Surface-links which bound immersed handlebodies

Kengo Kawamura (OCAMI)

Intelligence of Low-dimensional Topology, RIMS

May 19, 2016

Today's Talk

1. Singularities in 3- and 4-spaces
   - Ribbon singularity and Clasp singularity in 3-space
   - Ribbon singularity and Clasp singularity in 4-space
2. Surface-links
   - Ribbon surface-links
   - Ribbon-clasp surface-links
3. Normal forms
   - Normal forms for embedded surface-links
   - Normal forms for immersed surface-links

Today's Talk

1. Singularities in 3- and 4-spaces
   - Ribbon singularity and Clasp singularity in 3-space
   - Ribbon singularity and Clasp singularity in 4-space
2. Surface-links
   - Ribbon surface-links
   - Ribbon-clasp surface-links
3. Normal forms
   - Normal forms for embedded surface-links
   - Normal forms for immersed surface-links

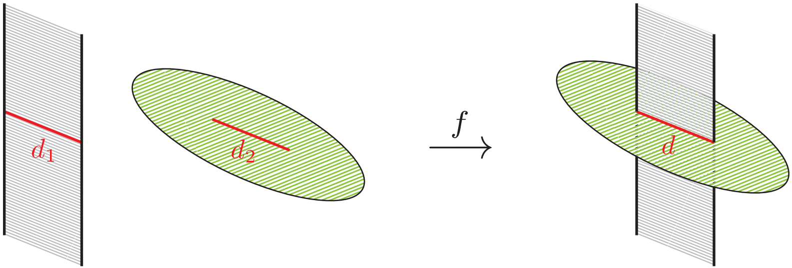
Ribbon singularity in the 3-space

M^2: a compact surface with non-empty boundary.

$f : M^2 \to \mathbb{R}^3$: an immersion.

d: a connected component of $\overline{\{x \in f(M) \mid \# f^{-1}(x) \geq 2\}}$.

d is a ribbon singularity $\overset{\text{def.}}{\Longleftrightarrow}$ $f^{-1}(d) = d_1 \cup d_2$ satisfies the following:



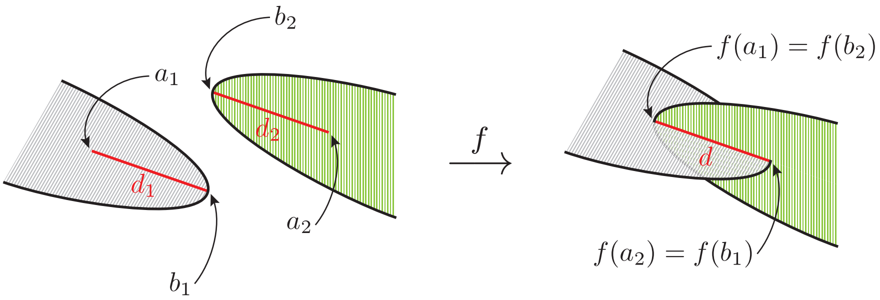
Clasp singularity in the 3-space

M^2: a compact surface with non-empty boundary.

$f : M^2 \to \mathbb{R}^3$: an immersion.

d: a connected component of $\overline{\{x \in f(M) \mid \# f^{-1}(x) \geq 2\}}$.

d is a clasp singularity $\overset{\text{def.}}{\Longleftrightarrow}$ $f^{-1}(d) = d_1 \cup d_2$ satisfies the following:



