# On annulus twists

Tetsuya Abe[1]

[1]OCAMI

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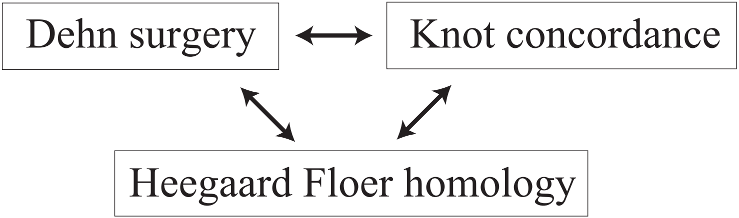
# Outline




1. Dehn surgery and annulus twists
2. Knot concordance and annulus twists
3. A construction of ribbon disks

# Adam Levin's question.

Dehn surgery $\longleftrightarrow$ Knot concordance

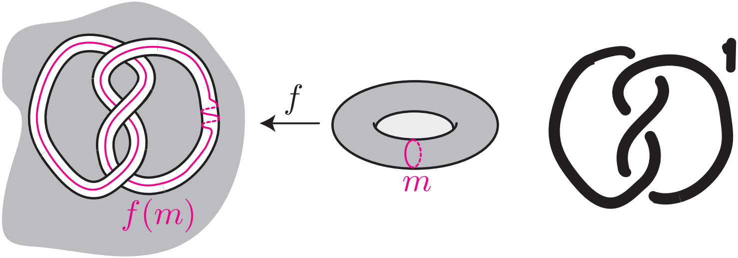
The following question asks a relation between Knot concordance and Dehn sugery.

**Question (Levine, 2016)**

*If $K$ is concordant to $K'$, then for all $n$, $S^3_n(K)$ is homology cobordant to $S^3_n(K')$. Is the converse true?*

# Dehn surgery 1.

Dehn surgery on knots is a long-standing technique for the construction of 3-manifolds.




The following existence theorem is important.

**Theorem (Lickorish, Wallace)**

*Any closed, oriented and connected 3-manifold is realized by integral surgery on a link in $S^3$.*

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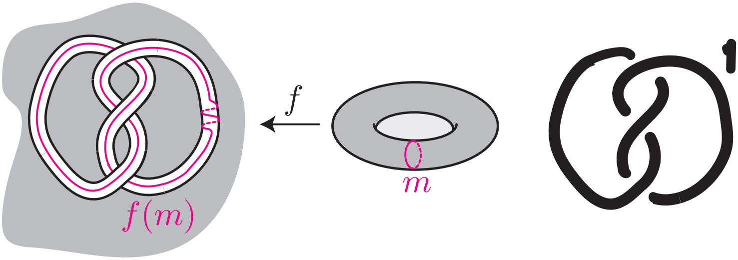


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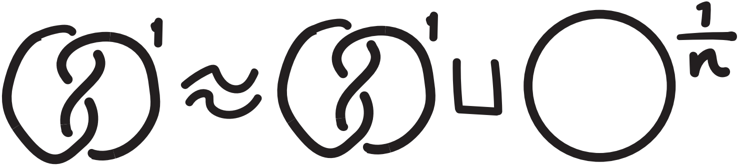
# Dehn surgery 2.

How about "uniqueness" ???

**Question**

*If two framed links $\mathcal{L}$ and $\mathcal{L}'$ gives the same 3-manifold, then*

$$\mathcal{L} = \mathcal{L}' \ ?$$

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Restriction 1 : Fix the knot type.
Restriction 2 : Consider only an orientation preserving hom.

Yi Ni and Zhongtao Wu.
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**Problem (Clark, Problem 3.6 in Kirby's problem list)**

*Let n be an integer.*
*Find a 3-manifold obtained by n-surgery on $\infty$-many knots.*

- In 2006, Osoinach solved this problem for $n = 0$.
- After the work of Teragaito, Takeuchi, Omae, Kohno, it was solved by Abe-Jong-Luecke-Osoinach in 2015.
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# Osoinach's result.

