Presentations of (immersed) surface-knots by marked graph diagrams

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Surface-links

A surface-link is the image $\mathscr{L}$ of the disjoint union of surfaces in the 4-space $\mathbb{R}^4$ by a smooth embedding. When it is connected, it is called a surface-knot.

When a surface-link is oriented, we call it an oriented surface-link.

Two surface-links $\mathscr{L}$ and $\mathscr{L}'$ are equivalent if there is an orientation preserving homeomorphism $h : \mathbb{R}^4 \to \mathbb{R}^4$ such that $h(\mathscr{L}) = \mathscr{L}'$ orientedly.

Normal forms of surface-links

Theorem (Kawauchi-Shibuya -Suzuki)

For any surface-link $\mathscr{L}$, there is a surface-link $\tilde{\mathscr{L}} \subset \mathbb{R}^3[-1,1]$ satisfying the following conditions:

(0) $\tilde{\mathscr{L}}$ is equivalent to $\mathscr{L}$ and has only finitely many critical points, all of which are elementary.

(1) All maximal points of $\tilde{\mathscr{L}}$ are in $\mathbb{R}^3[1]$.

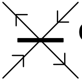
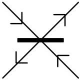
(2) All minimal points of $\tilde{\mathscr{L}}$ are in $\mathbb{R}^3[-1]$.

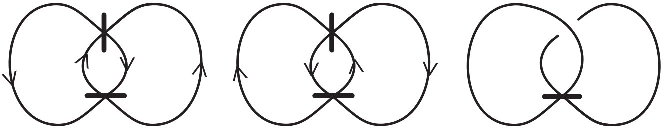
(3) All saddle points of $\tilde{\mathscr{L}}$ are in $\mathbb{R}^3[0]$.

We call $\tilde{\mathscr{L}}$ a normal form of $\mathscr{L}$.

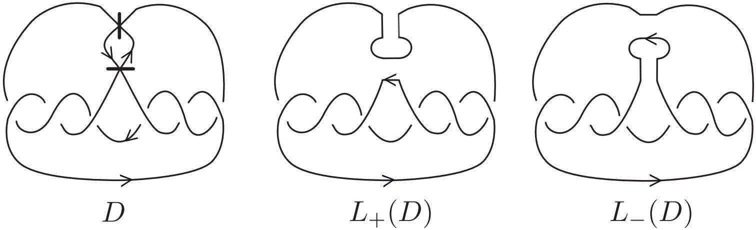
Marked graph diagrams

A marked graph diagram is a diagram of a finite spatial regular graph with 4-valent rigid vertices such that each vertex has a marker.

An orientation of a marked graph diagram D is a choice of an orientation for each edge of D in such a way that every rigid vertex in D looks like  or . A marked graph diagram is said to be orientable if it admits an orientation. Otherwise, it is said to be nonorientable.



A marked graph diagram D is admissible if both resolutions $L_+(D)$ and $L_-(D)$ are trivial links.

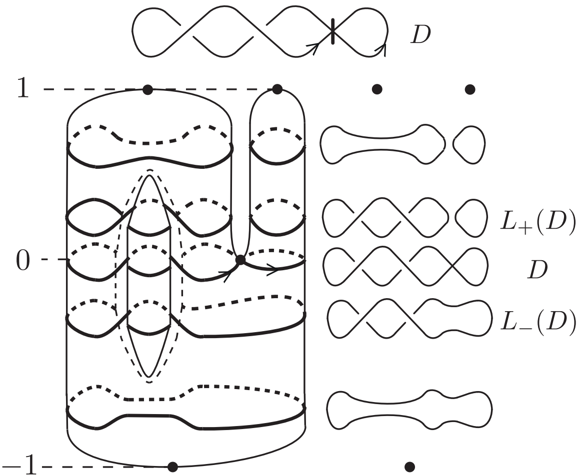


Theorem (Kawauchi-Shibuya-Suzuki, Yoshikawa)

(1) For an admissible marked graph diagram D, there is a surface-link $\mathscr{L}$ represented by D.

(2) Let $\mathscr{L}$ be a surface-link. Then there is an admissible marked graph diagram D such that $\mathscr{L}$ is represented by D.

