Problems on Low-dimensional Topology, 2017

Edited by Tomotada Ohtsuki[1] and Tetsuya Ito[2]

This is a list of open problems on low-dimensional topology with expositions of their history, background, significance, or importance. This list was made by editing manuscripts written by contributors of open problems to the problem session of the conference "Intelligence of Low-dimensional Topology" held at Research Institute for Mathematical Sciences, Kyoto University in May 24–26, 2017.

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[1]Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto, 606-8502, JAPAN
Email: tomotada@kurims.kyoto-u.ac.jp
The first editor is partially supported by JSPS KAKENHI Grant Numbers 16H02145 and 16K13754.
[2]Department of Mathematics, Graduate School of Science, Osaka University 1-1 Machikaneyama Toyonaka, Osaka 560-0043, JAPAN
Email: tetito@math.sci.osaka-u.ac.jp
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1 The AJ conjecture for cable knots

(Anh T. Tran)

The AJ conjecture relates the A-polynomial and the colored Jones polynomial of a knot in S^3. The A-polynomial $A_K(M,L)$ of a knot $K \subset S^3$ was introduced by Cooper et al. [11]; it describes the character variety of the knot complement as viewed from the boundary torus. For every knot K, Garoufalidis and Le [16] proved that the colored Jones polynomial $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$, with the color n standing for the irreducible $sl_2(\mathbb{C})$-module of dimension n, satisfy linear recurrence relations. The recurrence polynomial $\alpha_K(t,M,L)$ of K is the minimal linear recurrence relation for $J_K(n)$. Motivated by the work of Frohman, Gelca and LoFaro [12] on the non-commutative A-ideal, Garoufalidis [14] proposed the AJ conjecture which states that $\alpha_K \mid_{t=-1}$ is equal to $A_K(M,L)$ up to multiplication by a polynomial depending on M only.

Suppose K is a knot and r,s are two relatively prime integers. The (r,s)-cable knot $K^{(r,s)}$ of K is the (r,s)-curve on the torus boundary of K. A cabling formula for the A-polynomial has recently been given by Ni and Zhang [42]. For a pair of relatively prime integers (r,s) with $s \geq 2$, define $F_{(r,s)}(M,L) \in \mathbb{Z}[M,L]$ by

$$F_{(r,s)}(M,L) = \begin{cases} M^{2r}L + 1 & \text{if } s = 2, \\ M^{2rs}L^2 - 1 & \text{if } s > 2. \end{cases}$$

Then

$$A_{K^{(r,s)}}(M,L) = (L-1)F_{(r,s)}(M,L)\,\mathrm{Res}_\ell\left(\frac{A_K(M^s,\ell)}{\ell - 1}, \ell^s - L\right), \tag{1}$$

where Res_ℓ denotes the polynomial resultant eliminating the variable ℓ.

On the other hand, a cabling formula for the colored Jones polynomial was given by Morton [36]:

$$J_{K^{(r,s)}}(n) = t^{-rs(n^2-1)} \sum_{j=-(n-1)/2}^{(n-1)/2} t^{4rj(sj+1)} J_K(2sj+1). \tag{2}$$

We define 2 linear operators L, M acting on discrete functions $f : \mathbb{Z} \to \mathbb{C}[t^{\pm 1}]$ by

$$(Lf)(n) = f(n+1), \qquad (Mf)(n) = t^{2n}f(n).$$

Using (2), one can relate the colored Jones polynomials of $K^{(r,s)}$ and K as follows:

$$\begin{aligned} & t^{2rs}M^{rs}(L^2 - t^{-4rs}M^{-2rs})J_{K^{(r,s)}}(n) \\ = \; & t^{2r(n+1)}J_K(s(n+1)+1) - t^{-2r(n+1)}J_K(s(n+1)-1). \end{aligned}$$

Moreover, in case of $(r,2)$-cable knots one has

$$M^r(L + t^{-2r}M^{-2r})J_{K^{(r,2)}}(n) = J_K(2n+1).$$

These equalities should imply a relationship between the recurrence polynomials of K and its cable $K^{(r,s)}$. It is ideal is to have a similar formula to (1) such as

$$\alpha_{K^{(r,s)}}(-1,M,L) = (L-1)F_{(r,s)}(M,L)\,\mathrm{Res}_\ell\left(\frac{\alpha_K(-1,M^s,\ell)}{\ell - 1}, \ell^s - L\right) \tag{3}$$

up to multiplication by a polynomial depending on M only.

Problem 1.1 (A. T. Tran). *Prove equality* (3) *for all cable knots* $K^{(r,s)}$.

If K satisfies the AJ conjecture and (3) holds true for a cable knot $K^{(r,s)}$, then $K^{(r,s)}$ satisfies the AJ conjecture.

2 Loop finite type invariants of null-homologous knots in rational homology 3-spheres

(Delphine Moussard)

For loop finite type invariants of null-homologous knots in rational homology 3-spheres, there are two candidates to be universal loop finite type invariants, namely the Kricker rational lift of the Kontsevich integral [27, 15] and the Lescop invariant constructed by means of equivariant intersections in configuration spaces [31]. Lescop conjectured in [32] that these two invariants are equivalent for null-homologous knots in rational homology 3-spheres with a given Blanchfield pairing and a given cardinality for the first homology group of the manifold; we recall that the *Blanchfield pairing* is the equivariant linking pairing on the universal cyclic cover of the knot complement. By *equivalent*, we mean that one invariant distinguishes two knots if and only if the other invariant distinguishes them.

