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The state numbers of a knot

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“Min-type” inv. for a knot K

<u>The crossing number</u>

$c(K) := \min\{\#$ of crossings in $D\}$

<u>The unknotting number</u>

$u(K) := \min\{\#$ of c.c. for D to unknot$\}$

$\cdots$ and so on

$\rightarrow$ Complexity of K

The n-state number for a knot K $(n \in \mathbb{N})$

$\mathrm{s}_n(K) := \min\{\#$ of "n-states" for D $\}$

Problems :

0. Determine $\mathrm{s}_n(K)$ for a given K

1. Give upper/lower bounds by
   - the crossing number $c(K)$ of K
   - the Jones and the Miyazawa polynomial

2. Is $\mathcal{S}_n(i) := \{K \mid \mathrm{s}_n(K) = i\}$ finite?

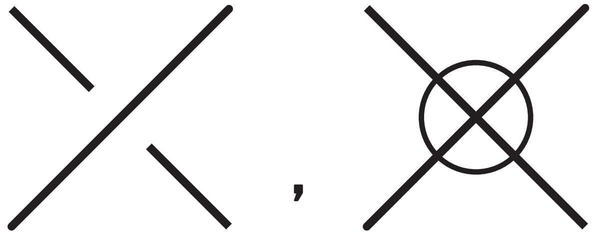
Contents

§ 1. Preliminaries

What is a virtual knot?

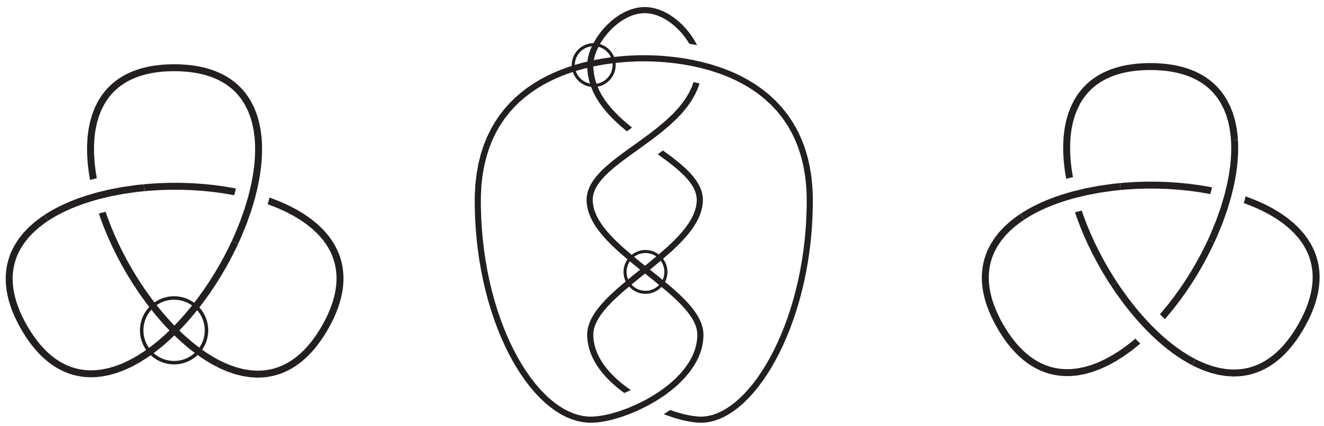
D : a virtual knot diagram

$\overset{\text{def}}{\Leftrightarrow}$ D : an immersed circle in $\mathbb{R}^2$ with



real , virtual

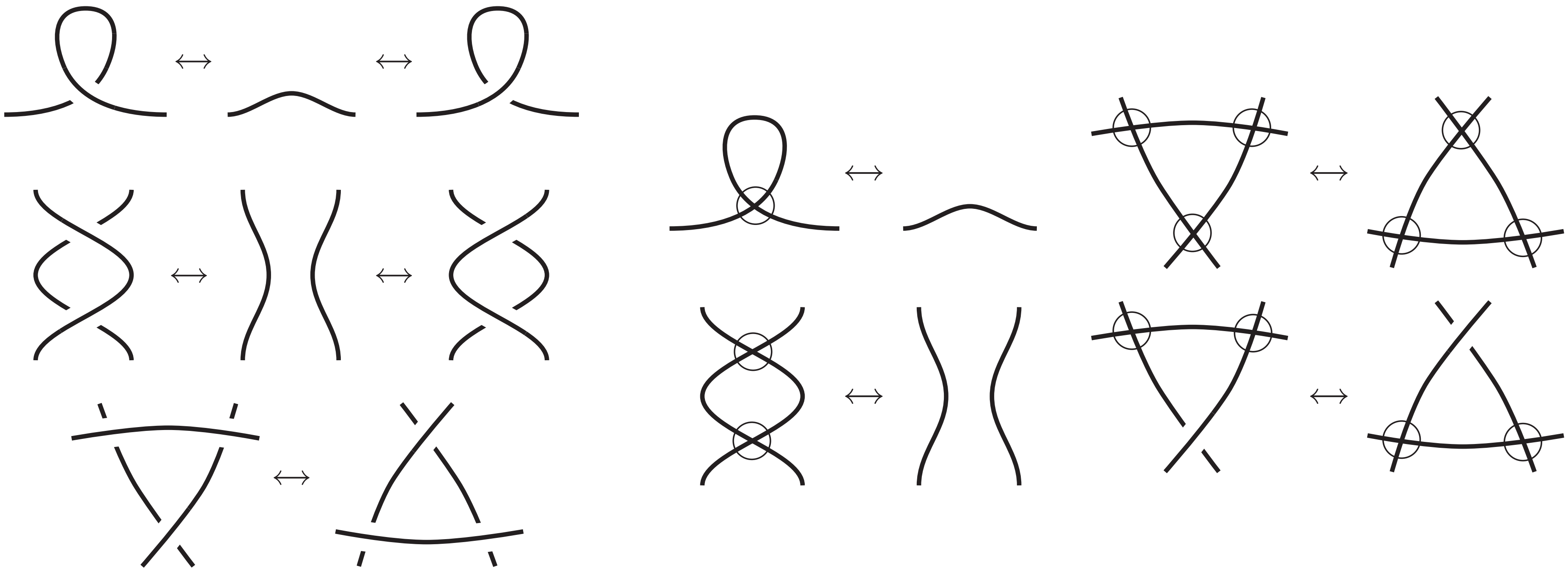
Examples :



classical

K : a virtual knot

$\overset{\text{def}}{\Leftrightarrow}$ $K = [D]$; **an equivalence class of virtual knot diagrams** D **under generalized Reidemeister moves**

