# An explicit relation between knot groups in lens spaces and those in $S^3$

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Intelligence of Low-dimensional Topology

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# Motivation

$K$: a knot in $S^3$.

**Problem**

*Does there exist a free action* $\mathbb{Z}_p \curvearrowright (S^3, K)$*?*

In other words,

**Problem**

*Does there exist a knot* $K' \subset L(p,q)$ *for some* $q$ *s.t.* $\pi^{-1}(K') \sim K$*?*

$\pi\colon S^3 \to L(p,q)$: the $p$-fold cyclic covering.
Fix a prime knot $K$ and $p \in \mathbb{Z}_{\geq 2}$.

**Remark (Sakuma '86, Boileau-Flapan '87)**

If $\exists K'$ & $\exists q$, then they are unique.

# Background

**Problem (Matveev '15 in ILDT, cf. Fox '61)**

*Do there exist non-equivalent knots in $\mathbb{R}P^3$ such that their lifts to $S^3$ are equivalent knots?*

**Remark**

Since $\mathrm{Diff}^+(\mathbb{R}P^3)$/diffeotopy $= \{[\mathrm{id}_{\mathbb{R}P^3}]\}$,
$K_0$ is ambient isotopic to $K_1$ iff $(\mathbb{R}P^3, K_0) \cong (\mathbb{R}P^3, K_1)$.

**In this talk...**

We do not focus on Uniqueness, but Existence.

**Theorem (Conner-Raymond '72 & Burde-Zieschang '66)**

*If* $\mathrm{Out}(G(K)) = 1$, *then the answer is NO for any* $p$.

**Remark (see Kawauchi (ed.) '96, Kodama-Sakuma '92)**

$\mathrm{Out}(G(K)) = 1$ for $9_{32}$, $9_{33}$ or $10_n$ ($n = 80$, $82$–$87$, $90$–$95$, $102$, $106$, $107$, $110$, $117$, $119$, $148$–$151$, $153$) (or their mirror images).

**Theorem (Hartley '81)**

*In the case of* $K = T_{m,n}$, *the answer is YES iff* $\gcd(mn, p) = 1$.
(*Moreover, the answer for prime knots with* $c(K) \leq 10$ *is given.*)

**The aim of this talk**

To deduce the above theorems from a single result.

# Main results

**Definition**

$G$: a group. $\mathrm{C}^p(G) := \langle \{g^p \mid g \in G\} \cup \{[g,h] \mid g,h \in G\} \rangle \lhd G$.
A group $H$ is called a $\mathrm{C}^p$-group if $\exists G$ s.t. $\mathrm{C}^p(G) \cong H$.

$\pi \colon \Sigma \to \Sigma'$: a $p$-fold cyclic covering, where $\Sigma$ is a $\mathbb{Z}\mathrm{HS}^3$.

**Theorem (N.)**

*$K'$: a knot in $\Sigma'$ with connected preimage $K := \pi^{-1}(K')$. Then,*

$$\mathrm{Im}[\pi_* \colon \pi_1(\Sigma \setminus K) \rightarrowtail \pi_1(\Sigma' \setminus K')] = \mathrm{C}^p(\pi_1(\Sigma' \setminus K')).$$

*In particular, $\pi_1(\Sigma \setminus K)$ is a $\mathrm{C}^p$-group.*

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# Basic properties of $C^p$

$$\mathrm{C}^p(G) := \langle \{g^p \mid g \in G\} \cup \{[g,h] \mid g,h \in G\} \rangle \lhd G.$$

## Remark

$\mathrm{C}^p(G) = \mathrm{Ker}[G \twoheadrightarrow G_{\mathrm{ab}}/pG_{\mathrm{ab}}]$. Indeed, we have $\mathrm{ab}^{-1}(pG_{\mathrm{ab}}) = \mathrm{C}^p(G)$, where $\mathrm{ab}\colon G \twoheadrightarrow G_{\mathrm{ab}}$.

## Example

- $G = \mathbb{Z}_n \rightsquigarrow G/\mathrm{C}^p(G) \cong \mathbb{Z}_{\gcd(n,p)}$.
- $G = \mathbb{R}_{>0} \rightsquigarrow G/\mathrm{C}^p(G) = 0$.
- $G = \mathbb{Q}_{>0} \rightsquigarrow G/\mathrm{C}^p(G) \cong \mathbb{Z}_p^{\oplus\{\text{prime numbers}\}}$.

# Proof of Theorem

$\pi\colon \Sigma \to \Sigma'$: a $p$-fold cyclic covering, where $\Sigma$ is a $\mathbb{Z}\mathrm{HS}^3$.
$G := \pi_1(\Sigma \setminus K)$, $G' := \pi_1(\Sigma' \setminus K')$.