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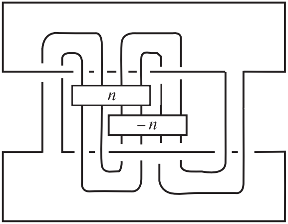


**Theorem (Osoinach, 2006)**

*(1) The sequence $\{K_n\}$ contains $\infty$-many hyperbolic knots.*
*(2) We have the following:*

$$S^3_0(K_0) \approx S^3_0(K_1) \approx S^3_0(K_2) \approx S^3_0(K_3) \approx \cdots .$$

- $K_0$ is the connected sum of two figure eight knots.
- Takioka proved that $K_n$ are mutually distinct for $n \geq 0$.

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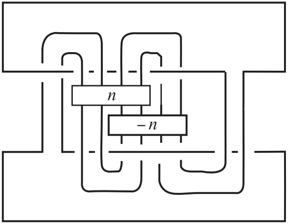
$K_n =$ 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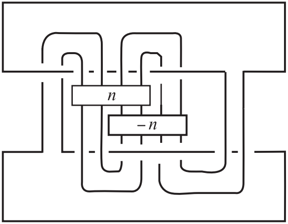


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# Basic lemmas

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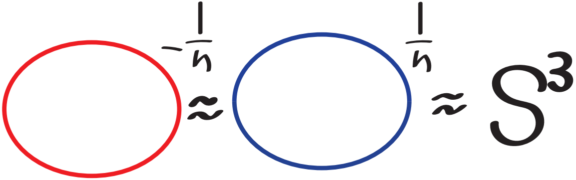


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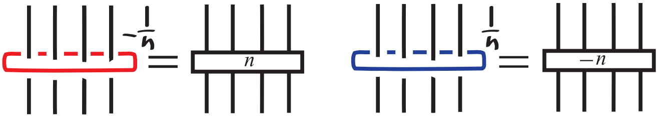


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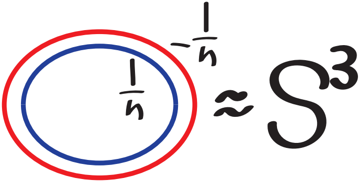
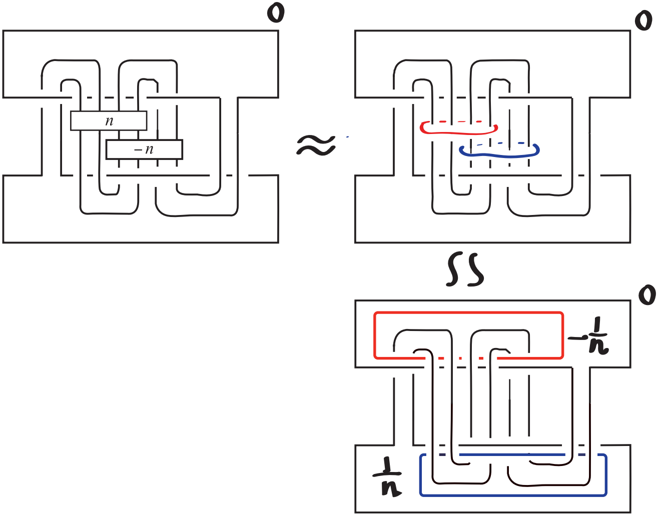


# Proof of $S^3_0(K_n) \approx S^3_0(K_0)$.



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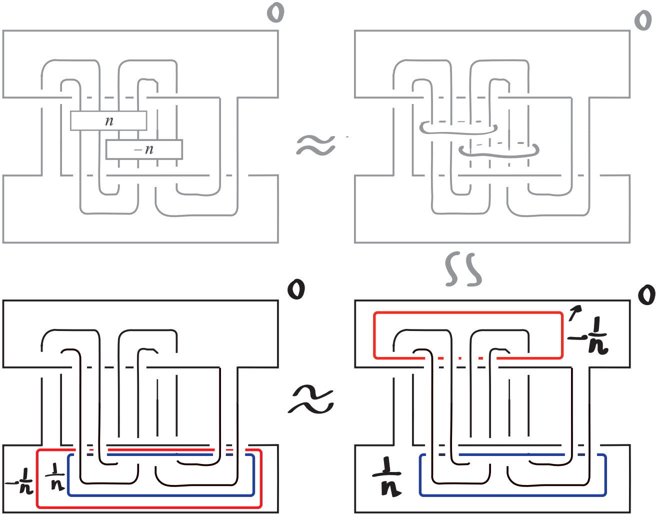

