Iwasawa invariants of links

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§ 1. Introduction

Low-dimensional topology	$\overset{\text{analogy}}{\longleftrightarrow}$	Number theory
Knots		Primes

[Kapranov], [Mazur], [Morishita], [Reznikov], ...

Alexander-Fox theory	$\longleftrightarrow$	Iwasawa theory

cf. [Morishita'12] *Knots and Primes*, Springer, for basic analogies.

Aims of this talk:

- Introducing "Iwasawa invariants" (defined by Morishita *et al.* for an infinite tower of cyclic branched covers of a link)
- How to calculate "Iwasawa invariants"
- A problem motivated by "Greenberg's conjecture" in Iwasawa theory

M : oriented connected closed 3-manifold $\longleftrightarrow$ k : number field, $\dim_{\mathbb{Q}} k < \infty$

$\downarrow$ finite cover branched over some link $\longleftrightarrow$ $|$ algebraic extension ramified over some prime numbers

S^3 $\qquad$ $\mathbb{Q}$

$H_1(M, \mathbb{Z}) \simeq \pi_1(M)^{ab}$ $\longleftrightarrow$ $Cl(k) \simeq \mathrm{Gal}\left(\text{maximal unramified extension} \middle/ k \right)^{ab}$

Assume $\# H_1(M, \mathbb{Z}) < \infty$. $\qquad$ ideal class group, $\# Cl(k) < \infty$

Fix a prime number p, both in knot theory side and in number theory side. (We will consider branched covers of degree p^n.)

$Cl(k) = \{1\} \ \Leftrightarrow$ The integer ring $\mathcal{O}_k$ is a Principal Ideal Domain.

e.g., $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})} = \mathbb{Z}[\sqrt{-1}]$ is PID.

$\mathcal{O}_{\mathbb{Q}(\zeta_{37})} = \mathbb{Z}[\zeta_{37}]$ is *not* PID, and $\# Cl(\mathbb{Q}(\zeta_{37})) \equiv 0 \pmod{37}$, where $\zeta_{37} = e^{\frac{2\pi i}{37}}$.

§ 2. Iwasawa invariants (λ, μ, ν)

$$L = K_1 \cup \cdots \cup K_r \subset M \quad \longleftrightarrow \quad S = \{\wp \mid p \in \wp \subset \mathcal{O}_k\}$$

r-component link $\longleftrightarrow$ prime ideals of $\mathcal{O}_k$ over p

X : the exterior of L

$$G_L = \pi_1(X) \quad \longleftrightarrow \quad G_S = \mathrm{Gal}\left(\begin{matrix}\text{maximal extension}\\ \text{unram. outside } S\end{matrix} \Big/ k\right)^{\text{pro-}p}$$

$$\sigma \downarrow \qquad\qquad \downarrow$$

$$\mathbb{Z} \simeq \mathrm{Aut}(X_\sigma/X) \qquad \mathbb{Z}_p \simeq \mathrm{Gal}(k_\infty/k) \quad \text{“}\mathbb{Z}_p\text{-extension”}$$

$$X \underset{\deg p^n}{\leftarrow} X_{\sigma,p^n} \leftarrow X_\sigma \quad \longleftrightarrow \quad k \underset{\deg p^n}{\subset} k_n \subset k_\infty$$

$$\cap \qquad\qquad \cap$$

$M \leftarrow M_{\sigma,p^n}$: the Fox completion (branched covers of L)

- $G^{\text{pro-}p} = \varprojlim \big(\text{quotient } p\text{-groups}\big)$: the pro-p completion of G
- $\mathbb{Z}_p = \mathbb{Z}^{\text{pro-}p}$: (the additive group of) the ring of p-adic integers, $\neq \mathbb{Z}/p\mathbb{Z}$

Iwasawa's class number formula.

$\exists\ (\lambda, \mu, \nu) = (\lambda_{k_\infty}, \mu_{k_\infty}, \nu_{k_\infty}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ such that

$$v_p\big(\#Cl(k_n)\big) = \lambda\, n + \mu\, p^n + \nu \quad \text{for } \forall\, n \gg 0,$$

where v_p is the p-adic additive valuation, normalized as $v_p(p) = 1$.

Theorem 1. [Morishita'04], [Hillman-Matei-Morishita'06], [KM'08]

Assume $\#H_1(M_{\sigma,p^n}, \mathbb{Z}) < \infty$ for all $n \geq 0$. Then

$\exists\ (\lambda, \mu, \nu) = (\lambda_{L,\sigma}, \mu_{L,\sigma}, \nu_{L,\sigma}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ such that

$$v_p\big(H_1(M_{\sigma,p^n}, \mathbb{Z})\big) = \lambda\, n + \mu\, p^n + \nu \quad \text{for } \forall\, n \gg 0.$$

- [Ueki] gave another proof (analogous to original proof of [Iwasawa'58]).

Problem 1. Remove the assumption, i.e., generalize to $\mathrm{Tor}\, H_1(M_{\sigma,p^n}, \mathbb{Z})$.

§ 3. Proof and calculation (Assume $M = S^3$ for simplicity.)