We first recall some definitions. Let (M, K) be a $\mathbb{Q}$*SK-pair*, *i.e.* a pair of a rational homology 3-sphere M and a null-homologous knot K in M. Let $A \subset M \setminus K$ be a rational homology handlebody ($\mathbb{Q}$-handlebody). Assume A is null in $M \setminus K$, *i.e.* the map $incl_* : H_1(A, \mathbb{Q}) \to H_1(M \setminus K, \mathbb{Q})$ has a trivial image. Let B be another $\mathbb{Q}$-handlebody with the same genus. Fix a homeomorphism $h : \partial A \to \partial B$. The $\mathbb{Q}$SK-pair obtained from (M, K) by the null Lagrangian-preserving surgery, or null LP-surgery, $(\frac{B}{A})$ is:

$$(M, K)(\frac{B}{A}) = ((M \setminus Int(A)) \cup_h B, K).$$

Let $\mathcal{F}_0$ be the rational vector space spanned by all $\mathbb{Q}$SK-pairs up to orientation-preserving homeomorphism. Let $\mathcal{F}_n$ be the vector subspace of $\mathcal{F}_0$ spanned by the

$$[(M, K); (\frac{B_1}{A_1}), \ldots, (\frac{B_n}{A_n})] = \sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} (M, K)((\frac{B_i}{A_i})_{i \in I})$$

for all $\mathbb{Q}$SK-pairs (M, K), all families of disjoint $\mathbb{Q}$-handlebodies $A_1, \ldots, A_n$ null in $M \setminus K$ and all families of $\mathbb{Q}$-handlebodies $B_1, \ldots, B_n$ with a fix identification $\partial B_i \cong \partial A_i$. Since $\mathcal{F}_{n+1} \subset \mathcal{F}_n$, this defines a filtration of $\mathcal{F}_0$.

Definition. A $\mathbb{Q}$-linear map $\lambda : \mathcal{F}_0 \to \mathbb{Q}$ is a *loop finite type invariant of degree* n of $\mathbb{Q}$SK-pairs if $\lambda(\mathcal{F}_{n+1}) = 0$.

The homeomorphism classes of $\mathbb{Q}$SK-pairs up to null LP-surgeries are characterized by the isomorphism classes of their Blanchfield pairings [37]. Therefore, the Blanchfield pairing dominates degree 0 invariants. The degree 1 invariants are given by the cardinality of the first homology group of the manifold.

Question 2.1 (D. Moussard)**.** *Are the Kricker lift of the Kontsevich integral and the Lescop invariant universal loop finite type invariants of $\mathbb{Q}$SK-pairs up to degree* 0 *and* 1 *invariants?*

Since these two invariants satisfy the same splitting formulas with respect to null LP-surgeries [32, 38], the answer should be the same for both invariants. Note that Question 2.1 implies Lescop's conjecture. The answer is known to be positive when the Blanchfield pairing is trivial [39].

Since the Blanchfield pairing dominates degree 0 invariants, the space $\mathcal{F}_0$ is the direct sum over all isomorphism classes of Blanchfield pairings $\mathfrak{B}$ of spaces $\mathcal{F}_0(\mathfrak{B})$, and the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ splits accordingly into filtrations $(\mathcal{F}_n(\mathfrak{B}))_{n\in\mathbb{N}}$ of each $\mathcal{F}_0(\mathfrak{B})$. Denote $\mathcal{G}_n(\mathfrak{B}) = \mathcal{F}_n(\mathfrak{B})/\mathcal{F}_{n+1}(\mathfrak{B})$ the quotients of the filtration. For a given Blanchfield pairing $\mathfrak{B}$, we have a map $C_n : \mathcal{G}_n(\mathfrak{B}^{\oplus k}) \to \mathcal{G}_n(\mathfrak{B}^{\oplus k+1})$ induced by the connected sum of $\mathbb{Q}$SK-pairs with a fixed $\mathbb{Q}$SK-pair (M_0, K_0) with Blanchfield pairing $\mathfrak{B}$.

Question 2.2 (D. Moussard)**.** *Given a Blanchfield pairing* $\mathfrak{B}$ *and integers* k, n *with* $1 \leq 2k < 3n$*, is the map* $C_n : \mathcal{G}_n(\mathfrak{B}^{\oplus k}) \to \mathcal{G}_n(\mathfrak{B}^{\oplus k+1})$ *injective?*

The bound on the value of k comes from the fact that $\mathcal{G}_n(\mathfrak{B}^{\oplus N})$ is combinatorially described in [39] when $N \geq \frac{3}{2}n$. In particular, for a Blanchfield pairing $\mathfrak{B}^{\oplus 3n}$, the Kricker lift of the Kontsevich integral and the Lescop invariant dominate loop finite type invariants of degree non-zero and at most $2n$. In [39], the following fact is also established:

Fact. For a given Blanchfield pairing, Question 2.1 is equivalent to Question 2.2 for all n and k.

About Question 2.2, we shall mention a result of existence and uniqueness of the decomposition of a $\mathbb{Q}$SK-pair as the connected sum of irreducible $\mathbb{Q}$SK-pairs [35]. Unfortunately, such a decomposition is not preserved by null LP-surgeries.

Remark (T. Ohtsuki). The *loop-degree* of a Jacobi diagram on S^1 is defined to be half of the number given by the number of trivalent vertices minus the number of univalent vertices of the uni-trivalent graph of the Jacobi diagram. The filtration of the space of Jacobi diagrams on S^1 given by loop-degrees is related to the filtration of loop finite type invariants through the rational lift of the Kontsevich integral. The theory of this filtration is developed in [17] (noting that this notion also appears in [26] and the September 1999 version of [28]); see also [44, Section 2.9].