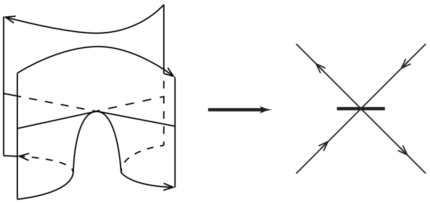
Example



Marked graph diagrams of surface-links

Let $\mathscr{L}$ be a surface-link, and $\tilde{\mathscr{L}}$ a normal form of $\mathscr{L}$. Then the cross-section $\tilde{\mathscr{L}} \cap \mathbb{R}^3[0]$ at $t = 0$ is a 4-valent graph in $\mathbb{R}^3[0]$.

We give a marker at each 4-valent vertex that indicates how the saddle point opens up above. Then the diagram D of resulting marked graph represents the surface-link $\mathscr{L}$. We call D a marked graph diagram of $\mathscr{L}$.

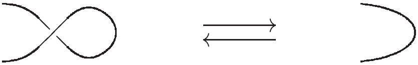


Yoshikawa moves for marked graph diagrams of surface-links

Theorem (Swenton, Kearton-Kurlin, Yoshikawa)

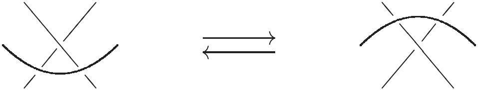
Two surface-links in $\mathbb{R}^4$ are equivalent if and only if their marked graph diagrams can be transformed into each other by a finite sequence of 8 types of moves, called the Yoshikawa moves.

Γ_1 :

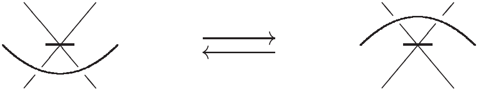


Γ_2 :

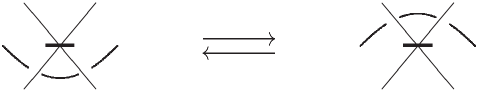
Γ_3 :



Γ_4 :

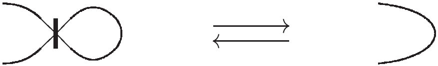


Γ'_4 :

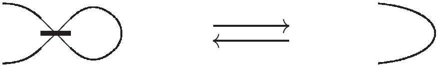


Γ_5 :

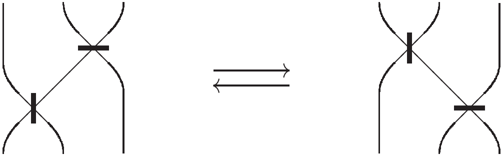
Γ_6 :



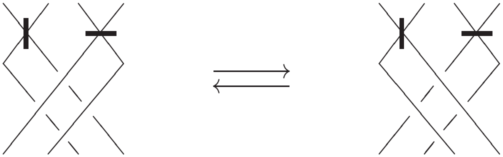
Γ_6' :



Γ_7 :



Γ_8 :



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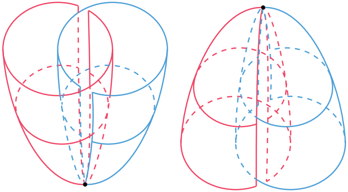
Immersed surface-links

An immersed surface-link is a closed surface generically immersed in $\mathbb{R}^4$. When $\mathscr{L}$ is connected, it is called an immersed surface-knot.

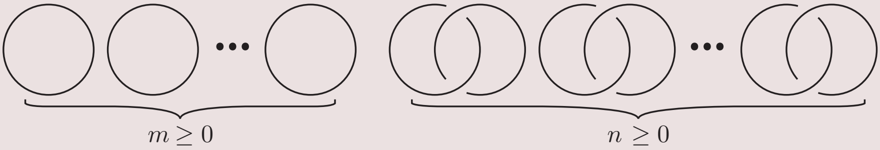
Two immersed surface-links $\mathscr{L}$ and $\mathscr{L}'$ are equivalent if there is an orientation preserving homeomorphism $h : \mathbb{R}^4 \to \mathbb{R}^4$ such that $h(\mathscr{L}) = \mathscr{L}'$ orientedly.

It is known that every double point singularity is constructed by a cone over a Hopf link.