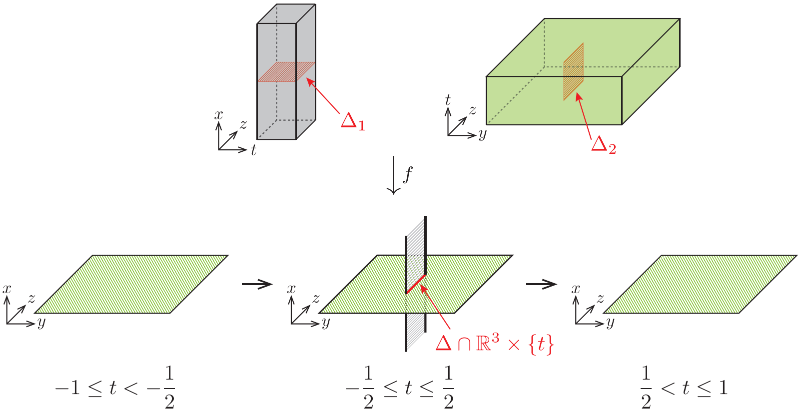
Ribbon singularity in the 4-space

M^3: a compact 3-manifold with non-empty boundary.

$f : M^3 \to \mathbb{R}^4$: an immersion.

Δ: a connected component of $\overline{\{x \in f(M) \mid \# f^{-1}(x) \geq 2\}}$.

Δ is a ribbon singularity $\overset{\text{def.}}{\Leftrightarrow}$ $f^{-1}(\Delta) = \Delta_1 \cup \Delta_2$ satisfies the following:



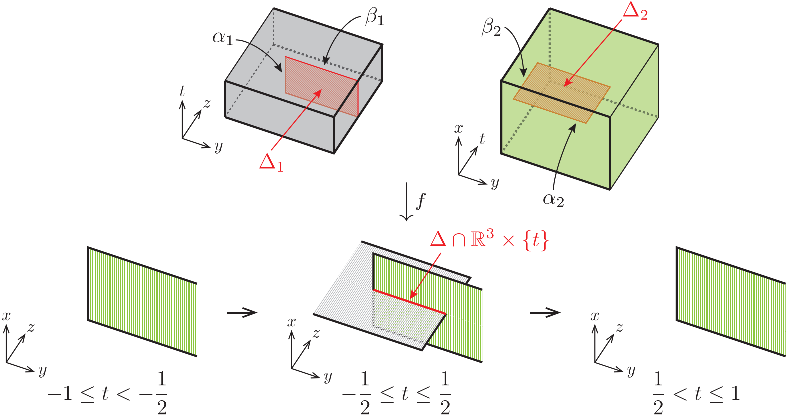
Clasp singularity in the 4-space

M^3: a compact 3-manifold with non-empty boundary.

$f : M^3 \to \mathbb{R}^4$: an immersion.

Δ: a connected component of $\overline{\{x \in f(M) \mid \# f^{-1}(x) \geq 2\}}$.

Δ is a clasp singularity $\overset{\text{def.}}{\Leftrightarrow}$ $f^{-1}(\Delta) = \Delta_1 \cup \Delta_2$ satisfies the following:



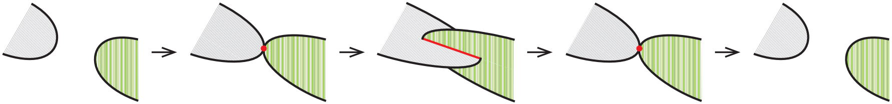
Clasp singularity in the 4-space

M^3: a compact 3-manifold with non-empty boundary.

$f : M^3 \to \mathbb{R}^4$: an immersion.

Δ: a connected component of $\overline{\{x \in f(M) \mid \# f^{-1}(x) \geq 2\}}$.

Δ is a clasp singularity $\overset{\text{def.}}{\Leftrightarrow}$ $f^{-1}(\Delta) = \Delta_1 \cup \Delta_2$ satisfies the following:



Today's Talk

Surface-links

- An (immersed) surface-link is a closed, oriented surface generically immersed in $\mathbb{R}^4$.
- When it is embedded, we also call it an embedded surface-link.
- Two surface-links are said to be equivalent if they are ambient isotopic in $\mathbb{R}^4$.
- A surface-link is called trivial if it is the boundary of a disjoint union of embedded handlebodies in $\mathbb{R}^4$.