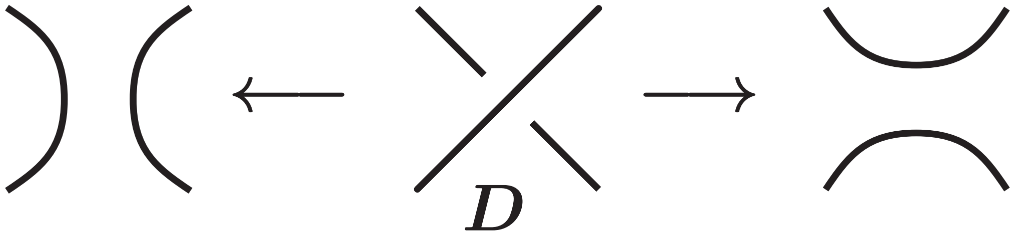


State of a virtual knot diagram

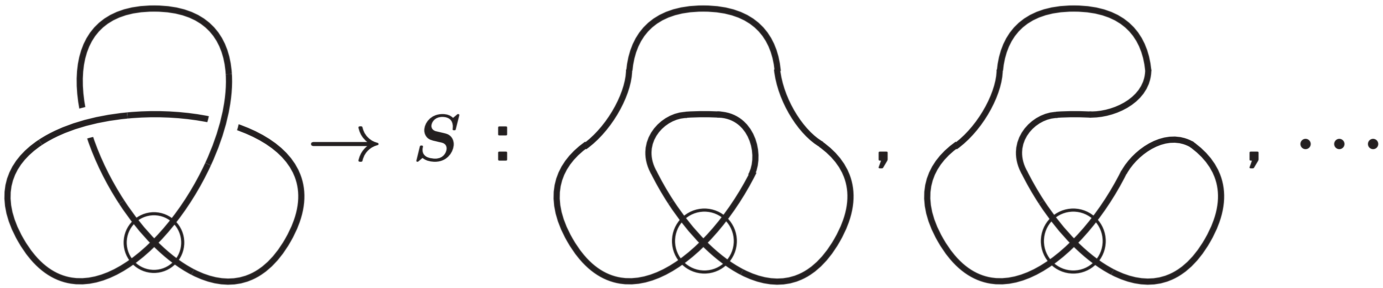
D : a virtual knot diagram

S : a state of D

$\overset{\text{def}}{\Leftrightarrow}$ a union of circles by splicing all real crossings in D



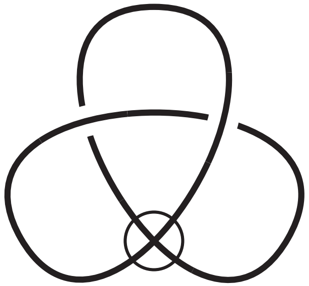
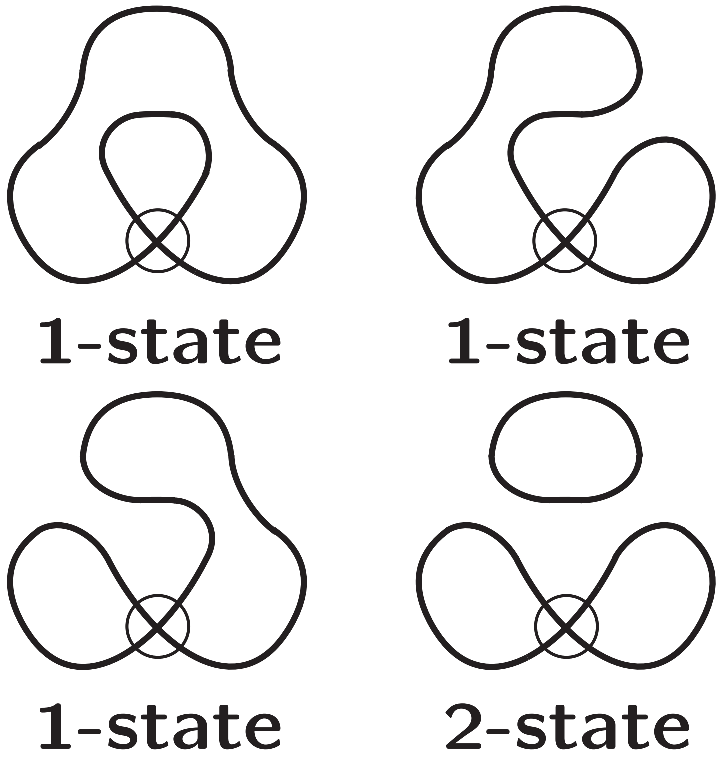
Example:

D : 

S : **an n-state** $\overset{\text{def}}{\Leftrightarrow}$ S consists of n circles ($n \geq 1$)

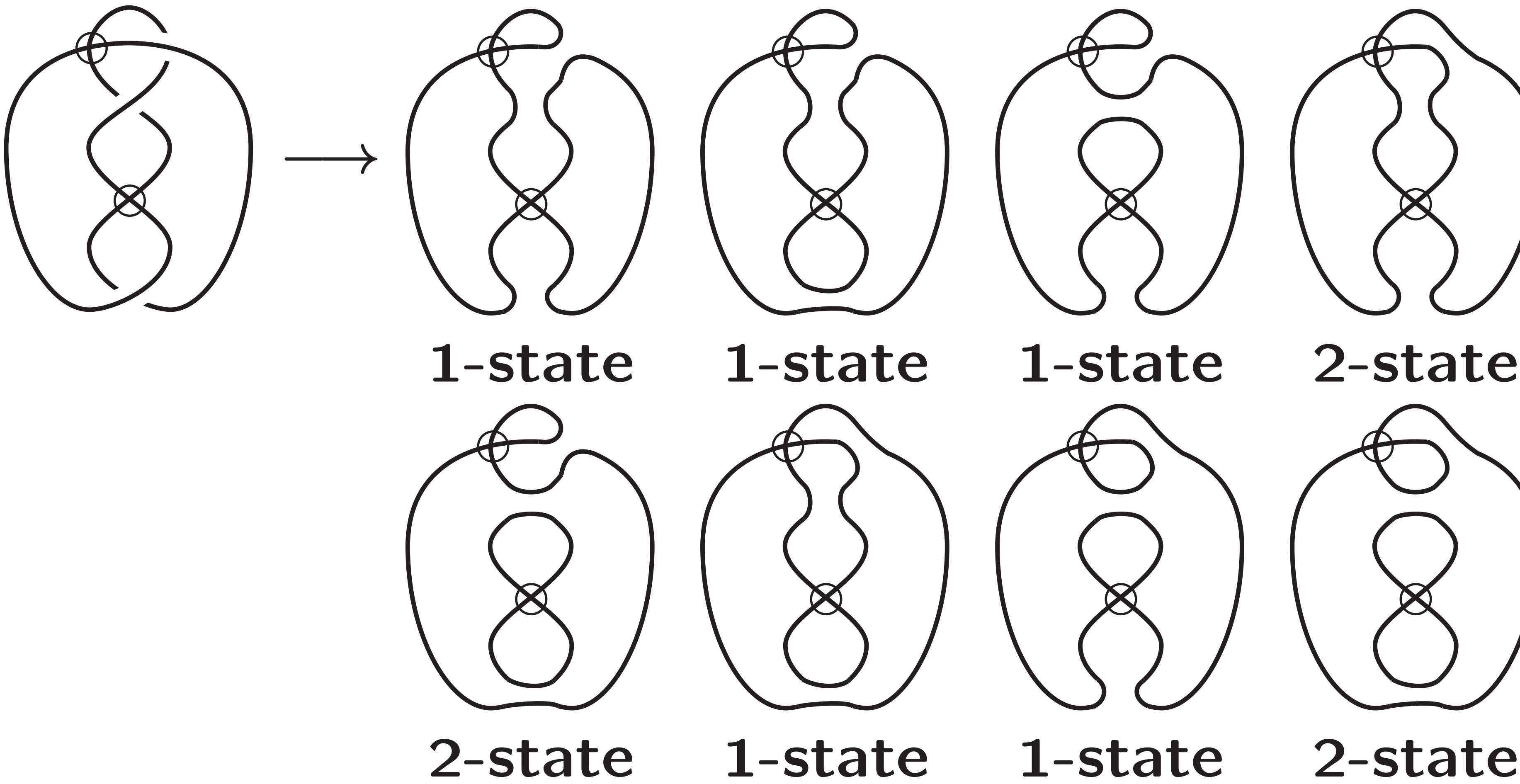
$s_n(D) := \#$ of n-state of D

Example:

D :  $\rightarrow$ 

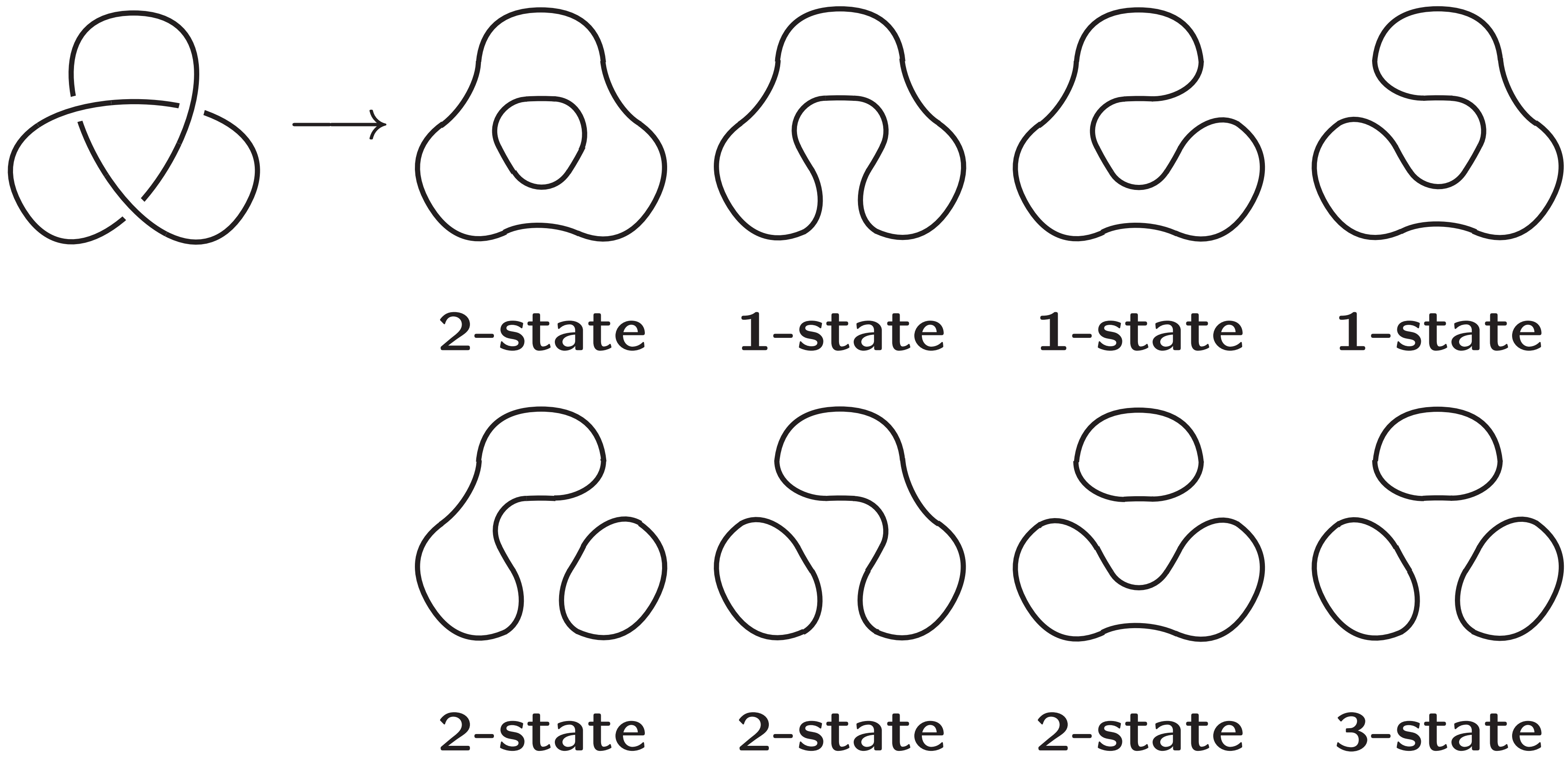
$s_1(D) = 3,\ s_2(D) = 1,\ s_i(D) = 0\ (i \geq 3)$

Example:



$$s_1(D) = 5,\ s_2(D) = 3,\ s_i(D) = 0\ (i \geq 3)$$

Example:



$$s_1(D) = 3,\ s_2(D) = 4,\ s_3(D) = 1,\ s_i(D) = 0\ \ (i \geq 4)$$

The n-state number for a virtual knot

K : a virtual knot

$\mathrm{s}_n(K) := \min\{s_n(D) \mid D$: a diagram for $K\}$;

<u>the n-state number for K</u>

Fact 1.1.

K : the trivial knot $\Rightarrow \mathrm{s}_1(K) = 1$,

$\mathrm{s}_i(K) = 0 \ (i \geq 2)$.

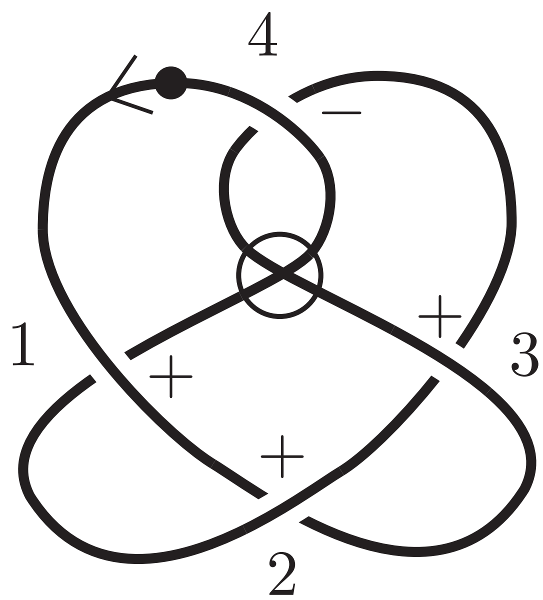
Gauss diagram

D : a virtual knot diagram

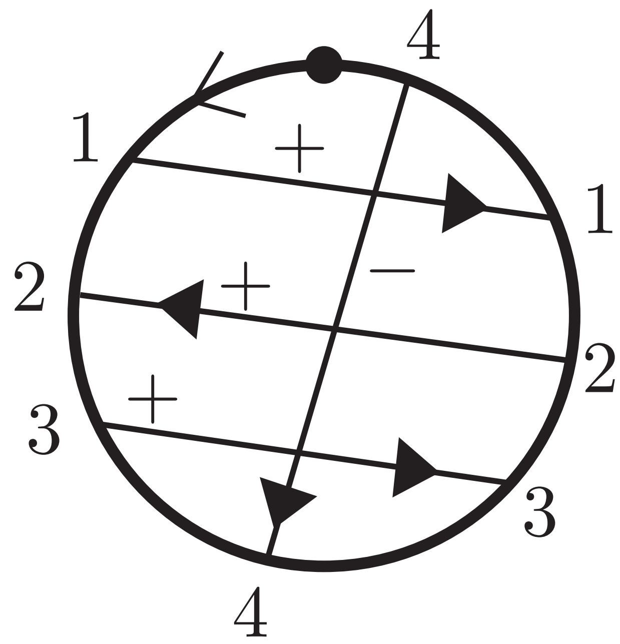
$G(D)$: the Gauss diagram of D

$\overset{\text{def}}{\Leftrightarrow}$ "*preimage*" of D;

a circle with signed and oriented chords.