3 The double covering method for twisted knots

(Naoko Kamada)[3]

Virtual knot theory was introduced by L. H. Kauffman [24] as a generalization of knot theory based on Gauss chord diagrams and link diagrams in closed oriented surfaces. A virtual link diagram is a link diagram possibly with virtual crossings, and a virtual link is an equivalence class of such diagrams by some Reidemeister type moves. Virtual links correspond to stable equivalence classes of links in oriented 3-manifolds which are line bundles over closed oriented surfaces ([6, 22]). Twisted knot theory was introduced by M. O. Bourgoin [1]. It is an extension of virtual knot theory. A twisted link diagram is a link diagram possibly with virtual crossings and bars on arcs, and a twisted link is an equivalence class of such diagrams by some Reidemeister type moves. Twisted links correspond to stable equivalence classes of links in oriented 3-manifolds which are line bundles over (possibly non-orientable) closed surfaces ([1, 6]).

An abstract link diagram is a link diagram D on a compact surface Σ such that $|D|$ is a deformation retract of Σ, where $|D|$ is the underlying immersed loops in Σ by forgetting over/under information from D. In [22] and [1], an equivalence relation on abstract link diagrams is defined. An abstract link is an equivalence classes of abstract link diagrams by the equivalence relation. There is a bijection from the set of virtual links and the set of abstract links on orientable surfaces [22], and there is a bijection from the set of twisted links and the set of abstract links on surfaces [1].

Definition. An abstract link diagram D on Σ is *normal* or *checkerboard colorable* if $\Sigma \setminus |D| \cup (\text{bars})$ is checkerboard colorable. (Namely, we can assign elements of $\mathbb{Z}/2\mathbb{Z}$ to regions of $\Sigma \setminus |D| \cup (\text{bars})$ such that two regions sharing an arc (or a bar) are assigned distinct values.) A virtual link diagram or a twisted link diagram is *normal* if the corresponding abstract link diagram (in the sense of [22]) is normal. A virtual link or a twisted link is *normal* if there is a representative which is a normal virtual (or twisted) link diagram.

Problem 3.1 (N. Kamada)**.** *Construct invariants of normal virtual links.*

For example, the signature is defined for normal virtual links [20].

In the talk at the conference, a method of obtaining a virtual link diagram from a twisted link diagram, called the double covering method ([23]). Using this method, we obtain a method of converting any virtual link diagram with a cut system to a normal virtual link diagram. (A *cut system* of a virtual link diagram D is a finite number of points C on arcs of D such that when we put bars at the points of C, we obtain a normal twisted link diagram. Then the double covering of such a normal twisted link diagram is a normal virtual link diagram. This is the method of the conversion. The resulting normal virtual link diagram depends on the cut system C for D, but its K-equivalence class is uniquely determined from D. Here K-equivalence

[3]Supported by JSPS KAKENHI Grant Number 15K04879.
Graduate School of Natural Sciences, Nagoya City University, 1 Yamanohata, Mizuho-cho, Mizuho-ku, Nagoya, Aichi 467-8501 Japan

is the equivalence relation on virtual link diagrams modulo the Reidemeister type moves for virtual links and an additional move called Kauffman's flype.)

Problem 3.2 (N. Kamada)**.** *Find another method of converting a virtual link to a normal virtual link. Using the method, construct a new invariant of virtual links or find an application.*

Definition. An abstract link diagram D on oriented Σ is *almost classical* or *region $\mathbb{Z}$-colorable* if $\Sigma \setminus |D|$ is $\mathbb{Z}$-colorable, namely, we can assign elements of $\mathbb{Z}$ to regions of $\Sigma \setminus |D|$ such that if the right side is assigned i with respect to the orientation of the arc then the left side is assigned $i+1$. A virtual link diagram is *almost classical* if the corresponding abstract link diagram is almost classical. A virtual link is *almost classical* if there is a representative which is an almost classical virtual link diagram.

By the projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, an almost virtual link is a normal virtual link.

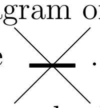
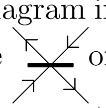
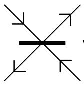
Problem 3.3 (N. Kamada)**.** *Construct invariants of almost classical virtual links.*

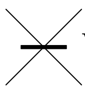
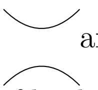
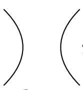
Problem 3.4 (N. Kamada)**.** *Find a method of converting a virtual link to an almost classical virtual link. Using the method, construct a new invariant of virtual links or find an application.*

4 Presentations of (immersed) surface-knots by marked graph diagrams

(Jieon Kim)[4]

An immersed surface-knot is a generically immersed closed and connected surface in the 4-space $\mathbb{R}^4$.

A *marked graph diagram* is a diagram of a 4-valent graph each of whose vertices is a vertex with a marker looks like . An *oriented marked graph diagram* is a marked graph diagram in which every edge has an orientation such that each marked vertex looks like  or .

For a given (oriented) marked graph diagram D, let $L_+(D)$ and $L_-(D)$ be classical (oriented) link diagrams obtained from D by replacing each marked vertex  with  and , respectively. A marked graph diagram D is said to be *H-admissible* if both resolutions $L_+(D)$ and $L_-(D)$ are H-trivial link diagrams, where an H-trivial link is a split union of trivial knots and Hopf links. See Fig. 1.