- A Montesinos twin is a surface-link which is the boundary of a pair of embedded oriented 3-disks B_1^3 and B_2^3 with a single clasp singularity between B_1^3 and B_2^3.

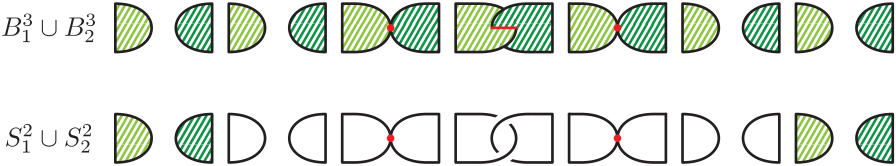


Figure: Montesinos twin $S_1^2 \cup S_2^2 = \partial B_1^3 \cup \partial B_2^3$

- An M-trivial 2-link is a surface-link which is a split union of a trivial 2-link and some (or no) Montesinos twins.

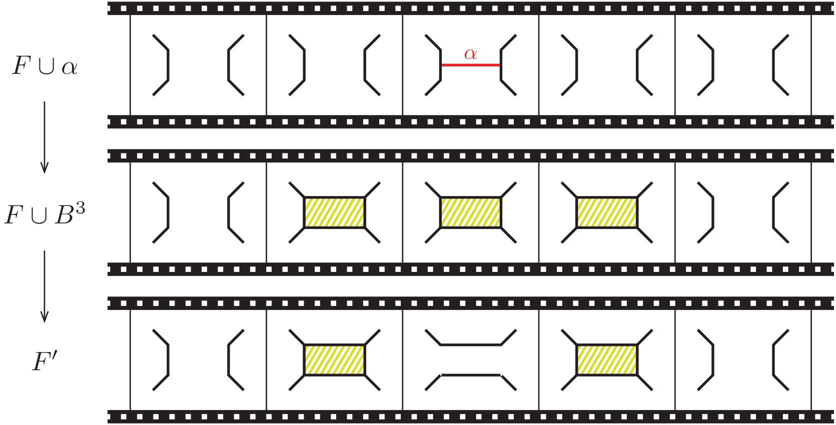
Chord attached to a surface-link

- A chord attached to a surface-link F means a simple arc α in $\mathbb{R}^4$ such that $F \cap \alpha = \partial\alpha$, which misses the double points of F.

- Two chords attached to F are equivalent if they are ambient isotopic in $\mathbb{R}^4$ keeping F setwise fixed.

1-handle surgery

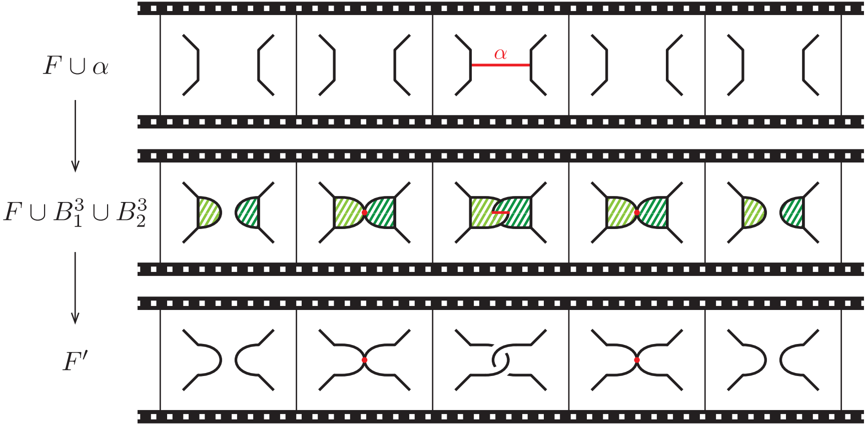
α: a chord attached to a surface-link F.



- F' is obtained from F by 1-handle surgery along α.
- $[F']$ is uniquely determined by $[F]$ and $[\alpha]$.