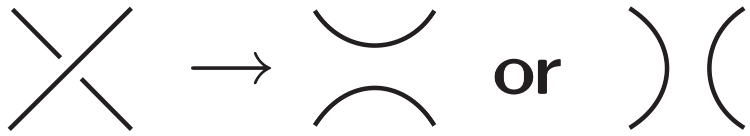


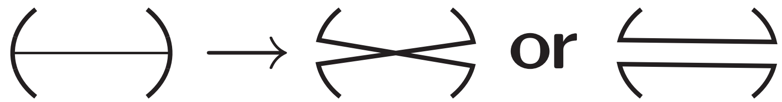
$\rightarrow$

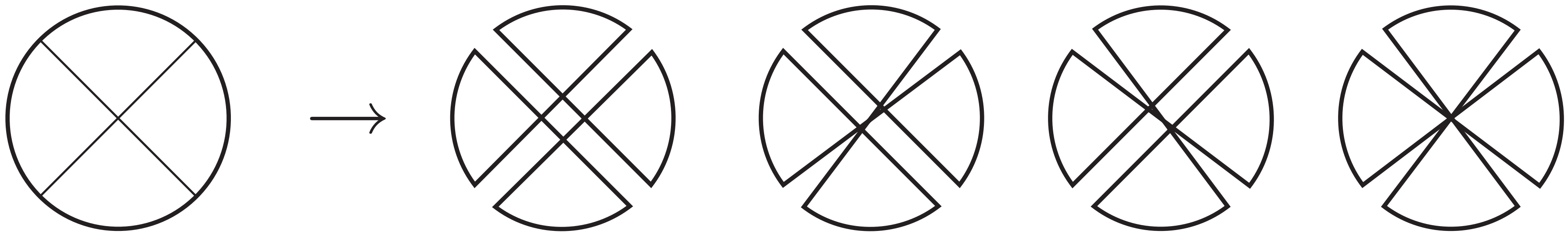


A chord γ is oriented from the upper to the lower.

State of a Gauss diagram

splice a crossing in D 

splice a chord in $G(D)$ 

G: 

underlying
Gauss diagram

Lemma 1.2.

D & D' have the same underlying Gauss diagram G.

$\Rightarrow s_n(D') = s_n(D)$

Notation: $s_n(G) := s_n(D)$

§ 2. Bounds for 1-,2-,3-state numbers

K : a virtual knot

$c(K)$: the minimal real crossing number for K

Theorem 2.1.

For any knot K, we have

(1) $1 \leq s_1(K) \leq \dfrac{2 \cdot 2^{c(K)} + (-1)^{c(K)}}{3}$,

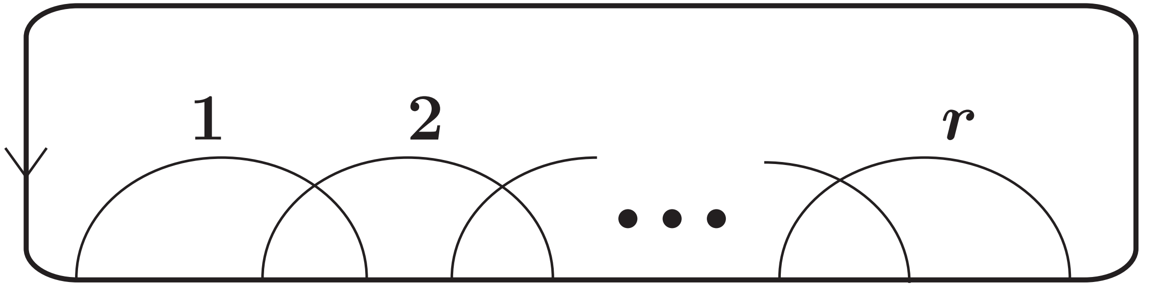
(2) $0 \leq s_2(K) \leq \dfrac{1}{2} \cdot 2^{c(K)}$,

(3) $0 \leq s_3(K) \leq \dfrac{3}{8} \cdot 2^{c(K)}$.

Example:

Gauss diagram

F_r:



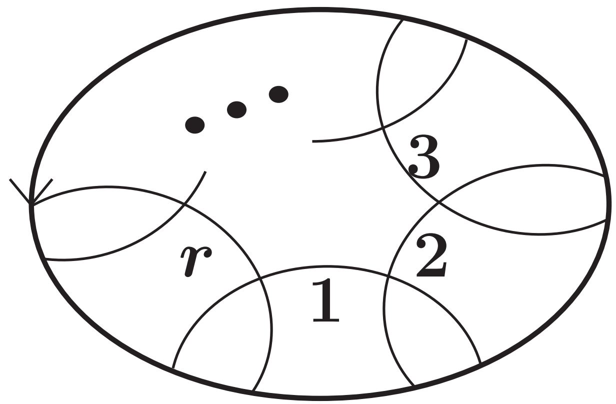
$$\Rightarrow s_1(F_r) = \frac{2 \cdot 2^r + (-1)^r}{3}$$

$F_r \xrightarrow{?}$ a knot K with $s_1(K) = \dfrac{2 \cdot 2^{c(K)} + (-1)^{c(K)}}{3}$

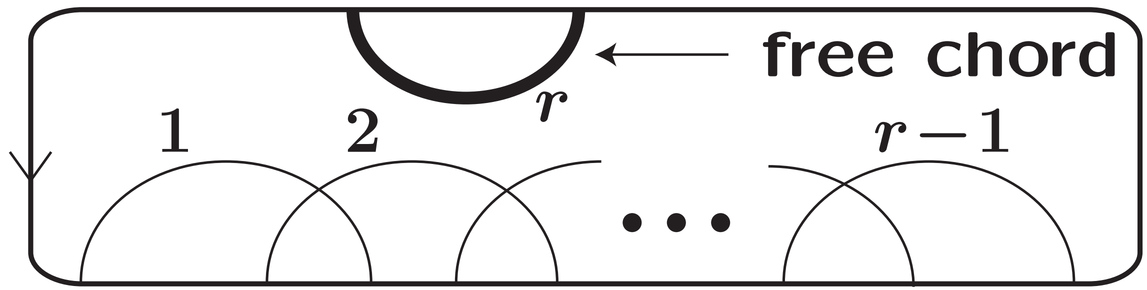
Example:

Gauss diagrams

F'_r:



$F_{r-1}+1$:



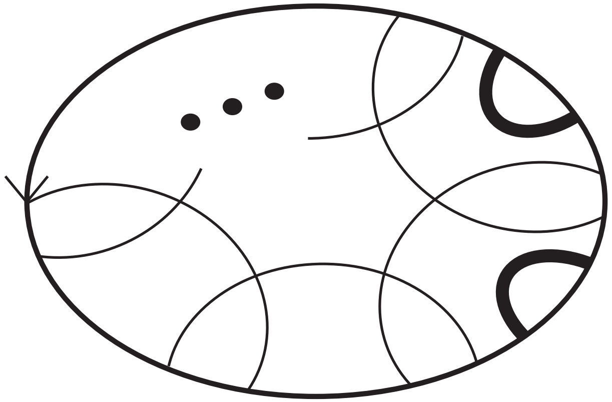
$$\Rightarrow s_2(F'_r) = s_2(F_{r-1}+1) = \frac{1}{2}\cdot 2^r$$

$$F'_r \overset{?}{\longrightarrow} \text{a knot } K \text{ with } s_2(K) = \frac{1}{2}\cdot 2^{c(K)}$$