**Lemma**

*$H_1(\Sigma') = \langle [K'] \rangle \cong \mathbb{Z}_p$, and $H_1(\Sigma' \setminus K') \cong \mathbb{Z}$.*

**Proof of $\mathrm{Im}[\pi_*\colon G \rightarrowtail G'] = \mathrm{C}^p(G')$.**

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \xrightarrow{\pi_*} & G' & \longrightarrow & \mathbb{Z}_p & \longrightarrow & 1 \\ &&&& \downarrow & \overset{\cong}{\nearrow} &&& \\ &&&& G'_{\mathrm{ab}}/pG'_{\mathrm{ab}} &&&& \end{array}$$

Therefore, $\mathrm{Im}\,\pi_* = \mathrm{Ker}[G' \twoheadrightarrow G'_{\mathrm{ab}}/pG'_{\mathrm{ab}}] \overset{\text{Remark}}{=\!=\!=} \mathrm{C}^p(G')$. □

# Proof of the 1st corollary

**Definition**

A group $G$ is complete if $Z(G) = 1$ & $\mathrm{Out}(G) = 1$.

**Lemma (Generalization of Haugh-MacHale '97)**

*If $G$ is complete & $\mathrm{C}^p(G) \neq G$, then $G$ is not a $\mathrm{C}^p$-group.*

**Lemma**

*If $G_{\mathrm{ab}} \cong \mathbb{Z}$, then $G/\mathrm{C}^p(G) \cong \mathbb{Z}_p$.*

**Proof of "$\mathrm{Out}(G(K)) = 1 \Rightarrow \nexists K'$".**

$\mathrm{Out}(G(K)) = 1 \rightsquigarrow K$ is not a torus knot

$\rightsquigarrow Z(G(K)) = 1 \rightsquigarrow G(K)$ is complete

$\xrightarrow{\text{Lemmas \& Theorem}} \nexists K'$. □

# Proof of the 2nd corollary

**Lemma**

*Let $m, n, p \in \mathbb{Z}_{\geq 2}$ with $\gcd(m, n) = 1$. If there is a group $G$ satisfies (a) $\mathrm{C}^p(G) \cong \mathbb{Z}_m * \mathbb{Z}_n$, (b) $G/\mathrm{C}^p(G) \cong \mathbb{Z}_p$ and (c) $H_*(G) \cong H_*(\mathbb{Z}_{mnp})$, then $\gcd(mn, p) = 1$.*

The key steps are as follows:

- $\mathrm{Ad}_g|_{\mathbb{Z}_m * \mathbb{Z}_n} \in \mathrm{Inn}(\mathbb{Z}_m * \mathbb{Z}_n)$.
- The five-term exact sequence for

$$1 \to \mathbb{Z}_m * \mathbb{Z}_n \hookrightarrow G \to \mathbb{Z}_p \to 1.$$

- $1 = \mathbb{Z}_m * \mathbb{Z}_n / \mathrm{C}^p(\mathbb{Z}_m * \mathbb{Z}_n) \twoheadrightarrow \mathbb{Z}_{mn}/\mathrm{C}^p(\mathbb{Z}_{mn}) = \mathbb{Z}_{\gcd(mn,p)}$.

We can obtain a group $G$ as above from $K'$.

**Proof of "$\exists K' \subset L(p,q) \Rightarrow \gcd(mn,p)=1$".**

$$1 \to G(T_{m,n}) \xrightarrow{\pi_*} \pi_1(L(p,q) \setminus K') \to \mathbb{Z}_p \to 1 \quad \text{(exact)}.$$

Taking the quotients by $Z := Z(G(T_{m,n})) \cong \mathbb{Z}$,

$$1 \to \mathbb{Z}_m * \mathbb{Z}_n \to \pi_1(L(p,q) \setminus K')/Z \to \mathbb{Z}_p \to 1 \quad \text{(exact)}.$$

Set $G := \pi_1(L(p,q) \setminus K')/Z$. Theorem $\rightsquigarrow$ (a) & (b).
Lyndon-Hochschild-Serre spectral sequence for

$$1 \to Z \to \pi_1(L(p,q) \setminus K') \to G \to 1 \quad \text{(exact)}.$$

$\rightsquigarrow$ (c) $H_*(G) \cong H_*(\mathbb{Z}_{mnp})$. □

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# The $n$th symmetric group $\mathfrak{S}_n$

Let $n \geq 3$. We have

$$\mathrm{C}^p(\mathfrak{S}_n) = \begin{cases} \mathfrak{S}_n & \text{if } p \text{ is odd,} \\ \mathfrak{A}_n & \text{if } p \text{ is even.} \end{cases}$$

Hence, $\mathfrak{S}_n$ is a $\mathrm{C}^p$-group for odd $p$.