Finger move

α: a chord attached to a surface-link F.



- F' is obtained from F by finger move along α.
- $[F']$ is uniquely determined by $[F]$ and $[\alpha]$.

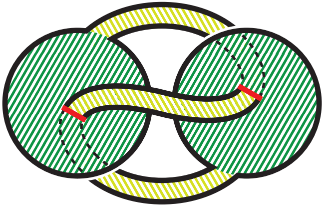
Ribbon 2-knot (Generalization of ribbon 1-knot)

Ribbon 1-knot

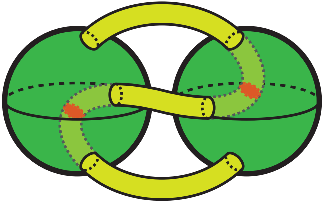
$K \subset \mathbb{R}^3$: a ribbon 1-knot
$\overset{\text{def.}}{\Longleftrightarrow} \exists D^2$: a ribbon singular 2-disk in $\mathbb{R}^3$ s.t. $\partial D^2 = K$.

Ribbon 2-knot

$F \subset \mathbb{R}^4$: a ribbon 2-knot
$\overset{\text{def.}}{\Longleftrightarrow} \exists D^3$: a ribbon singular 3-disk in $\mathbb{R}^4$ s.t. $\partial D^3 = F$.



$\partial D^2 = K$



$\partial D^3 = F$

Ribbon surface-link

Definition

An embedded surface-link is ribbon if it is the boundary of an immersed 3-manifold M^3 in $\mathbb{R}^4$ such that M^3 is a disjoint union of handlebodies and the singularity is a union of ribbon singularities.

Theorem ([Yanagawa '69], [Kawauchi-Shibuya-Suzuki '83])

An embedded surface-link is ribbon if and only if it is obtained from a trivial 2-link by 1-handle surgeries.

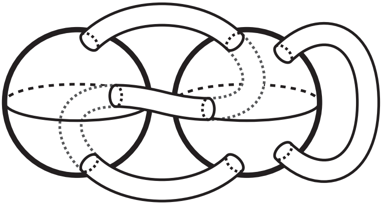


Figure: Ribbon torus-knot

Ribbon-clasp surface-link

Definition

A surface-link is ribbon-clasp if it is the boundary of an immersed 3-manifold M^3 in $\mathbb{R}^4$ such that M^3 is a disjoint union of handlebodies and the singularity is a union of ribbon singularities and clasp singularities.

Theorem (Kamada-Kawamura)

For a surface-link F, the following conditions are equivalent.

(1) F is a ribbon-clasp surface-link.

(2) F is obtained from a ribbon surface-link by finger moves.

(3) F is obtained from a trivial 2-link by 1-handle surgeries and finger moves.

(4) F is obtained from an M-trivial 2-link by 1-handle surgeries.

Today's Talk

1. Singularities in 3- and 4-spaces
   - Ribbon singularity and Clasp singularity in 3-space
   - Ribbon singularity and Clasp singularity in 4-space
2. Surface-links
   - Ribbon surface-links
   - Ribbon-clasp surface-links
3. Normal forms
   - Normal forms for embedded surface-links
   - Normal forms for immersed surface-links

The realizing surface

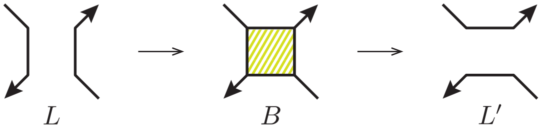


Figure: A band surgery on a link: $L \to L'$

The realizing surface of a band surgery $L \to L'$ by bands $B_1, \ldots, B_m$ is a compact oriented surface, say F, in $\mathbb{R}^3[a, b]$ defined by:

$$F \cap \mathbb{R}^3 \times \{t\} = \begin{cases} L' \times \{t\} & \text{for } t \in ((a+b)/2, b] \\ (L \cup B_1 \cup \cdots \cup B_m) \times \{t\} & \text{for } t = (a+b)/2 \\ L \times \{t\} & \text{for } t \in [a, (a+b)/2). \end{cases}$$

We denote the realizing surface by $F(L \to L')_{[a,b]}$.