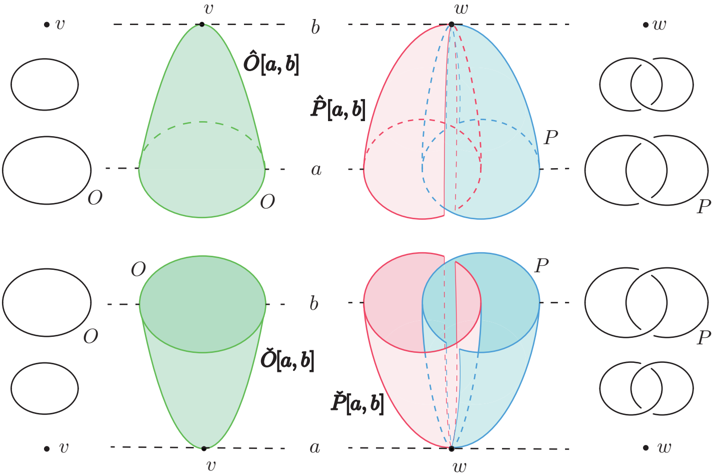


Definition

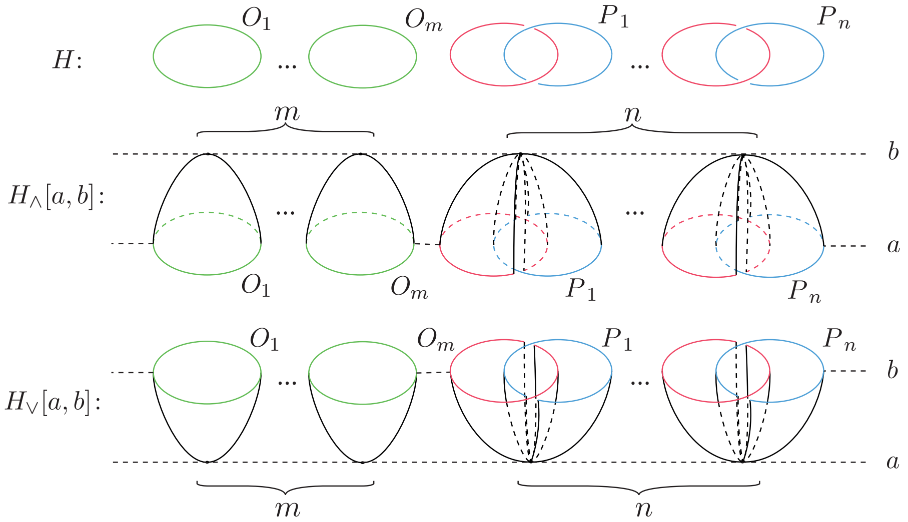
A link L is H-trivial if L is a split union of a finite number of trivial knots and Hopf links.



Trivial knot cones $\hat{O}[a,b]$ & $\check{O}[a,b]$, and Hopf link cones $\hat{P}[a,b]$ & $\check{P}[a,b]$



H-trivial link cones $H_\wedge[a,b]$ & $H_\vee[a,b]$



Theorem (Kamada-Kawamura)

For any immersed surface-link $\mathscr{L}$, there is an immersed surface-link $\tilde{\mathscr{L}} \subset \mathbb{R}^3[-2,2]$ satisfying the following conditions:

(0) $\tilde{\mathscr{L}}$ is equivalent to $\mathscr{L}$ and has only finitely many critical points, all of which are elementary.

(1) The cross-sections $H = \tilde{\mathscr{L}} \cap \mathbb{R}^3[1]$ and $H' = \tilde{\mathscr{L}} \cap \mathbb{R}^3[-1]$ of $\tilde{\mathscr{L}}$ are H-trivial links.

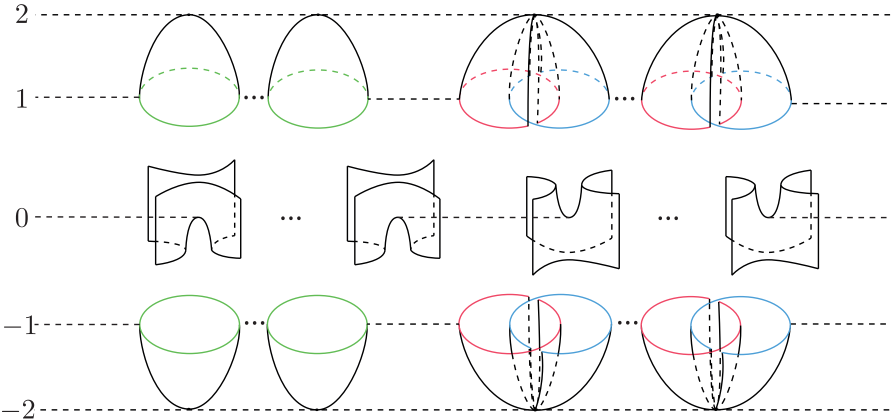
(2) All maximal points of $\tilde{\mathscr{L}}$ are in $\mathbb{R}^3[2]$.