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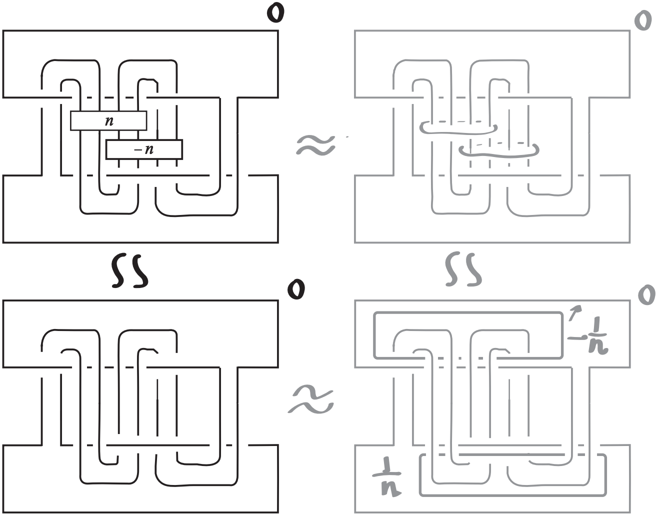
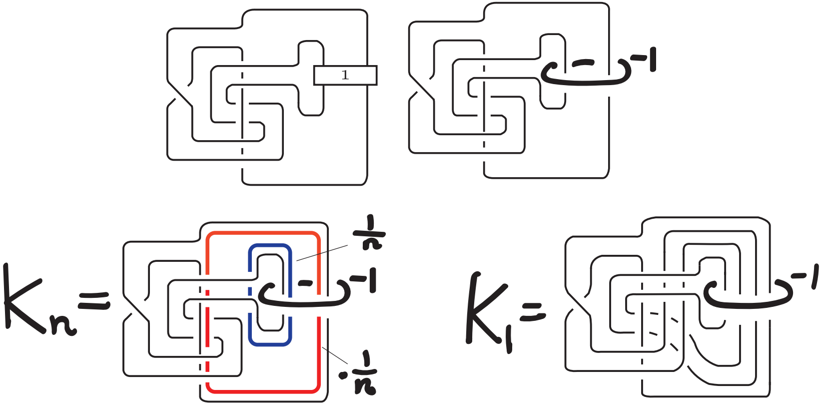


# How to generalize?




**Lemma (Osoinach)**

$$S^3_0(K'_n) \approx S^3_0(K'_0).$$

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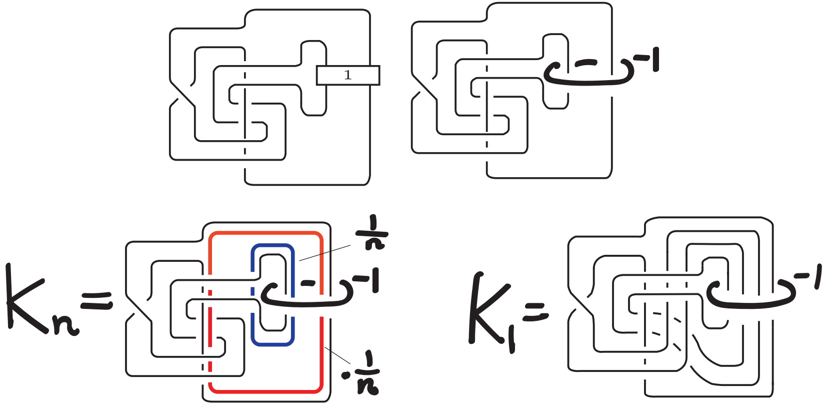