Let

$$\mathcal{L} : L_1 \to L_2 \to \cdots \to L_m$$

be a band surgery sequence.

The realizing surface $F(\mathcal{L})_{[a,b]}$ of $\mathcal{L}$ in $\mathbb{R}^3 \times [a, b]$ with a division $a = t_1 < t_2 < \cdots < t_m = b$ is the union of the realizing surfaces $F(L_i \to L_{i+1})_{[t_i,t_{i+1}]}$ for $i = 1, 2, \ldots, m-1$.

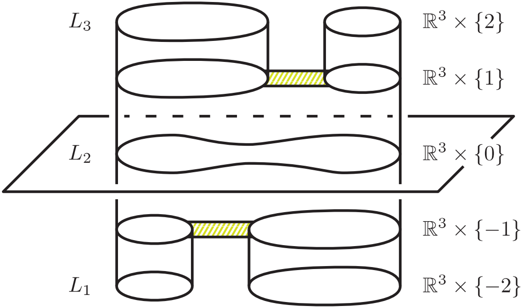


Figure: The realizing surface $F(L_1 \to L_2 \to L_3)_{[-2,2]}$

The closed realizing surface

For the realizing surface $F(\mathcal{L})_{[a,b]}$ of a band surgery sequence

$$\mathcal{L} : L_1 \to L_2 \to \cdots \to L_m$$

with trivial links L_1 and L_m, let $\mathcal{D}$ and $\mathcal{D}'$ be any disk systems bounded by L_1 and L_m respectively.

The closed oriented surface

$$\overline{F}(\mathcal{L})_{[a,b]} = F(\mathcal{L})_{[a,b]} \cup \mathcal{D} \times [a] \cup \mathcal{D}' \times [b]$$

is called the closed realizing surface of $\mathcal{L}$ in $\mathbb{R}^3[a,b]$.

Note that by Horibe-Yanagawa's lemma, $\overline{F}(\mathcal{L})_{[a,b]}$ does not depend on choices of disk systems $\mathcal{D}$ and $\mathcal{D}'$.

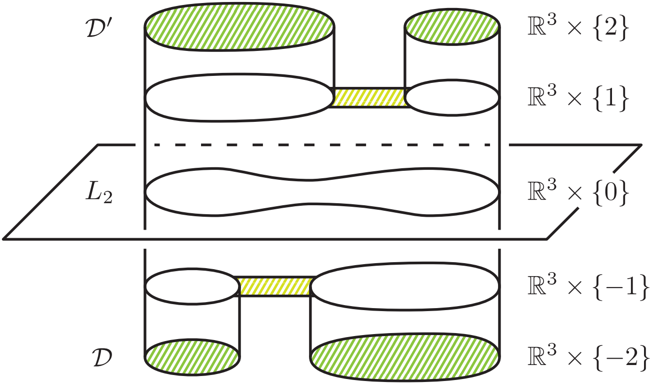


Figure: The closed realizing surface $\overline{F}(L_1 \to L_2 \to L_3)_{[-2,2]}$

Normal forms for embedded surface-links

Theorem (Normal form, [Kawauchi-Shibuya-Suzuki '83])

Every embedded surface-link with μ components and g total genus is equivalent to the closed realizing surface of a band surgery sequence

$$O \to L_- \to L_0 \to L_+ \to O',$$

where O and O' are trivial links, L_- and L_+ are μ-component links and L_0 is a $(\mu + g)$-component link.