(3) All minimal points of $\tilde{\mathscr{L}}$ are in $\mathbb{R}^3[-2]$.

(4) All saddle points of $\tilde{\mathscr{L}}$ are in $\mathbb{R}^3[0]$.

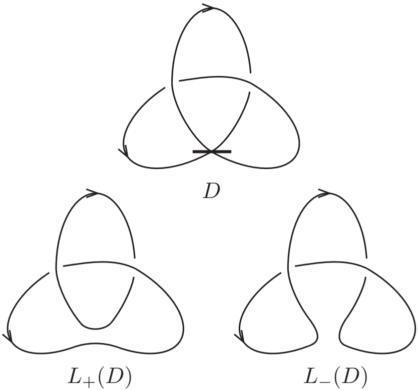
(5) $\tilde{\mathscr{L}} \cap \mathbb{R}^3[1,2] = H_\wedge[1,2]$ and $\tilde{\mathscr{L}} \cap \mathbb{R}^3[-2,-1] = H'_\vee[-2,-1]$.

We call $\tilde{\mathscr{L}}$ a normal form of $\mathscr{L}$.



2
1
0
−1
−2

A marked graph diagram D is H-admissible if both resolutions $L_+(D)$ and $L_-(D)$ are H-trivial links.

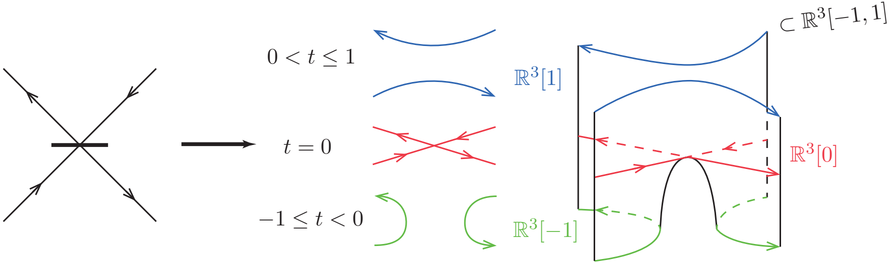


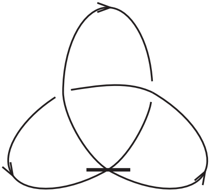
Theorem (Kamada-Kawauchi-K.-Lee)

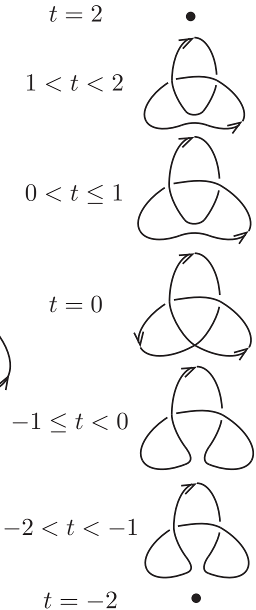
(1) For an H-admissible marked graph diagram D, there is an immersed surface-link $\mathscr{L}$ represented by D.

(2) Let $\mathscr{L}$ be an immersed surface-link. Then there is an H-admissible marked graph diagram D such that $\mathscr{L}$ is represented by D.