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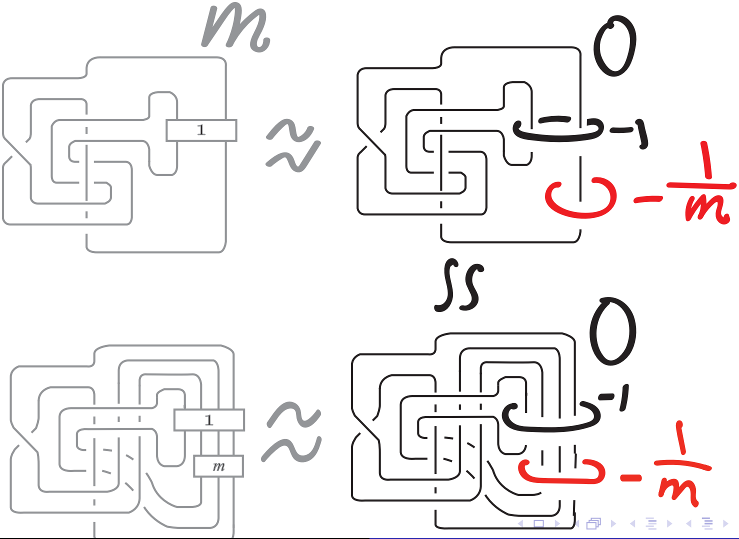
# A general case.



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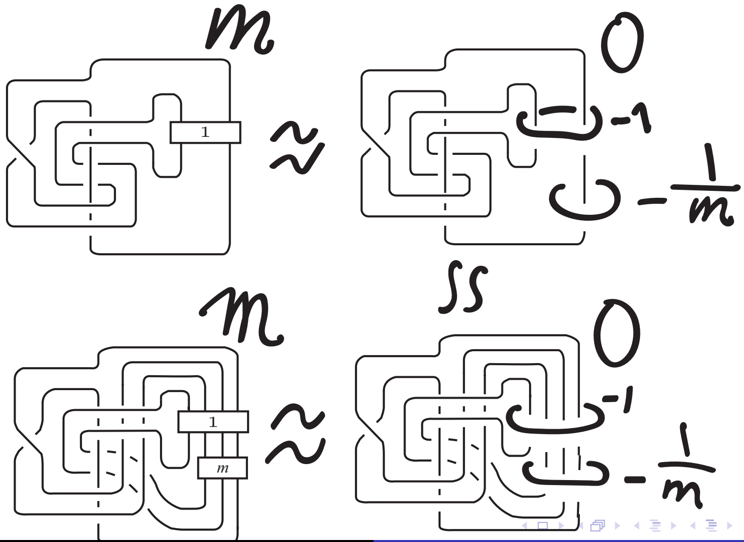

# Knot concordance group $\mathcal{C}$.

The knot concordance group was introduced by Fox and Milnor in 1966.

Two oriented knots $K$ and $K'$ are concordant $\overset{def}{\Longleftrightarrow}$ they cobound a properly embedded annulus in $S^3 \times I$.

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# Akbulut-Kirby's conjecture.

**Question (Levine, 2016)**

*If $K$ is concordant to $K'$, then for all $n$, $S^3_n(K)$ is homology cobordant to $S^3_n(K')$. Is the converse true?*

**Conjecture (Akbulut-Kirby)**

*If two knots $K$ and $K'$ have the same 0-surgery, then $K$ and $K'$ are concordant.*

After Abe-Tagami's work, Yasui solved this conjecture.

Notation: $X_n(K)$ is the 2-handleboby obtained from $K \subset S^3$ with framing $n$.