An *unlinking crossing point c* of a classical link diagram L is a crossing of L such that L is transformed into a diagram of an unknotted link by switching over arc and under arc of c.

[4](Osaka city University/JSPS)

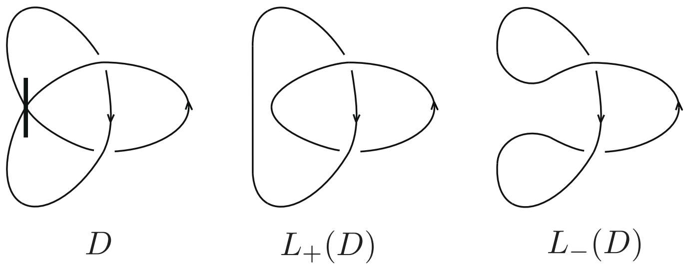


Figure 1: H-admissible marked graph diagram

Definition. A crossing point p (in a marked graph diagram D) is an *upper singular point* if p is an unlinking crossing point in the resolution $L_+(D)$, and a *lower singular point* if p is an unlinking crossing point in the resolution $L_-(D)$, respectively.

Remark. The moves (a) of Γ_9 and Γ_9' in Fig. 4 need the conditions that the components l^+ (in the resolution $L_+(D)$) and l^- (in the resolution $L_-(D)$) are trivial, respectively. The moves (b) of Γ_9 and Γ_9' need the conditions that p are an upper singular point and a lower singular point, respectively.

It is known that two marked graph diagrams present equivalent surface-links if and only if they are related by the Yoshikawa moves of type I and type II (see [49]). The *generalized Yoshikawa moves* for marked graph diagrams are the deformations $\Gamma_1, \ldots, \Gamma_5$ (Type I), $\Gamma_6, \ldots, \Gamma_8$ (Type II), and $\Gamma_9, \Gamma_9', \Gamma_{10}$ (Type III) as shown in Fig. 2, Fig. 3, and Fig. 4, respectively.

Definition. A set $\mathcal{S}$ of moves are *independent* if x is not generated by finite sequences of moves in $\mathcal{S} \setminus \{x\}$ for every $x \in \mathcal{S}$.

Question 4.1 (S. Kamada, A. Kawauchi, J. Kim, S. Y. Lee [21]). *Is the set of generalized Yoshikawa moves independent?*

Lemma. Let $\mathcal{L}$ and $\mathcal{L}'$ be immersed surface-links, and D and D' their marked graph diagrams, respectively. If D and D' are related by a finite sequence of generalized Yoshikawa moves, then $\mathcal{L}$ and $\mathcal{L}'$ are equivalent.

Problem 4.2 (J. Kim). *Find the set $\mathcal{S}$ of local moves of marked graph diagrams such that the marked graph diagrams are related by $\mathcal{S}$ if and only if their immersed surface-links are equivalent.*

Problem 4.3 (J. Kim). *Create a table of H-admissible marked graph diagrams representing immersed surface-links under the equivalence of $\mathcal{S}$ in the previous Problem with ch-index 10 or less, where the ch-index of a marked graph diagram is the sum of the number of crossings and that of vertices.*