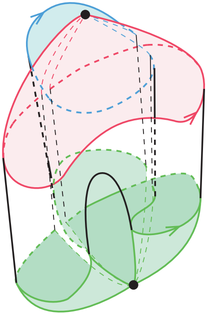
Construction of immersed surface-links from H-admissible marked graph diagrams







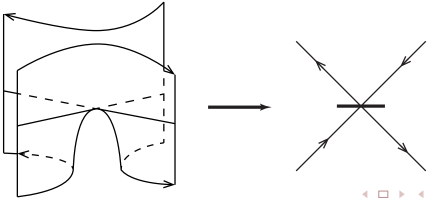
$t = 2$
$1 < t < 2$
$0 < t \leq 1$
$t = 0$
$-1 \leq t < 0$
$-2 < t < -1$
$t = -2$



Marked graph diagrams of immersed surface-links

Let $\mathscr{L}$ be an immersed surface-link, and $\tilde{\mathscr{L}}$ a normal form of $\mathscr{L}$. Then the cross-section $\tilde{\mathscr{L}} \cap \mathbb{R}^3[0]$ at $t = 0$ is a 4-valent graph in $\mathbb{R}^3[0]$.

We give a marker at each 4-valent vertex that indicates how the saddle point opens up above. Then the diagram D of a resulting marked graph presents the surface-link $\mathscr{L}$. We call D a marked graph diagram of $\mathscr{L}$.

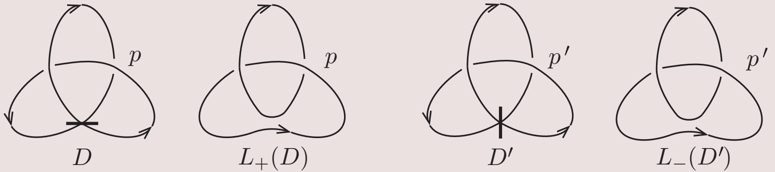


Further moves for immersed surface-links

Definition

A crossing point p (in a marked graph diagram D) is an upper singular point if p is an unlinking crossing point of a Hopf link diagram in the resolution $L_+(D)$, and a lower singular point if p is an unlinking crossing point in the resolution $L_-(D)$, resp.

Example

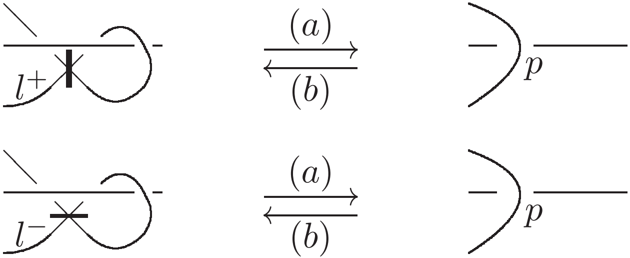


p : upper singular point, p' : lower singular point.

The following moves are new entries on marked graph diagrams.

Γ_9 :

Γ'_9 :

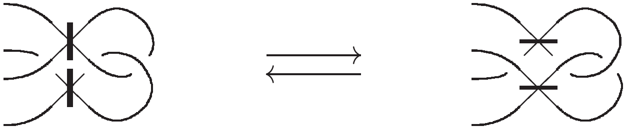


- In Γ_9 , the component containing l^+ in $L_+(D)$ is a trivial knot.
- In Γ_9 , p is an upper singular point.

- In Γ'_9, the component containing l^- in $L_-(D)$ is a trivial knot.
- In Γ'_9, p is a lower singular point.

The following move is a new entry on marked graph diagrams.

Γ_{10} :



Note

Let D be an H-admissible marked graph diagram. Let $h_+(D)$ and $h_-(D)$ be the numbers of Hopf-links in $L_+(D)$ and $L_-(D)$, resp.

- The ordered pair $(h_+(D), h_-(D))$ is an invariant except Γ_{10}.
- If D and D' are related by a single Γ_{10} move, then $(h_+(D'), h_-(D')) = (h_+(D) + \varepsilon, h_-(D) - \varepsilon)$ for $\varepsilon \in \{1, -1\}$.

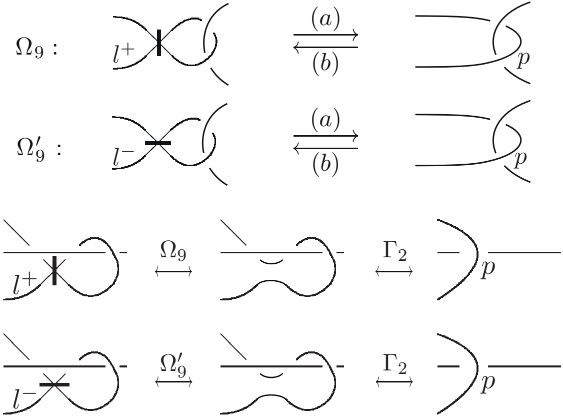
Definition

The generalized Yoshikawa moves for marked graph diagrams are the deformations $\Gamma_1, \ldots, \Gamma_8$, Γ_9, Γ_9', and Γ_{10}.