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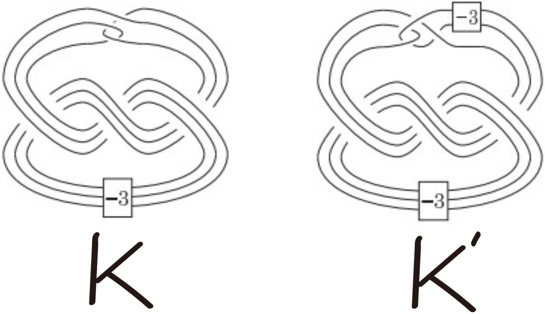
# Yasui's result.




(A picture from Yasui's paper)

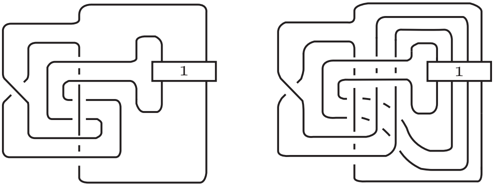
**Theorem (Yasui, 2015)**

*(1) $X_0(K)$ and $X_0(K')$ are exotic (home. but not diffeo.).*
*(2) $\tau(K) \neq \tau(K')$.*

# A remaining conjecture.

## Conjecture

*If $X_0(K) \approx X_0(K')$, then $K$ and $K'$ are concordant.*




$K_0$ $K_1$

## Theorem (Abe-Jong-Omae-Takeuchi, 2013, Abe-Tagami, 2015)

*(1) The 4-manifold $X_0(K_0)$ and $X_0(K_1)$ are diffeomorphic.*
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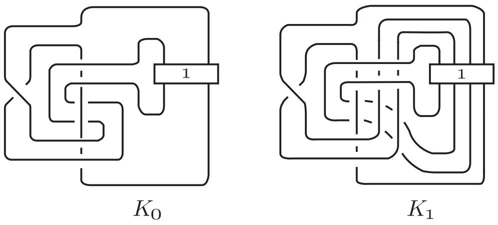


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# Another consequence of the slice-ribbon conjecture.

**Observation (Abe-Tagami, 2015)**

*If the slice-ribbon conjecture is true, the set of prime tight fibered knots is linearly independent in* $Con(S^3)$.

If the slice-ribbon conjecture is true, then following is true.

- The set of the algebraic knots is linearly independent.
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# The definition of a slice knot.

$B^4$ : the 4-ball s.t $\partial B^4 = S^3$

A knot $K \subset S^3$ is (smoothly) slice

$\overset{def}{\Longleftrightarrow}$ it bounds a smoothly embedded disk in $B^4$.

- A knot $K$ is slice iff [K] is the unit element of $\mathcal{C}$.

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A knot in $S^3$ is ribbon

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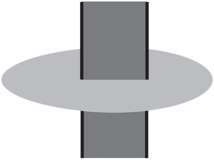
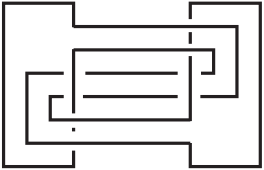

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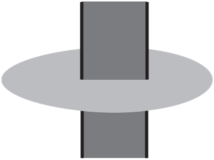

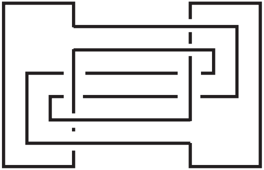

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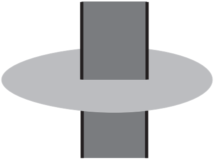

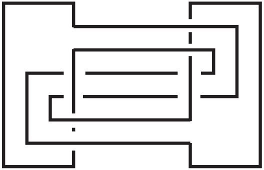

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Using annulus twists technique, we can construct potential counterexamples of slice-ribbon conjecture.

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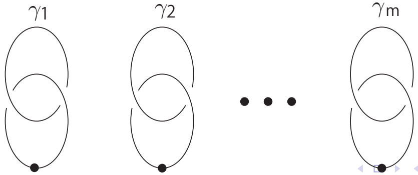
T. Abe and M. Tange, *A construction of slice knots via annulus twists*, accepted by the Michigan Mathematical Journal.