Theorem (Ribbon normal form, [KSS '83])

An embedded surface-link is ribbon if and only if it is equivalent to the closed realizing surface of a band surgery sequence

$$O \to L \to O,$$

where O is a trivial link and the band surgery $L \to O$ is the inverse of $O \to L$.

H-trivial link and Hopf-splitting deformation

- An H-trivial link is a split union of a trivial link and some (or no) Hopf links.
- A Hopf-splitting deformation is a crossing change deformation from an H-trivial link into a trivial link such that each crossing change occurs for a Hopf link to change it into a trivial 2-component link.

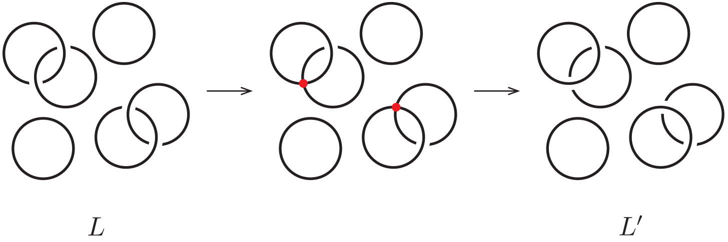


Figure: A Hopf-splitting deformation: $L \to L'$

Let

$$\mathcal{L} : L_1 \to L_2 \to \cdots \to L_m$$

be a band surgery sequence with H-trivial links L_1 and L_m.

Let

$$\mathcal{L}' : L_1' \to L_1 \to L_2 \to \cdots \to L_m \to L_m'$$

denote a sequence of links, where $L_1' \to L_1$ is the inverse operation of a Hopf-splitting deformation and $L_m \to L_m'$ is a Hopf-splitting deformation.

Then, we can construct the "realizing surface" $F(\mathcal{L}')_{[a,b]}$ of $\mathcal{L}'$ in $\mathbb{R}^3[a,b]$.

The closed oriented surface

$$\overline{F}(\mathcal{L})_{[a,b]} = F(\mathcal{L}')_{[a,b]} \cup \mathcal{D} \times [a] \cup \mathcal{D}' \times [b]$$

is called the closed realizing surface of $\mathcal{L}$ in $\mathbb{R}^3[a,b]$.

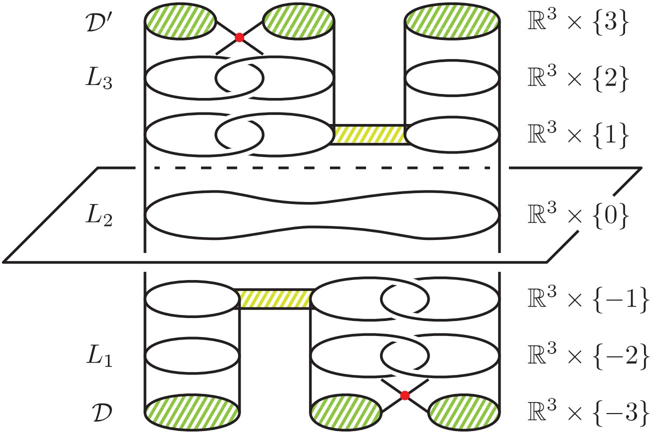


Figure: The closed realizing surface $\overline{F}(L_1 \to L_2 \to L_3)_{[-3,3]}$

Normal forms for immersed surface-links

Theorem (Normal form, [Kamada-Kawamura])

Every immersed surface-link with μ components and g total genus is equivalent to the closed realizing surface of a band surgery sequence

$$O \to L_- \to L_0 \to L_+ \to O',$$

where O and O' are H-trivial links, L_- and L_+ are μ-component links and L_0 is a $(\mu + g)$-component link.

Theorem (Ribbon-clasp normal form, [K-K])

An immersed surface-link is ribbon-clasp if and only if it is equivalent to the closed realizing surface of a band surgery sequence

$$O \to L \to O,$$

where O is an H-trivial link and the band surgery $L \to O$ is the inverse of $O \to L$.