Theorem (Kamada-Kawauchi-K.-Lee)

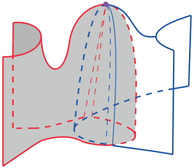
Let $\mathscr{L}$ and $\mathscr{L}'$ be immersed surface-links, and D and D' their marked graph diagrams, resp. If D and D' are related by a finite sequence of generalized Yoshikawa moves, then $\mathscr{L}$ and $\mathscr{L}'$ are equivalent.

Sketch of Proof. The moves Γ_9 (or Γ_9') can be generated by Ω_9 (or Ω_9') and Γ_2, resp.

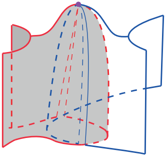


We need to show that if two marked graphs are related by Ω_9, Ω_9', and Γ_{10}, then their immersed surface-links are equivalent.

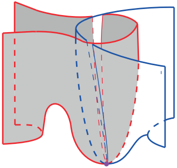
Ω_9 :



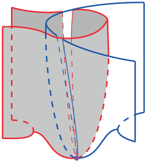
$$\underset{(b)}{\overset{(a)}{\rightleftarrows}}$$



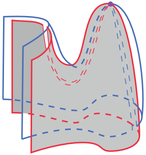
Ω_9' :



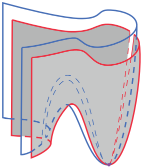
$$\underset{(b)}{\overset{(a)}{\rightleftarrows}}$$



Γ_{10} :

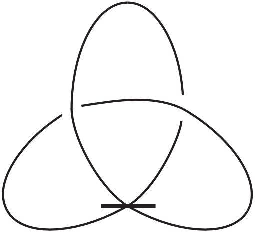


$$\rightleftarrows$$

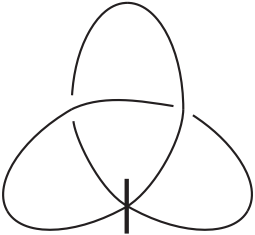


Example

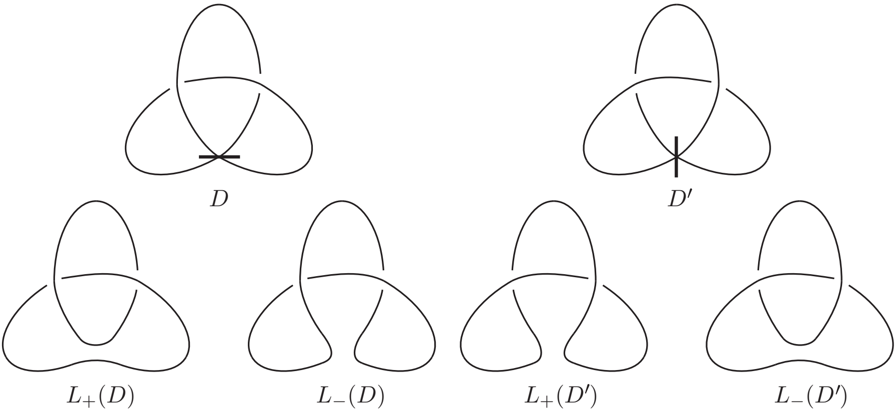
The following marked graph diagrams D and D' are related by a finite sequence of generalized Yoshikawa moves.



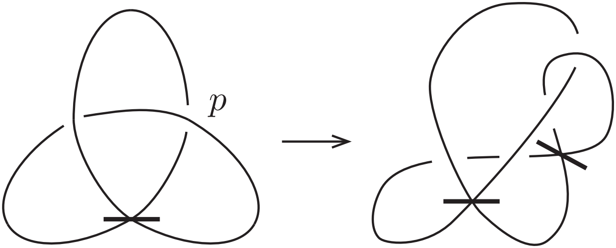
D



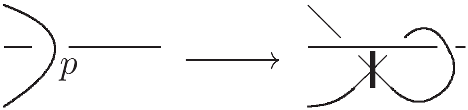
D'