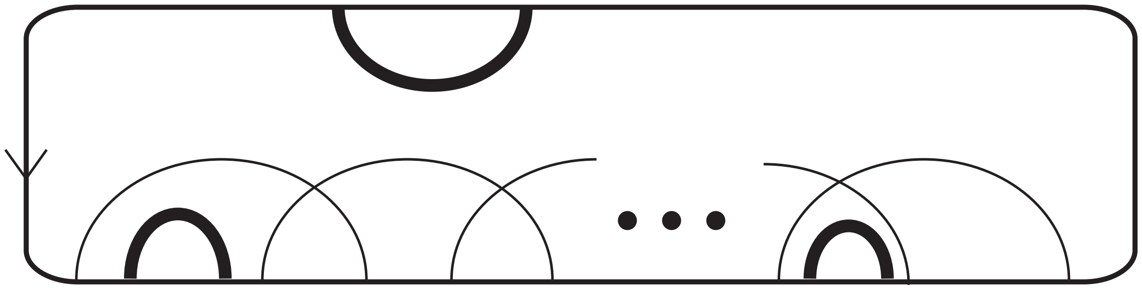
Example:

Gauss diagrams

F'_{r-2}+2:



F_{r-3}+3:



$$\Rightarrow s_3(F'_{r-2}+2) = s_3(F_{r-3}+3) = \frac{3}{8} \cdot 2^r$$

§3. Finiteness problem

A set of virtual knots with $\mathrm{s}_n(K) = i$;

$\mathcal{S}_n(i) := \{K \mid \mathrm{s}_n(K) = i\}$

Proposition 3.1.

(1) $\mathcal{S}_1(1) = \{$trivial knot$\}$

(2) $\mathcal{S}_1(2k) = \emptyset \ (k \geq 0)$

(3) $\mathcal{S}_2(0) = \{$trivial knot$\}$

Is $\mathcal{S}_n(i)$ finite?

Theorem 3.2.

$\mathcal{S}_2(i)$ is **finite** for any i.

For any virtual knot K with $c(K) \geq 3$, we see that $\boldsymbol{c(K) \leq \mathrm{s}_2(K)}$. □

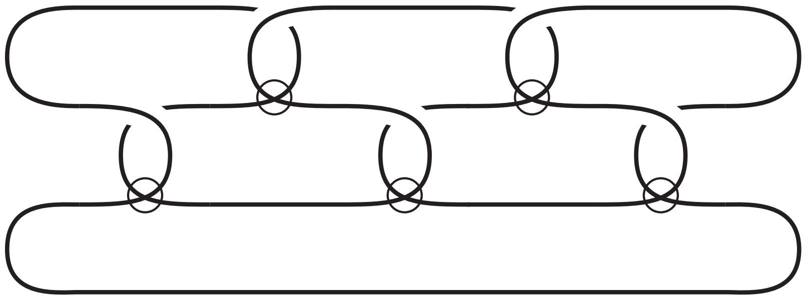
Theorem 3.2.

$\mathcal{S}_2(i)$ is **finite** for any i.

For any virtual knot K with $c(K) \geq 3$, we see that $c(K) \leq s_2(K)$. □

Theorem 3.3.

$\mathcal{S}_n(0)$ is **infinite** for $n \geq 3$.

 $\in \mathcal{S}_3(0)$

Remark: Any non-trivial $K \in \mathcal{S}_3(0)$ is non-classical.

Proposition 3.4.

$\mathcal{S}_1(i)$ is finite for $i \leq 7$.

In particular, $\mathcal{S}_1(3) = \{2.1, 3.5, 3.6, 3.7\}$.

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Theorem 3.5.

$\mathcal{S}_1(9)$ is **infinite**.

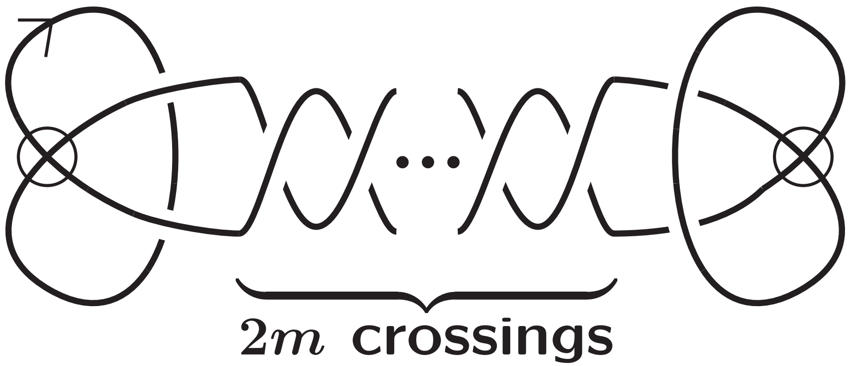
Proposition 3.4.

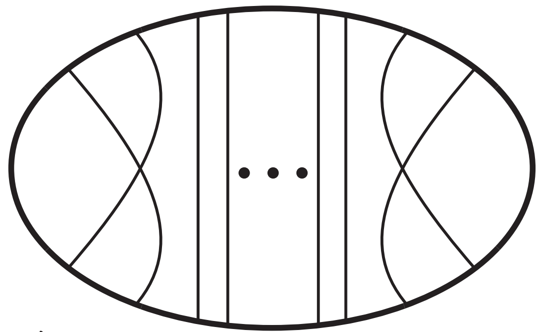
$\mathcal{S}_1(i)$ is **finite** for $i \leq 7$.

In particular, $\mathcal{S}_1(3) = \{2.1, 3.5, 3.6, 3.7\}$.

Theorem 3.5.

$\mathcal{S}_1(9)$ is **infinite**.

K_m: 

$\to G_m$: 

$s_1(G_m) = 9$

$K_m \neq K_{m'}$ if $m \neq m'$ by the Miyazawa polynomial.