$\sigma : G_L \twoheadrightarrow G_L^{ab} \twoheadrightarrow \mathbb{Z}$: [meridian m_i of K_i] $\mapsto t_i \mapsto z_i$

- We may assume $\prod_{i=1}^{r} z_i \neq 0$ by removing unbranched components.

$\Delta_L(t_1, \cdots, t_r)$: Alexander polynomial of L

$$\Delta_{L,\sigma}(t) := (t-1)^{\min\{1,r-1\}} \Delta_L(t^{z_1}, \ldots, t^{z_r}) \in \mathbb{Z}[t^{\pm 1}]$$

$$\| \qquad \qquad \qquad \qquad \cap$$

$$\Delta_{L,\sigma}(1+T) \underset{\uparrow}{=} p^{\mu_{L,\sigma}} P_{L,\sigma}(T)\, u(T) \in \mathbb{Z}_p[[T]]$$

p-adic Weierstrass preparation theorem

$\exists !$ monic $P_{L,\sigma}(T) \equiv T^{\lambda_{L,\sigma}} \pmod p$, $u(T) \in \mathbb{Z}_p[[T]]^\times$

$$\Delta_{L,\sigma}(t) \in \mathbb{Z}[t^{\pm 1}] \quad \longleftrightarrow \quad p^{\mu_{k_\infty}} \left(T^{\lambda_{k_\infty}} + p\,(\text{lower deg.})\right) \in \mathbb{Z}_p[T]$$

is the characteristic poly. of $H_1(X_\sigma, \mathbb{Z})$ over $\mathbb{Z}[t^{\pm 1}]$ $\longleftrightarrow$ char. poly. of $\mathfrak{X} = \mathrm{Ker}(G_S \twoheadrightarrow \mathbb{Z}_p)^{ab,ur}$ over $\mathbb{Z}_p[[T]] \simeq \mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$

- By taking v_p of the following, we obtain Theorem 1 for $M = S^3$.

Theorem. [Sakuma'79], [Mayberry-Murasugi'82], [Porti'04]

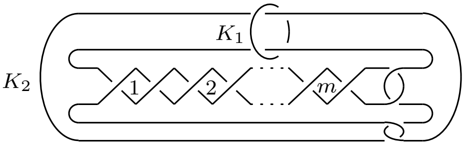
$$|H_1(M_{\sigma,p^n}, \mathbb{Z})| = |H_1(M_{\sigma,p^v}, \mathbb{Z})| \cdot \Bigg| \prod_{\substack{\zeta^{p^n}=1 \\ \zeta^{p^v}\neq 1}} \Delta_{L,\sigma}(\zeta) \Bigg|$$

for all $n \geq v := \max_i v_p(z_i)$, where $|H| = \#H$ if $\#H < \infty$, and 0 otherwise.

- By this formula, one can check whether $\#H_1(M_{\sigma,p^n}, \mathbb{Z}) < \infty$ or not.
- [Iwasawa'72] gave another proof of (a part of) his formula in this way.
- For $M \neq S^3$, we need [Sakuma'81] generalizing the above formula.

$$\begin{array}{ccc} H_1(M_{\sigma,p^n}, \mathbb{Z}) & & Cl(k_n)^{\text{pro-}p} \\ \sim \mathrm{Tor}\big(H_1(X_\sigma, \mathbb{Z})/(t^{p^n}-1)\big) & \longleftrightarrow & \sim \mathfrak{X}/((1+T)^{p^n}-1) \end{array}$$

Example [KM'13] Let $L = K_1 \cup K_2 \subset S^3$ be the following link.



Then $\Delta_L(t_1, t_2) = m(t_1 - 1)(t_2 - 1)^3$, and hence

$$\begin{aligned} \Delta_{L,\sigma} &= m(t-1)(t^{z_1}-1)(t^{z_2}-1)^3 \\ &= p^{v_p(m)} T\big((1+T)^{p^{v_p(z_1)}} - 1\big)\big((1+T)^{p^{v_p(z_2)}} - 1\big)^3 u(T). \end{aligned}$$

Since $\gcd\big(\Delta_{L,\sigma}(t),\ p^n\text{th cyclotomic poly.}\big) = 1$ for $\forall\, n > v = v_p(z_1 z_2)$,

we have $\# H_1(M_{\sigma,p^n}, \mathbb{Z}) < \infty$ for $\forall\, n \geq 0$, and

$$\lambda_{L,\sigma} = 1 + p^{v_p(z_1)} + 3p^{v_p(z_2)}\ , \qquad \mu_{L,\sigma} = v_p(m)\ .$$

§ 4. Existence of L and σ with prescribed Iwasawa invariants

Problem 2. Determine the possible values of $(\lambda_{L,\sigma}, \mu_{L,\sigma}, \nu_{L,\sigma})$.