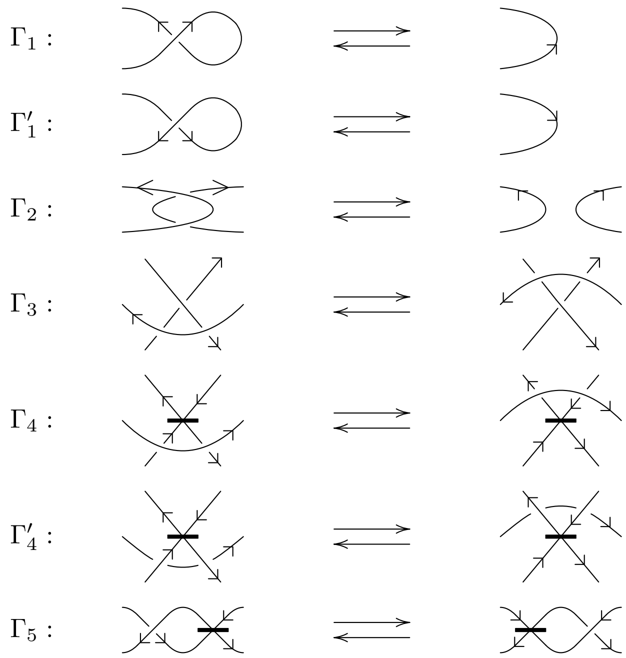


Figure 2: Moves of Type I

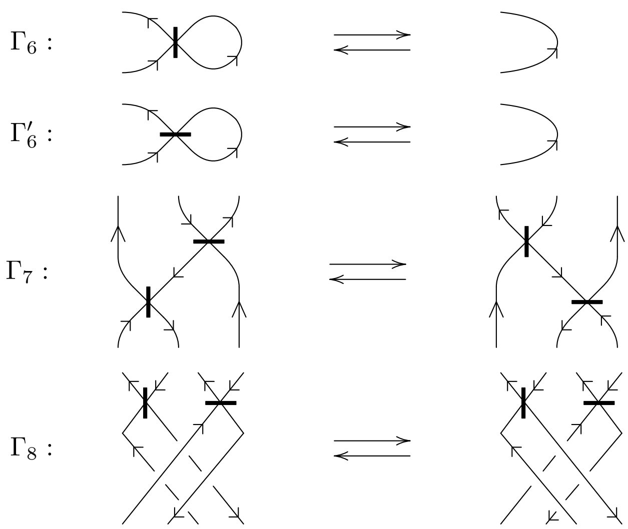


Figure 3: Moves of Type II

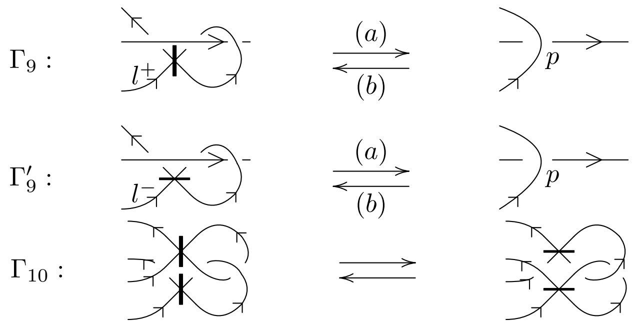


Figure 4: Moves of Type III

5 Knotted 2-foams and quandle homology

(J. Scott Carter)

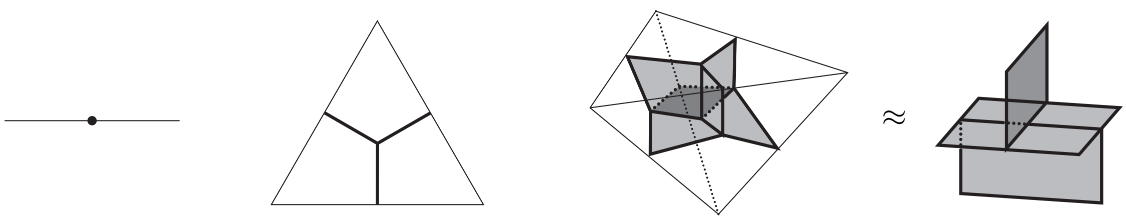
Let

$$\Delta^{n+1} = \left\{\vec{x} = (x_0, x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+2} \;\middle|\; \sum x_i = 1 \ \&\ 0 \leq x_i\right\}$$

denote the standard simplex. The space $Y^n \subset \Delta^{n+1}$ is defined as follows: $Y^0 = (\frac{1}{2}, \frac{1}{2})$. Take $\Delta^n_j = \{\vec{x} \in \Delta^{n+1} \,|\, x_j = 0\}$. Embed a copy, $Y^{n-1}_j \subset \Delta^n_j$. Cone $\cup_{j=1}^{n+2} Y^{n-1}_j$ to the barycenter $b = \frac{1}{n+2}(1, 1, \ldots, 1)$ of Δ^{n+1}. We put

$$Y^n = C\left(\cup_{j=1}^{n+2} Y^{n-1}_j\right).$$

An *n-foam* is a compact topological space X for which each point $x \in X$ has a neighborhood $N(x)$ that is homeomorphic to a neighborhood of a point in Y^n.



Y_0 Y_1 Y_2

In particular, a 2-foam is a compact topological space such that any point has a neighborhood that is homeomorphic to a neighborhood of a point in Y_2. A *knotted 2-foam* is an embedded 2-foam in a 4-dimensional space. A knotted 2-foam is a 2-dimensional generalization of a spatial trivalent graph. See [4, 3] for details.

Problem 5.1 (J. S. Carter, see [3, Problem 3.5]). *Construct interesting examples of knotted 2-foams by using movie techniques.*

This problem is related to the next problem.

In a group, let $a \triangleleft b = b^{-1}ab$. Then we have,

YY	$(ab)c = a(bc)$
YI	$(ab) \triangleleft c = (a \triangleleft c)(b \triangleleft c)$
IY	$(a \triangleleft b) \triangleleft c = a \triangleleft (bc)$
III	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

Local crossings of 2-foams are geometric representations of chains in the homology that is associated to the algebraic structure encoded by these four identities. There is a homology theory that encompasses both group and quandle homology. (See for example, this volume or my talk at RIMS).

Problem 5.2 (J. S. Carter)**.** *Develop methods of computing homology and cohomology classes for foam homology.*

6 3-dimensional braids

(J. Scott Carter)

Carter and Kamada [5] described a method for constructing simple braided embeddings or immersions in 5-space of the 2-fold and 3-fold branched covers of S^3 branched along a given knot or link.

Let D^2 be a 2-disk. We denote by $pr_2 : D^2 \times S^3 \to S^3$ the second factor projection. A 3-*dimensional braid* in $D^2 \times S^3$ of degree d is a 3-manifold M such that the restriction map $pr_2|_M : M \to S^3$ is a simple branched covering map of degree d branched along a link in S^3. Here, a *simple* branched covering map is a branched covering map whose monodromy takes each meridian of the link of the branch set to a transposition. See [5] for details of 3-dimensional braids.

We consider 3-dimensional braids of degree 2 or 3.

Question 6.1 (J. S. Carter)**.**
(1) *When are these knotted?*
(2) *In particular, are the embeddings that are constructed via the Seifert algorithm for a given diagram knotted?*
(3) *What are the relationships among the immersions of the* 2*-fold branched covers that are parametrized by the bracket smoothing cube? Explicitly modify the Seifert algorithm to construct (possibly non-orientable) surfaces using any combination of* A *and* B *smoothings. Can the immersions in* 5*-space be isotopic?*

7 Volume conjecture for 3-manifolds

(Qingtao Chen)

In my paper with T. Yang [10], we proposed the Volume Conjecture for Reshetikhin-Turaev and Turaev-Viro invariants at roots of unity $q(2)$, where $q(s) = e^{s\pi\sqrt{-1}/r}$. The example of Turaev-Viro invariant of non-orientable 3-manifold N_{2_1} (Callahan-Hildebrand-Weeks census [2]) vanishes at roots of unity $q(s)$, where r and s are both odd numbers and $(r,s)=1$ (required by a condition from the definition of the Turaev-Viro invariant). Numerical evidence shows that it is nonzero at $q(s)$ and also goes exponentially large as $r \to \infty$, when s is an odd number other than 1 but r is an even number.