a virtual knot invariant

§4. Lower bounds for $\mathrm{s}_1(K)$ by polynomials

Lower bounds for $\mathrm{s}_1(K)$ by Jones polynomial

$$f_K(A) = (-A^{-3})^{w(D)} \sum_S A^{a(S)-b(S)} (-A^2 - A^{-2})^{|S|-1}$$

$w(D)$: writhe of D, $|S| := \#\{$circles in a state S of $D\}$

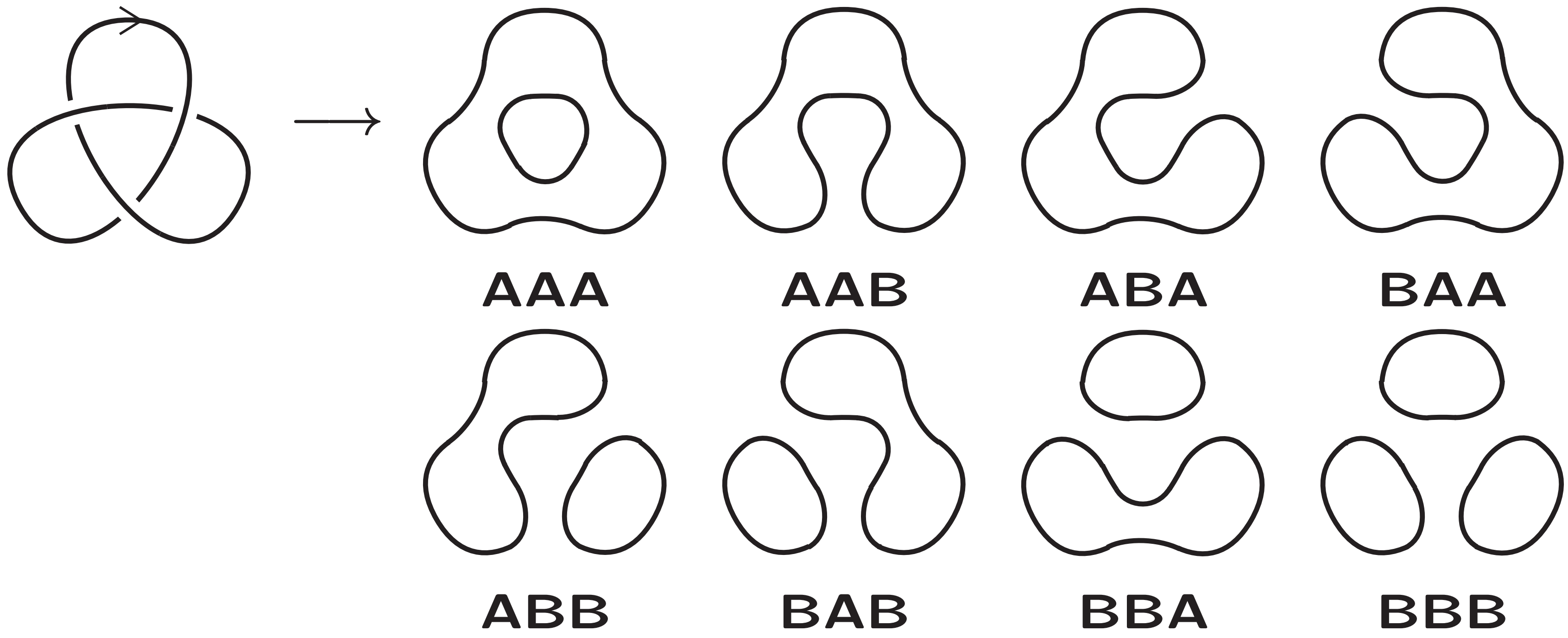
$a(S) := \#\{A$-splice $D \to S\}$, $b(S) := \#\{B$-splice $D \to S\}$

$f_K(t^{-\frac{1}{4}}) = V_K(t)$: Jones polynomial of K

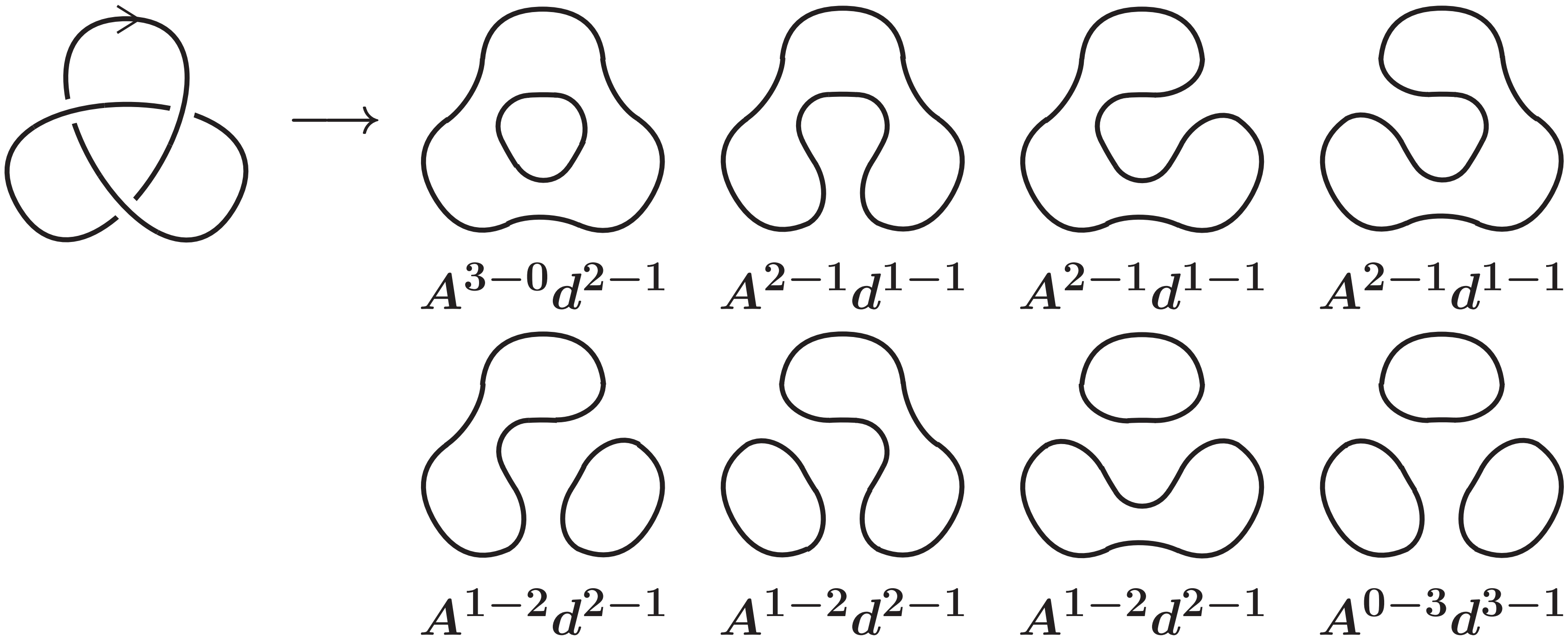
Proposition 4.1.

$\mathrm{s}_1(K) \geq |f_K(\xi)| = |V_K(-1)|$, where $\xi = e^{\frac{\pi}{4}i}$

Example:



Example: $d = -A^2 - A^{-2}$



$$f_K(A) = A^{-4} + A^{-12} - A^{-16}$$

$$\xrightarrow{A=\xi} |f_K(\xi)| = 3 \leq \mathrm{s}_1(K) \leq 3$$

Miyazawa polynomial $R_K(A, \vec{x})$

$$R_K(A, \vec{x}) = F_0(A) + F_1(A)x_1 + F_{01}(A)x_2 + F_2(A)x_1^2 + F_{001}(A)x_3 + F_{11}(A)x_1x_2 + F_3(A)x_1^3 + \cdots$$

: a generalization of the Jones polynomial

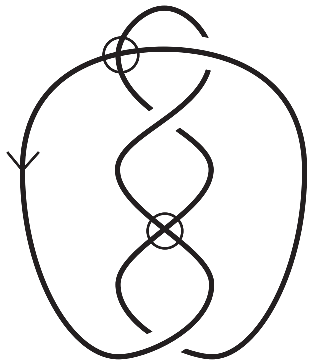
$F_{\underbrace{0\ldots0}_{k-1}1}(A)$ is derived from 1-states.

$$\widetilde{F_0}(A) = F_0(A)/(-A^2 - A^{-2})$$

Theorem 4.2.

$$s_1(K) \geq \left|\widetilde{F_0}(\xi)\right| + \sum_{k=1}^{\infty} \left|F_{\underbrace{0\ldots0}_{k-1}1}(\xi)\right|.$$

Example

K:  virtual knot 3.3 in Green's table

$\exists D$ s.t. $s_1(D) = 5$

$$V_K(t) = -t^{-\frac{5}{2}} + t^{-\frac{3}{2}} + t^{-1} \Rightarrow |V_K(-1)| = \sqrt{5}$$

$$R_K(A, \vec{x}) = (-A^2 - A^{-2})\,(-A^8 + A^4) + (-A^{10} + A^6)\,x_1 - A^{10}x_1^2 + (-A^{12})\,x_2$$

