1^r(K_2 \times H) \cong \pi_1^r(H)_{ev}.$$

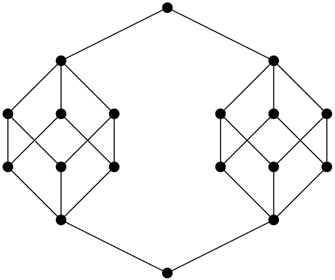
Since $\pi_1^2(N(G))_{ev} \cong \pi_1^2(G)_{ev}$, we have that if $K_2 \times G \cong K_2 \times H$, then $\pi_1(N(G)) \cong \pi_1(N(H))$. Indeed, we can say that if $K_2 \times G \cong K_2 \times H$, then $N(G) \cong N(H)$ and $B(G) \cong B(H)$ as poset.



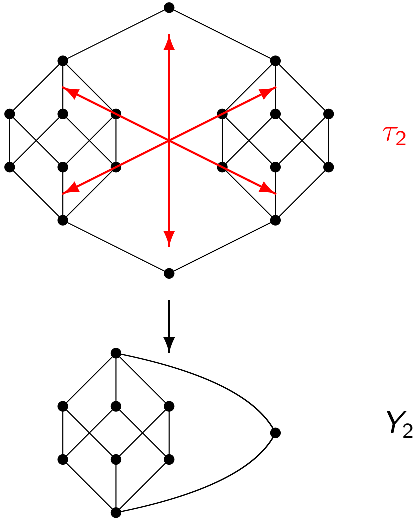
$Y_1,\ \chi(Y_1) = 4.$

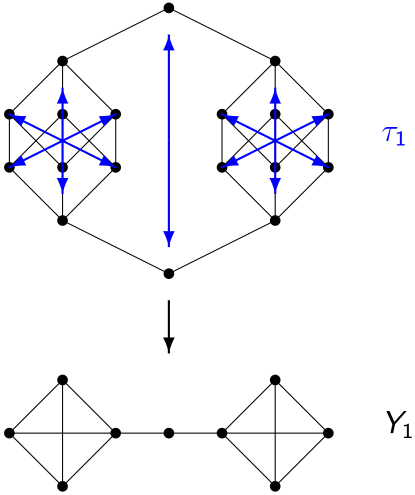


$Y_2,\ \chi(Y_2) = 3.$



$$K_2 \times Y_1 \cong K_2 \times Y_2 \cong Z$$


τ_2
Y_2


τ_1
Y_1

For further researches

Let n be a positive integer, and G a connected graph s.t. $\#N(v) = n$ for $v \in V(G)$. Consider the following property.

(∗) For $v, w \in V(G)$ with $N(v) \cap N(w) \neq 0$, then $\#(N(v) \cap N(w)) > n/2$.

Then the diameter of G is smaller than 4. (Especially, G is finite.)

For further researches

Definition 13

A graph property (P) is said to be *r-local* if for a surjective r-covering map $p : G \to H$, G satisfies (P) if and only if H satisfies (P).

Then the condition $(*)$ is a 2-local property.
Suppose a 2-local property (P) implies the finiteness of connected graphs. Suppose a connected graph G satisfies (P). Then the universal 2-covering of G satisfies it and is finite. Hence $\pi_1^2(G)$ is finite. This implies that $\chi(G) \neq 3$.

For further researches

Problem 2

Find the r-local property s.t. a connected graph satisfying such a property is finite.