# A construction of ribbon disks.

Let $U \subset S^3$ be the unknot, and $\Delta \subset S^3 = \partial B^4$ a spanning disk for $U$. Regard $B^4$ as a 0-handle $h^0$. Consider an $m$-component unlink $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_m$ such that $L \cap \Delta = \emptyset$. Regard the unlink $L$ as dotted 1-handles $h^1_i (i = 1, 2, \cdots, m)$. Attach 2-handles $h^2_i (i = 1, 2, \cdots, m)$ along circles in $S^3 \setminus (L \cup U)$ so that, the resulting 2-handlebody

$$h^0 \bigcup h^1_1 \cup \cdots \cup h^1_m \bigcup h^2_1 \cup \cdots \cup h^2_m$$

is represented by the handle diagram in Figure 1 after isotopy, where $\gamma_i$ is the framing of a 2-handle $h^2_i (i = 1, 2, \cdots, m)$.



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By the handle canceling theorem, the 2-handlebody

$$h^0 \bigcup h^1_1 \cup \cdots \cup h^1_m \bigcup h^2_1 \cup \cdots \cup h^2_m$$

is diffeomorphic to $B^4$. Note that

$$\Delta \subset h^0 \subset h^0 \bigcup h^1_1 \cup \cdots \cup h^1_m \bigcup h^2_1 \cup \cdots \cup h^2_m \approx B^4.$$

Therefore we obtain a new slice disk $D$ in $B^4$ (which is $\Delta$ in $h^0 \bigcup h^1_1 \cup \cdots \cup h^1_m \bigcup h^2_1 \cup \cdots \cup h^2_m$).

**Theorem (Abe-Tange, cf. Hitt, Asano-Marumto-Yanagawa)**

*Any slice disk obtained by the above construction is ribbon. Conversely, any ribbon disk is obtained by this construction.*

# A schematic picture of this construction for $m = 1$.

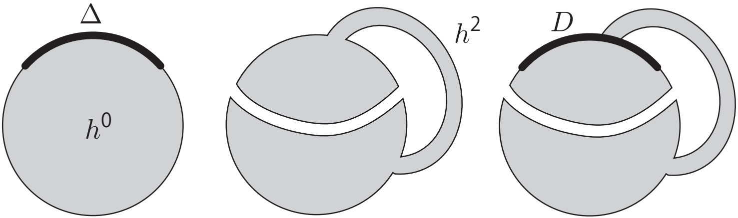
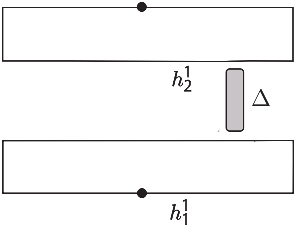
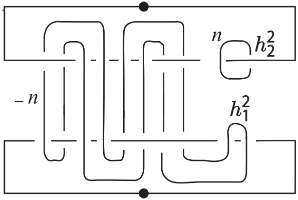


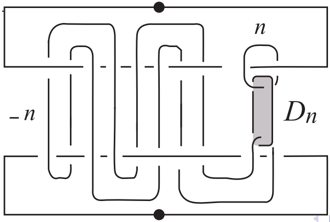

Figure: The spanning disk $\Delta$ in $S^3 = \partial B^4$, the 2-handlebody $h^0 \cup h^1 \cup h^2$ which is diffeomorphic to $B^4$, and a new slice disk $D$ in $B^4$.

# A concrete example of this construction for $m = 2$.

Let $n$ be an integer.



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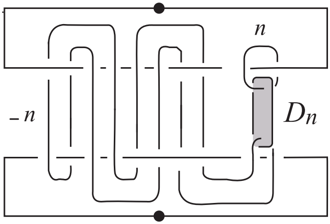


**Theorem (Abe-Tange)**

*(1) The exteriors of $D_n$ are the same.*
*(2) $D_n$ is a ribbon disk.*
*(2) $\partial D_n = K_n$.*