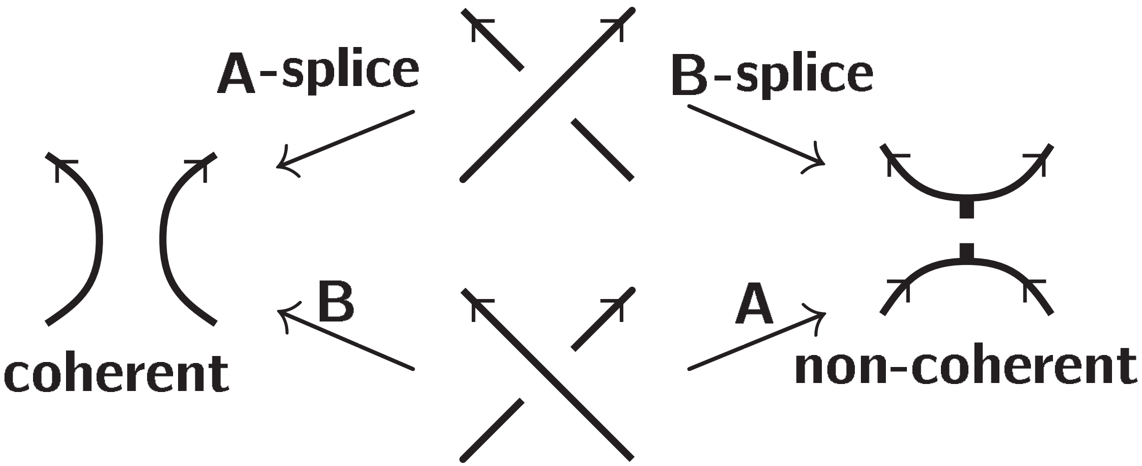
$$\left|\widetilde{F_0}(\xi)\right| + \sum_{k=1}^{\infty} |F_{I_k}(\xi)| = \left|\widetilde{F_0}(\xi)\right| + |F_1(\xi)| + |F_{01}(\xi)| = 5$$

$$\Rightarrow 5 \le \mathrm{s}_1(K) \le s_1(D) = 5$$

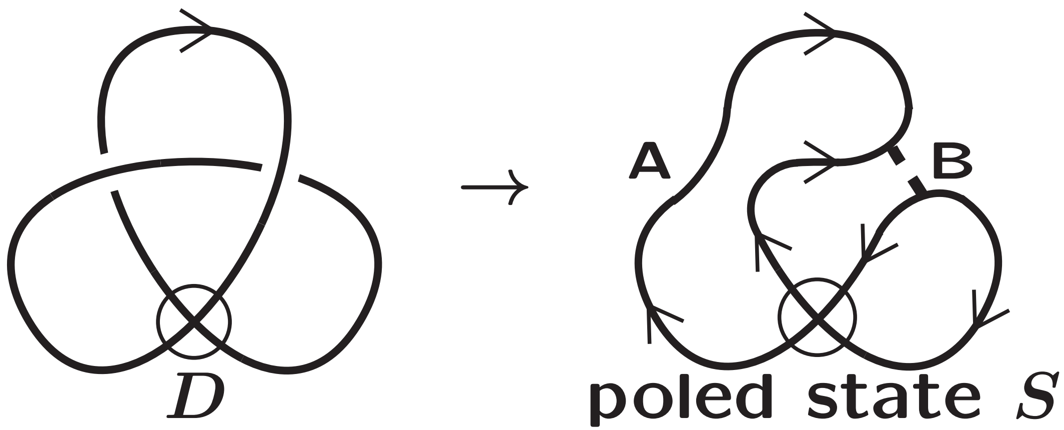
Definition of Miyazawa polynomial $R_K(A, \vec{x})$

D an oriented virtual diagram

S : a poled state of D obtained by

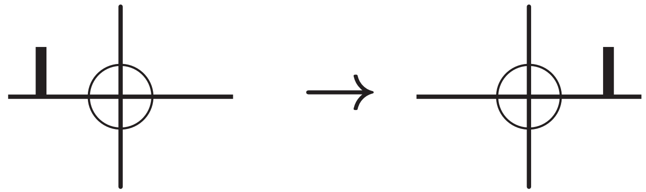


Example:



Reduction of the number of poles

(1) Sliding a pole through virtual crossings.



(2) Remove two successive poles on the same side

$\widetilde{C}$: the circle obtained from C by reductions

$\Rightarrow \widetilde{C}$:

$$p(C) := \frac{\#(\text{poles in } \widetilde{C})}{2}$$

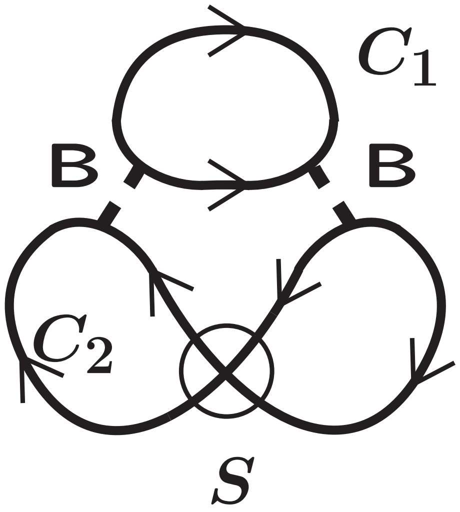
$a(S) := \#\{A\text{-splice } D \to S\}$,

$b(S) := \#\{B\text{-splice } D \to S\}$

$c_i(S) := \#\{C \text{ in } S \mid p(C) = i\}$

$$\langle S \mid D \rangle := A^{a(S)-b(S)}(-A^{-2} - A^2)^{c_0(S)} x_1^{c_1(S)} x_2^{c_2(S)} \cdots$$

Example



$a(S) = 0,\ b(S) = 2,$
$p(C_1) = 0,\ p(C_2) = 1,$

$c_0(S) = 1,\ c_1(S) = 1,\ c_i(S) = 0\ (i \geq 2)$

$$\Rightarrow \langle S \mid D \rangle = A^{0-2} x_0 x_1 = A^{-2}(-A^{-2} - A^2) x_1$$

$$R_K(A, \vec{x}) := (-A^3)^{-w(D)} \sum_S \langle S \mid D \rangle \in \mathbb{Z}[A, A^{-1}, x_1, x_2, \ldots]$$