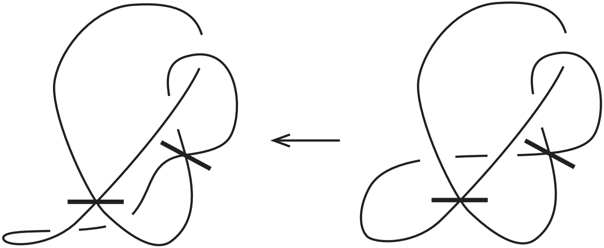


H-admissibility

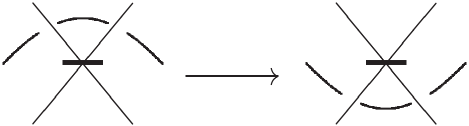


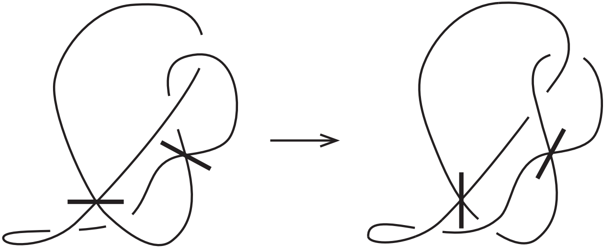
Γ_9 :



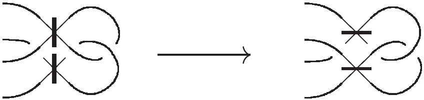


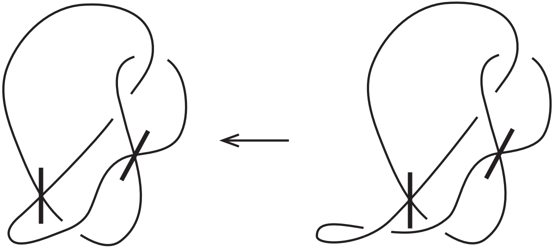
Γ'_4 :



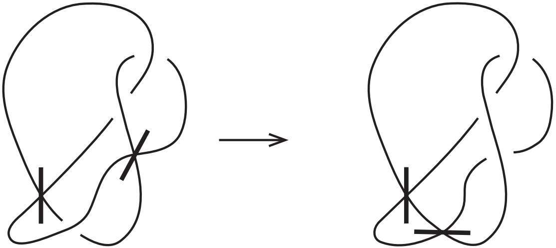


$\Gamma_{10}:$

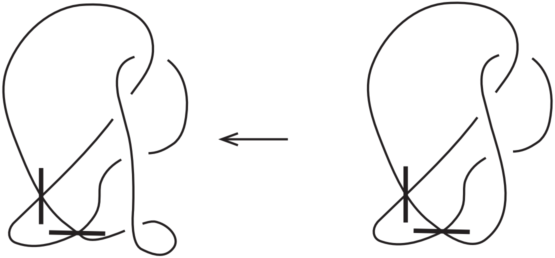




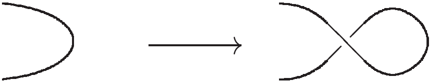
Γ_1 :

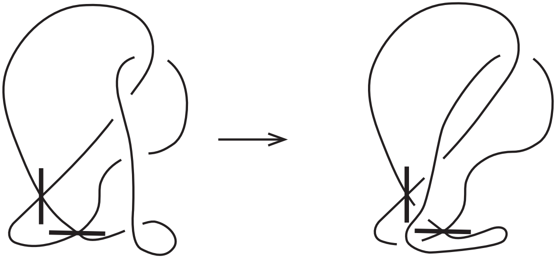


Γ_5 :

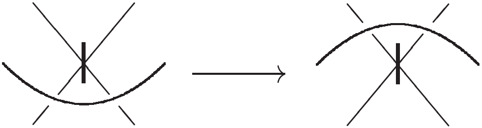


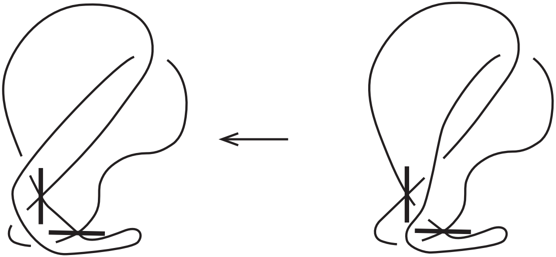
Γ_1 :