Thus it is natural to propose the following Volume Conjecture,

Conjecture 7.1 (Q. Chen)**.** *Let M be an (orientable or non-orientable) hyperbolic 3-manifold with cusps or totally geodesic boundary. For a fixed odd number s other than 1 and such integer r that condition $(*)$ $TV_r(M,q(s)) \neq 0$ is satisfied, then we have*

$$\lim_{\substack{r\to\infty \\ (r,s)=1 \\ r \text{ satisfies } (*)}} \frac{s\pi}{r} \log \big| TV_r(M,q(s)) \big| \;=\; \mathrm{Vol}(M).$$

Remark. If $TV_r(M,q(s)) \neq 0$ for all any even integer r and any 3-manifold M with boundary, we could change condition "r satisfies $(*)$" to "r is even".

Remark. When M has cusps, as in [10], we consider ideal tetrahedral decomposition of M, and consider Turaev-Viro invariant of this tetrahedral decomposition.

When M has totally geodesic boundary, we consider the singular 3-manifold obtained from M by collapsing each boundary component, and consider Turaev-Viro invariant of a triangulation of this singular 3-manifold.

Similar phenomenon happens to the Reshetikhin-Turaev invariants also. The Reshetikhin-Turaev invariants of closed 3-manifold M obtained from a $4k+2$-surgery along a knot K, $RT_r(M,q(s))$, vanishes at roots of unity $q(s)$, where r and s are both odd numbers and $(r,s)=1$ (see [25] and see also [9]). Numerical evidence shows that Reshetikhin-Turaev of certain examples we tested are nonzero at $q(s)$ and also goes exponentially large as $r \to \infty$, when s is an odd number other than 1 but r is an even number.

Thus it is natural to propose the following Volume Conjecture,

Conjecture 7.2 (Q. Chen)**.** *Let M be a closed hyperbolic hyperbolic 3-manifold. For a fixed odd number s other than 1 and such integer r that condition $(**)$ $RT_r(M,q(s)) \neq 0$ is satisfied, then we have*

$$\lim_{\substack{r\to\infty \\ (r,s)=1 \\ r \text{ satisfies } (**)}} \frac{2s\pi}{r} \log \big(RT_r(M,q(s)) \big) \;=\; \mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M) \pmod{\sqrt{-1}\pi^2\mathbb{Z}},$$

with a suitable choice of a branch of the log.

Remark. If $RT_r(M,q(s)) \neq 0$ for all any even integer r and any 3-manifold closed manifold M, we could change condition "r satisfies $(**)$" to "r is even".

Habiro [18, 19] proved that the colored Jones polynomial (reduced colored $SU(2)$ invariants) of a knot K can be expanded in the following form, called the *cyclotomic* *expansion*,

$$\begin{aligned}
J_0(K) &= 1,\\
J_1(K) &= 1+\{1\}\{3\}H_1(K),\\
J_2(K) &= 1+\{2\}\{4\}H_1(K)+\{1\}\{2\}\{4\}\{5\}H_2(K),\\
&\vdots\\
J_N(K) &= 1\ +\ \{N\}\{N+2\}H_1(K)\ +\ \dots\\
&\qquad +\ \{1\}\cdots\{N\}\{N+2\}\cdots\{2N+1\}H_N(K),
\end{aligned}$$

for some $H_i(K)\in\mathbb{Z}[q]$, where $\{n\}=q^n-q^{-n}$ and $J_N(K)$ denotes the colored Jones polynomial associated with the N-th symmetric tensor product of the fundamental representation of $\mathfrak{sl}_2$ (*i.e.*, the $(N+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$). Careful readers who are familiar with Volume Conjecture may find that root of unity employed in Volume Conjecture proposed by Kashaev-Murakami-Murakami is exactly $q=e^{\frac{\pi}{N+1}}$, which is a solution of "gap equation" $\{N+1\}=0$ in cyclotomic expansion.

The *colored $SU(n)$ invariant* $J_N^{SU(n)}(K)$ of a knot K is the quantum $SU(n)$ invariant associated with the N-th symmetric tensor product of the fundamental representation of $\mathfrak{sl}_n$.

Conjecture 7.3 (Chen-Liu-Zhu [9]). *The colored $SU(n)$ invariants of a knot K can be expanded in the following form,*

$$\begin{aligned}
J_0^{SU(n)}(K) &= 1,\\
J_1^{SU(n)}(K) &= 1+\{1\}\{n+1\}H_1^{SU(n)}(K),\\
J_2^{SU(n)}(K) &= 1+\{2\}\{n+2\}H_1^{SU(n)}(K)+\{1\}\{2\}\{4\}\{5\}H_2^{SU(n)}(K),\\
&\vdots\\
J_N^{SU(n)}(K) &= 1\ +\ \{N\}\{N+n\}H_1^{SU(n)}(K)\ +\ \dots\\
&\qquad +\ \{1\}\cdots\{N\}\{N+n\}\cdots\{2N+n-1\}H_N^{SU(n)}(K),
\end{aligned}$$

for some $H_i^{SU(n)}(K)\in\mathbb{Z}[q]$.