Theorem 2. [KM'13] Assume $M = S^3$. Put

$$\mathbf{P}_r = \left\{ (\lambda_{L,\sigma}, \mu_{L,\sigma}) \,\middle|\, L \text{ is } r\text{-component}, \prod_{i=1}^{r} z_i \neq 0, \forall\, n \; ; \#H_1(M_{\sigma,p^n}, \mathbb{Z}) < \infty \right\}.$$

Then

(1) If $r = 1$ then $\mathbf{P}_1 = \{(0,0)\}$

(2) If $p \neq 2$ and $r \geq 2$ then $\mathbf{P}_r = (r - 1 + 2\mathbb{Z}_{\geq 0}) \times \mathbb{Z}_{\geq 0}$

(3) If $p = 2$ and $r = 2$ then $\mathbf{P}_2 = \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$

Proved by

($\subset$-part) Torres conditions,

($\supset$-part) [Hosokawa'58], [Levine'88] on the existence of L with prescribed Δ_L.

§ 5. Analogies (Suppose $M = S^3$ for simplicity.)

Assume $\prod_{i=1}^{r} z_i \neq 0$. $\longleftrightarrow$ Assume $\forall \wp \in S$ ramifies in k_∞/k.

$\lambda_{L,\sigma} \geq r-1$ $\longleftrightarrow$ $\lambda_{k_\infty} \geq r_2 = \#\{k \hookrightarrow \mathbb{C},\, k \not\hookrightarrow \mathbb{R}\}/2$ if $\#S = \dim_{\mathbb{Q}} k$.

Problem 3. What is "r" of primes ? (r_2+1 ? Not $\#S$?)

Suppose $L = K_1 \cup K_2$. $\longleftrightarrow$ Suppose $S = \{\wp_1, \wp_2\}$ and $\dim_{\mathbb{Q}} k = 2$, $r_2 = 1$.

$(\lambda_{L,\sigma}, \mu_{L,\sigma}) = (1,0) \Leftrightarrow$		If $v_p(\#Cl(k)) = 0$, $(\lambda_{k_\infty}, \mu_{k_\infty}) = (1,0) \Leftrightarrow$
$\mathrm{lk}(K_1, K_2) \not\equiv 0 \pmod{p}$	$\longleftrightarrow$	$\pi_2^{p-1} \not\equiv 1 \pmod{\wp_1^2}$, where $\wp_2^{\#Cl(k)} = \pi_2 \mathcal{O}_k$.
linking number		"power residue symbol" [Gold'74]

In previous example, $\sup_\sigma\{\lambda_{L,\sigma}\} = \infty$, $\mu_{L,\sigma} = v_p(m)$. $\not\longleftrightarrow$ If "Greenberg's conjecture" holds, $(\lambda_{k_\infty}, \mu_{k_\infty}) = (1,0)$ for almost all k_∞. [Ozaki'01]

$$G_L/G'_L = \prod_{i=1}^{r} t_i^{\mathbb{Z}} \simeq \mathbb{Z}^r \quad \longleftrightarrow \quad (G_S)^{ab}/_{\mathrm{Tor}} \simeq \mathbb{Z}_p^{r_2+1+\delta}$$

Put $\Lambda = \mathbb{Z}[G_L/G'_L]$ and the differential module $\mathfrak{A}_L = \sum_{g \in G_L} \Lambda\, dg$.

Crowell sequence: $0 \to (G'_L)^{ab} \xrightarrow{\theta} \mathfrak{A}_L \to \Lambda \to \mathbb{Z} \to 0$ ($\exists$ pro-p version for G_S)

The strict analogue of Greenberg's conjecture is

Is $(G'_L)^{ab} = (G'_L)^{ab}/\theta^{-1}\Big(\sum_{g \in \bigcup_i \langle [m_i] \rangle \cap G'_L} \Lambda\, dg \Big)$ a *pseudonull* Λ-module ?

Almost not ! One of the modifications is

Problem 4. When is $Y_L := (G'_L)^{ab}/\theta^{-1}\Big(\sum_{i=1}^{r} \Lambda\, d[m_i] \Big)$ *pseudonull* ?

i.e., the minimal principal ideal containing $\mathrm{Ann}_\Lambda Y_L$ is Λ ?

Example [KM'13] Y_L is pseudonull for $L = K_1 \cup K_2$ of previous example.

§ 6. More analogies

Riemann-Hurwitz for $\lambda_{L,\sigma}$ [Ueki]	$\longleftrightarrow$	Riemann-Hurwitz for $\lambda_{k_\infty^{cyc}}$ [Kida'80], [Iwasawa'81]

[Iwasawa'63] "$\lambda_{k_\infty^{cyc}}$ is an analogue of the genus of algebraic curve."

Alexander invariant ∼ Ruelle zeta [Sugiyama'07]	$\longleftrightarrow$	"Iwasawa Main Conjecture" char. poly. of $\mathfrak{X}$ ∼ p-adic zeta [Mazur-Wiles'84]

Growth of Betti numbers in p-adic Lie towers $\rho : G_L \to GL_d(\mathbb{Z}_p)$ [Calegari-Emerton'11] *et.al.*	←- - - - →	Iwasawa type formula for p-adic Lie extensions $\rho : \mathrm{Gal} \to GL_d(\mathbb{Z}_p)$ [Perbet'11] *et.al.*

Problem 5. Give Iwasawa type formulas for p-adic Lie towers over L.