Example: K : the virtual knot 3.3 in Green's table

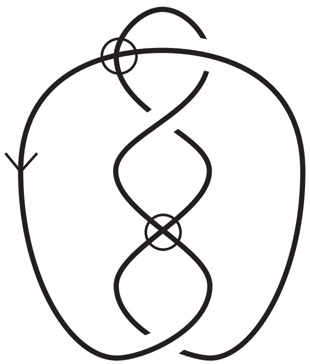
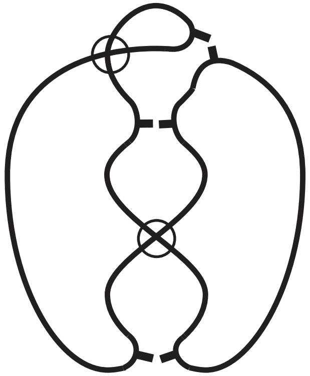
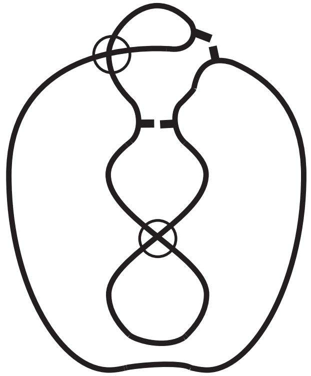


diagram D

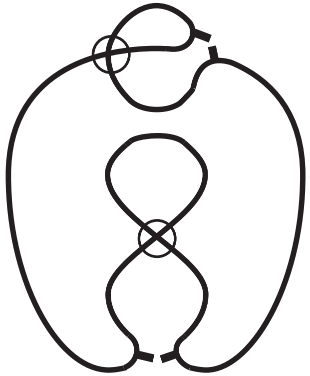
$\longrightarrow$



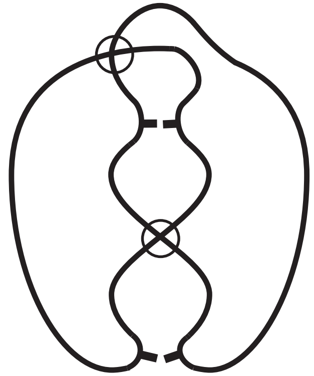
AAA



AAB

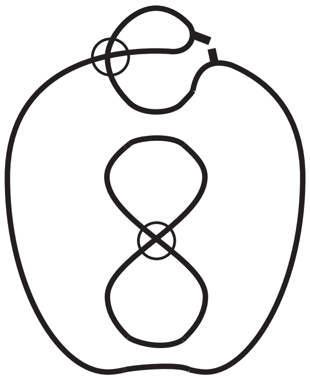


ABA

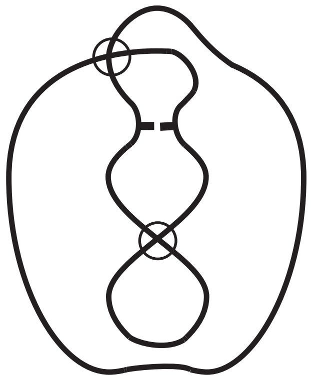


BAA

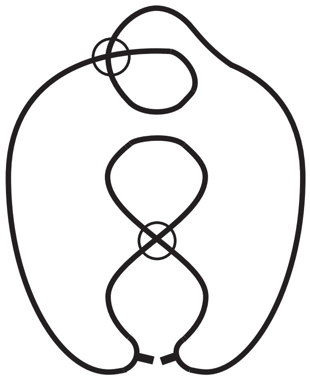
$w(D) = -3$



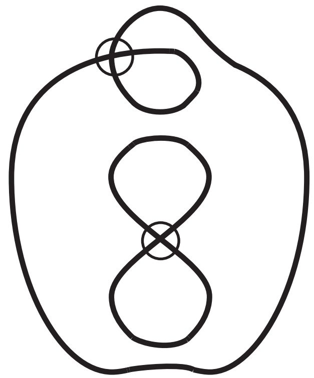
ABB



BAB



BBA



BBB

Example: K : the virtual knot 3.3 in Green's table

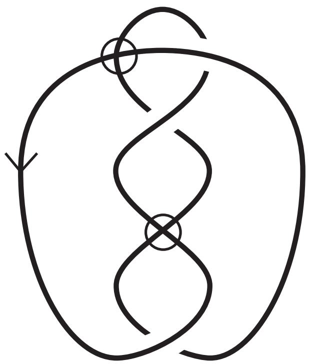
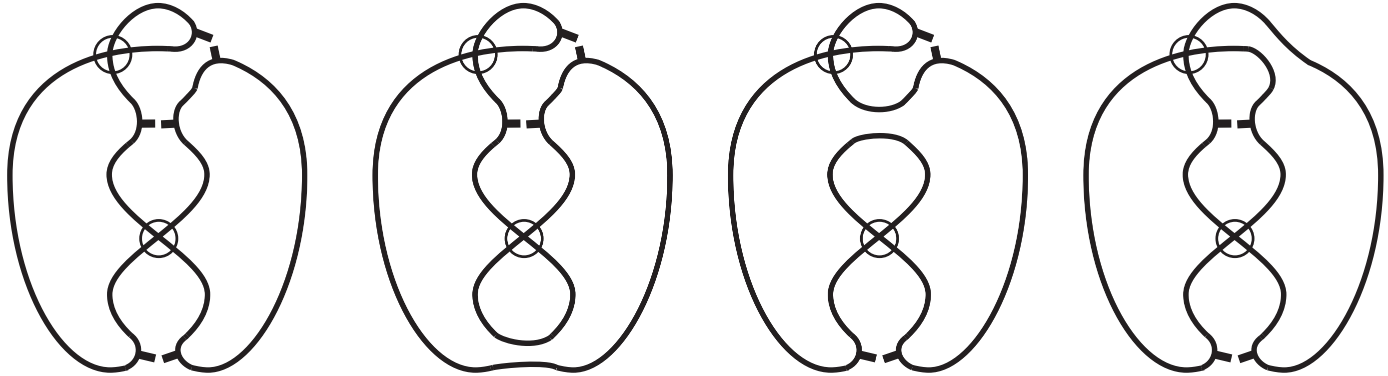


diagram D

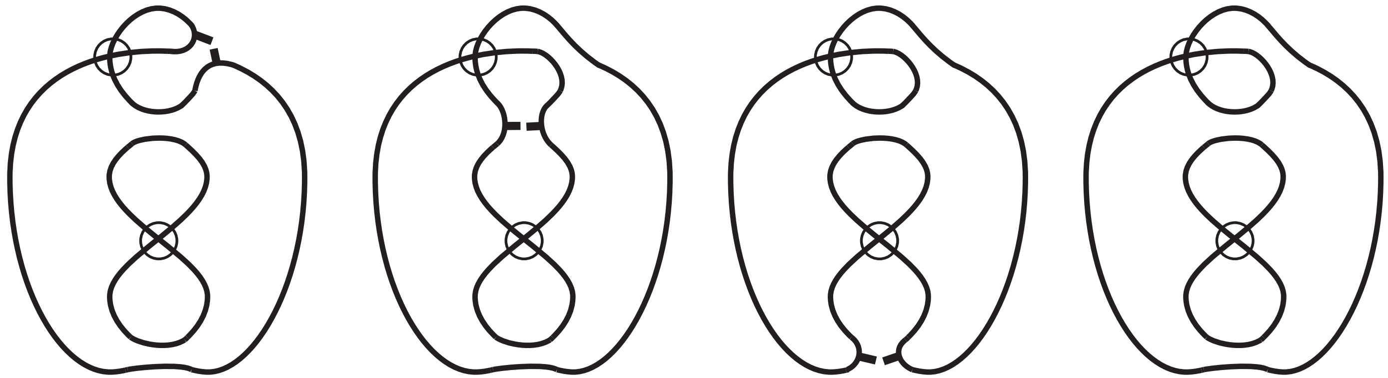
$\longrightarrow$



$A^{3-0}x_2$ $\quad A^{2-1}x_1$ $\quad A^{2-1}x_1$ $\quad A^{2-1}x_1^2$

$w(D) = -3$

$x_0 = -A^2 - A^{-2}$



$A^{1-2}x_0x_1$ $\quad A^{1-2}x_0$ $\quad A^{1-2}x_0$ $\quad A^{0-3}x_0^2$

$$R_K(A, \vec{x}) = (-A^2 - A^{-2})(-A^8 + A^4) + (-A^{10} + A^6)x_1 - A^{10}x_1^2 - A^{12}x_2$$