Conjecture 7.4 (Chen-Liu-Zhu [9]). *Volume Conjecture holds for the colored $SU(n)$ invariants of a knot K at roots of unity $q=e^{\frac{\pi}{N+a}}$ (with a wider choices), where $a=1,...,n-1$.*

We proved the conjecture in the case of the figure-eight knot. We note that these roots corresponds to the "gap equation" $\{N+a\}=0$ in the above cyclotomic expansion.

Chen [7] applied this philosophy to the (reduced) superpolynomial of HOMFLY-PT homology and Kauffman homology and thus obtained both cyclotomic expansion and Volume Conjectures (proved in the case of the figure-eight knot). All the examples in existence literature have been verified. Roots of unity employed in this Volume Conjecture are solutions of a two variable "gap equation".

Because even categorified invariants seem to have Habiro type cyclotomic expansion and Volume Conjecture, thus now it is natural to ask whether all quantum invariants of a knot have cyclotomic expansions and Volume Conjectures. (I believe that the answer is positive.)

A cyclotomic expansion conjecture of the superpolynomial of colored HOMFLY-PT homology is formulated in [7], as follows.

Conjecture 7.5 (Q. Chen [7])**.** *For any knot $\mathcal{K}$, there exists an integer valued invariant $\alpha(K) \in \mathbb{Z}$, such that the reduced superpolynomial $\mathcal{P}_N(K;a,q,t)$ of the colored HOMFLY-PT homology of a knot K has the following cyclotomic expansion formula*

$$\begin{aligned}&(-t)^{N\alpha(K)}\mathcal{P}_N(K;a,q,t)\\&= 1+\sum_{k=1}^{N} H_k(K;a,q,t)\left(A_{-1}(a,q,t)\prod_{i=1}^{k}\Big(\frac{\{N+1-i\}}{\{i\}}B_{N+i-1}(a,q,t)\Big)\right)\end{aligned}$$

with coefficient functions $H_k(K;a,q,t) \in \mathbb{Z}[a^{\pm 1},q^{\pm 1},t^{\pm 1}]$, where $A_i(a,q,t) = aq^i + t^{-1}a^{-1}q^{-i}$, $B_i(a,q,t) = t^2aq^i + t^{-1}a^{-1}q^{-i}$ and $\{p\} = q^p - q^{-p}$.

Remark. The above Conjecture-Definition for the invariant $\alpha(K)$ should be understood in this way. If the above conjecture of a knot K is true for $N = 1$, then $\alpha(K)$ is defined.

We tested many homologically thick knots up to 10 crossings to illustrate this conjecture as well as many examples with higher representation. Based on highly nontrivial computations of torus knots/links studied, in [7], we are able to prove the above conjecture of $N = 1$ for torus knots, and show that $\alpha\big(T(m,n)\big) = -(m-1)(n-1)/2$ for the torus knot $T(m,n)$.

Now we are considering a problem relating to the sliceness of a knot. The *smooth 4-ball genus* $g_4(K)$ of a knot K is the minimum genus of a surface smoothly embedded in the 4-ball B^4 with boundary the knot. In particular, a knot $K \subset S^3$ is called *smoothly slice* if $g_4(K) = 0$. It is known (Milnor Conjecture, proved by Kronheimer-Mrowka [29] and Rasmussen [45]) that the smooth 4-ball genus for the torus knot $T(m,n)$ is $(m-1)(n-1)/2$. Based on all the above results, we are able to propose the following conjecture.

Conjecture 7.6 (Q. Chen [7])**.** *The invariant $\alpha(K)$ (determined by the cyclotomic expansion conjecture for $N = 1$) is a lower bound for the smooth 4-ball genus $g_4(K)$, i.e. $|\alpha(K)| \le g_4(K)$.*

Remark. Rasmussen [45] introduced a knot concordant invariant $s(K)$, which is a lower bound for the smooth 4-ball genus for knots. For all the knots we tested, it is identical to the Ozsváth-Szabó's τ invariant and Rasmussen's s invariant (up to a factor of 2).

8 Gauge theory on 4-manifolds with $\mathbb{Z}$-actions

(Masaki Taniguchi)

The gauge theory has been an important tool for the study of 4-dimensional manifolds since the early 1980s, when Donaldson solved long-standing problems in topology. There are two fundamental invariants which are called Donaldson invariant and Seiberg-Witten invariant in gauge theory, but these invariants can not be defined for closed oriented 4-manifolds with $b^2_+ = 0$.

We consider a closed oriented 4-manifold X whose $\mathbb{Z}$-homology is isomorphic to $H_*(S^3 \times S^1; \mathbb{Z})$; we call such a 4-manifold a *homology* $S^3 \times S^1$. We assume that the homology of $\mathbb{Z}$ covering of X is isomorphic to the homology of S^3. From the viewpoint of the Donaldson gauge theory, an invariant $\lambda_{\mathrm{FO}}(X)$ is defined in [13], by algebraically counting gauge equivalence classes of irreducible flat SU(2) connections on X with signs determined from the orientation of the moduli space of anti-self-dual connections in the Donaldson gauge theory. From the viewpoint of the Seiberg-Witten gauge theory, another invariant $\lambda_{\mathrm{MRS}}(X)$ is defined in [40], by algebraically counting gauge equivalence classes of solutions of the Seiberg-Witten equations. It is conjectured [40] that these invariants are essentially equal.