**Lemma (Recall)**

*If $G$ is complete & $\mathrm{C}^p(G) \neq G$, then $G$ is not a $\mathrm{C}^p$-group.*

**Corollary**

*$\mathfrak{S}_n$ is not a $\mathrm{C}^p$-group for even $p$.*

**Remark**

The case $n = 6$ requires an additional argument.

# The $n$th braid group $B_n$

**Definition**

$H \leq G$ is characteristic if $f(H) = H$ for $\forall f \in \mathrm{Aut}(G)$.

**Lemma (Generalization of Sun '79)**

*If $G$: a $\mathrm{C}^p$-group, $f\colon G \twoheadrightarrow G'$: a homomorphism, $\mathrm{Ker}\, f$: characteristic, then $G'$ is also a $\mathrm{C}^p$-group.*

$\mathrm{Ker}[f\colon B_n \twoheadrightarrow \mathfrak{S}_n] = P_n$ is characteristic in $B_n$ (Artin '47).

**Corollary**

*$B_n$ is not a $\mathrm{C}^p$-group for even $p$.*

In particular, $G(T_{3,2})$ is not a $\mathrm{C}^p$-group if $2 \mid p$. On the other hand, $G(T_{3,2})$ is a $\mathrm{C}^p$-group if $\gcd(6, p) = 1$ (Hartley '69).
(The case $3 \mid p$ & $2 \nmid p$ is unknown.)

$G$: a group. $p \in \mathbb{Z}_{\geq 2}$ (not necessarily prime).

**Definition (Cochran-Harvey '08)**

The derived $p$-series of $G$ is defined by

$$G^{(0)} := G, \quad G^{(n+1)} := \mathrm{C}^p(G^{(n)}).$$

(cf. Stallings ('63) introduced the $p$-lower central series.)

**Theorem (Cochran-Harvey '08)**

*$p$: a prime. $A, B$: finitely generated groups. If $\phi\colon A \to B$ induces an isomorphism (resp. monomorphism) on $H_1(-;\mathbb{Z}_p)$ and an epimorphism on $H_2(-;\mathbb{Z}_p)$, then for each $n \in \mathbb{Z}_{\geq 0}$, it induces an isom (resp. monom) $A/A^{(n)} \to B/B^{(n)}$, and a monom $A/A^{(\omega)} \rightarrowtail B/B^{(\omega)}$.*

$G^{(\omega)} := \bigcap_{n\geq 0} G^{(n)} \leq G$.

### Corollary

*$p$: a prime. $G$: a f.g. group with $H_1(G)$: free & $H_2(G) = 0$.*
*Then $G^{(\omega)} = [G, G]$.*

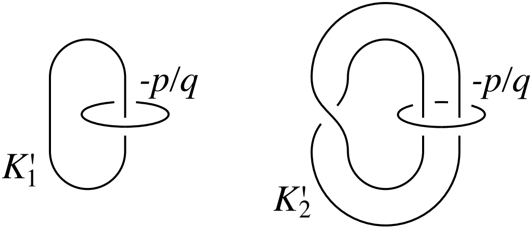
$G(K)$ and $B_n$ satisfies the above conditions ($p \neq 2$ is required when $n \geq 4$).

### Remark

Let $6 \mid p$ and $G := G(T_{3,2}) = B_3$. Then $f\colon G \twoheadrightarrow \mathfrak{S}_3$ induces $G/G^{(\omega)} \twoheadrightarrow \mathfrak{S}_3/\mathfrak{S}_3^{(\omega)} = \mathfrak{S}_3$. Hence, $G^{(\omega)} \lneq [G, G]$.

If $G(K)$ surjects onto $G(T_{3,2})$, then $G(K)^{(\omega)} \lneq [G(K), G(K)]$ $(6 \mid p)$.

Let $p > 3$ be an odd and $q := (p \pm 1)/2$ ($\rightsquigarrow \gcd(p, q) = 1$).




**Theorem (Manfredi '14)**

*$K_1'$ is **not** isotopic to $K_2'$, but their preimages are the unknot.*

**Proof.**

$2[K_1'] = [K_2'] \in H_1(L(p, q)) \rightsquigarrow [K_1'] \neq [K_2']$. □

**Remark**

$(L(p, q), K_1')$ is diffeomorphic to $(L(p, q), K_2')$.

## Future research

I would like to

- know whether the converse of "$\exists K' \subset L(p,q) \Rightarrow G(K)$ is a $\mathrm{C}^p$-group" is true.
- replace knots $K$ with links.
- study $\mathrm{C}^p$ from an algebraic point of view.
- find a relation with some invariants of knots.



Y. Nozaki, An explicit relation between knot groups in lens spaces and those in $S^3$, arXiv:1602.05884