When $X = Y \times S^1$ for a $\mathbb{Z}$-homology 3-sphere Y, it is known that these invariants coincide with the Casson invariant of Y. In fact, $\lambda_{\mathrm{FO}}(Y \times S^1)$ is an invariant obtained by algebraically counting gauge equivalence classes of irreducible flat SU(2) connections on Y with signs of the Donaldson gauge theory, and it is known [47] that the Casson invariant can be obtained in such a way. Further, it is known, see [40], that $\lambda_{\mathrm{MRS}}(Y \times S^1)$ is equal to the Seiberg-Witten invariant of Y, and it is known [33] that this invariant is equal to the Casson invariant of Y.

It is known, see *e.g.* [30, 46], that the Casson invariant has combinatorial descriptions such as surgery formulas.

Question 8.1 (M. Taniguchi)**.** *Are there combinatorial descriptions such as surgery formulas for* $\lambda_{\mathrm{FO}}(X)$ *and* $\lambda_{\mathrm{MRS}}(X)$*?*

It is known [47] that the instanton Floer homology is a categorifications of the Casson invariant. This is shown by reconstructing the Casson invariant from the viewpoint of the gauge theory.

Problem 8.2 (M. Taniguchi)**.** *Construct categorifications of* $\lambda_{\mathrm{FO}}(X)$ *and* $\lambda_{\mathrm{MRS}}(X)$ *by developing the gauge theory on the* $\mathbb{Z}$ *covering space of* X*.*

Considering this problem leads to developing the gauge theory on non-compact 4-manifolds and constructing diffeomorphism invariants of 4-manifolds with $b^2_+ = 0$. To solve such a problem, it is important to describe the divergence of any sequence in the moduli space. We now introduce more explicit setting of the compactness problem in the case of Donaldson theory.

Let X be an oriented $\mathbb{Z}$-homology $S^3 \times S^1$, and let $p : \widetilde{X} \to X$ be the $\mathbb{Z}$ covering space of X. We denote the product SU(2) bundles on X and $\widetilde{X}$ by P_X and $P_{\widetilde{X}}$ respectively. If we have Riemannian metric g_X on X, we have a periodic Riemannian metric $g_{\widetilde{X}}$ on $\widetilde{X}$ by the pull-back. There is a natural orientation of $\widetilde{X}$ induced by the orientation of X. We fix such Riemannian metric and orientation. We also fix two SU(2) connections on P_X and denote them by a, b. Let $f : X \to S^1$ be a smooth map which corresponds to $[f] = 1 \in [X, S^1] \cong H^1(X)$ and $\widetilde{f} : \widetilde{X} \to \mathbb{R}$ be its lift. Then we can consider the following moduli space for $q > 3$:

$$M(a,b) := \left\{ A_{a,b} + c \;\middle|\; c \in L^2_q,\ (1+*)F(A_{a,b}+c) = 0 \right\} / \mathcal{G}_{a,b}$$

where $*$ is the Hodge star operator, $A_{a,b}$ is an SU(2)-connection with $A_{a,b}|_{\widetilde{f}^{-1}(-\infty,-1]} = p^*a$ and $A_{a,b}|_{\widetilde{f}^{-1}[1,\infty)} = p^*b$, and $\mathcal{G}_{a,b}$ is defined by

$$\mathcal{G}_{a,b} := \left\{ g \in \mathrm{Aut}(P_{\widetilde{X}}) \subset \mathrm{End}(\mathbb{C}^2)_{L^2_{q+1,loc}} \;\middle|\; \nabla_{A_{a,b}} g \in L^2_q \right\}.$$

The action of $\mathcal{G}_{a,b}$ on $\left\{ A_{a,b} + c \;\middle|\; c \in L^2_q,\ (1+*)F(A_{a,b}+c) = 0 \right\}$ is the pull-backs of connections. We can show that $||F(A)||^2_{L^2}$ is equal to the extended Chern-Simon invariant which we will define in the lecture (independent of A) for A in $M(a,b)$. Now we extend the chain convergence in the case of $\mathbb{Z}$ covering.

Definition. Let $c_1, \dots, c_m$ be SU(2) flat connections on P_X, and let $\{A_n\}$ be a sequence in $M(a,b)$. If there exist m sequences $\{s^j_n\}_{1 \leq j \leq m}$ in $\mathbb{Z}$ which satisfy

$$T^{s^j_n *} A_n \to B_j \in \ L^2_{q,loc}$$

where $(B_1, \dots, B_n)$ is an element in $M(c_1, c_2) \times \cdots \times M(c_{m-1}, c_m)$ and T is the deck transformation of $\widetilde{X}$, then we say that the $\{A_n\}$ chain converges to $(B_1, \dots, B_n)$.

In the instanton Floer theory, any sequence in $M(a,b)$ has a chain convergent subsequence under the assumption that all flat connections on Y are non-degenerate.

Question 8.3 (M. Taniguchi)**.** *What is a topological condition of X satisfying that any sequence in $M(a,b)$ has a chain convergent subsequence?*

We consider an oriented closed 4-manifold X with a homomorphism $\pi_1(X) \to \mathbb{Z}$. Let $p : \widetilde{X} \to X$ be the corresponding $\mathbb{Z}$ covering space of X. We consider a fundamental domain X_0 of the $\mathbb{Z}$ action on $\widetilde{X}$. We regard X_0 as a cobordism from $-Y$ to Y, and expect that the relative Donaldson invariant gives a linear transformation on the Floer homology of Y.

Question 8.4 (M. Taniguchi)**.** *Under some appropriate assumption, can we construct "$\mathbb{Z}$-equivariant Donaldson invariant" of $\widetilde{X}$ as the characteristic polynomial of such a linear transformation?*

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