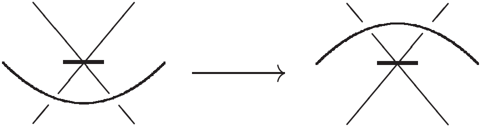


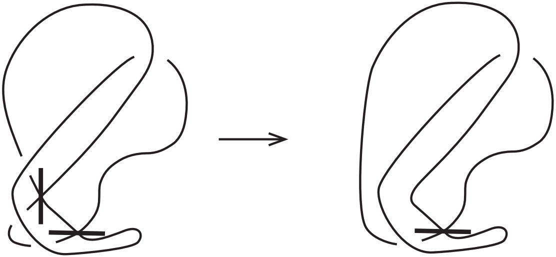


Γ_4 :

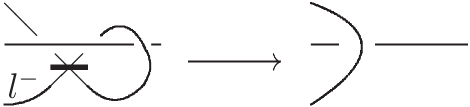




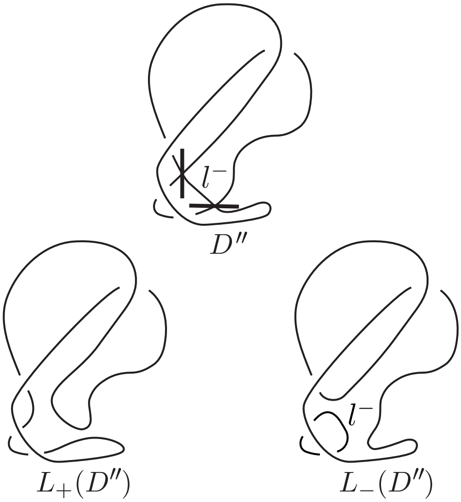
Γ_4 : 



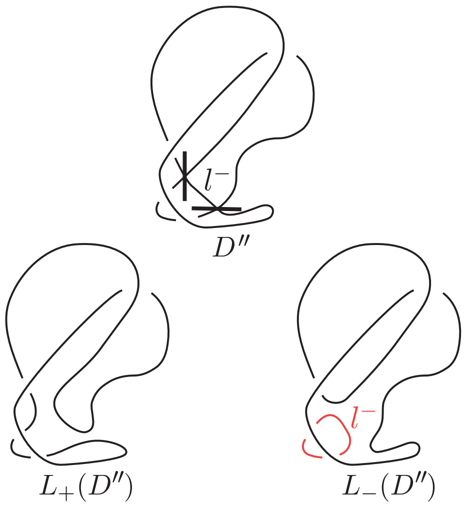
Γ'_9 :

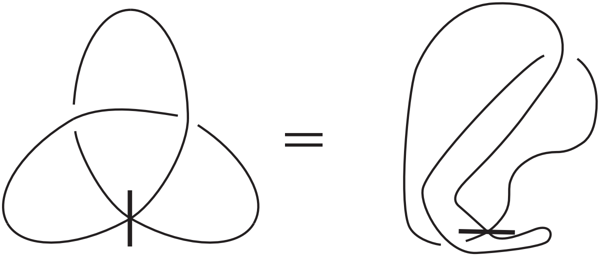


Well-definedness of the move Γ_9' :



Well-definedness of the move Γ_9' :

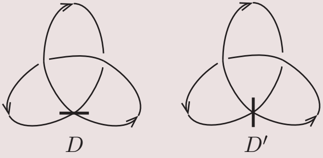




Contents

Definition

A positive (or negative) standard singular 2-knot, denoted by $S(+)$ (or $S(-)$) is the immersed 2-knot of D (or D'), resp. An unknotted immersed sphere is defined to be the connected sum $mS(+)\#nS(-)$ for $m,n \in \mathbb{Z}_{\geq 0}$ with $m+n>0$.



Definition

A double point singularity p of an immersed 2-knot S is inessential if S is the connected sum of an immersed 2-knot and an unknotted immersed sphere such that p belongs to the unknotted immersed sphere. Otherwise, p is essential.

I answer the following question.

Question

For any integer $n \geq 1$, is there an immersed 2-knot with n double point singularities every of which is essential?

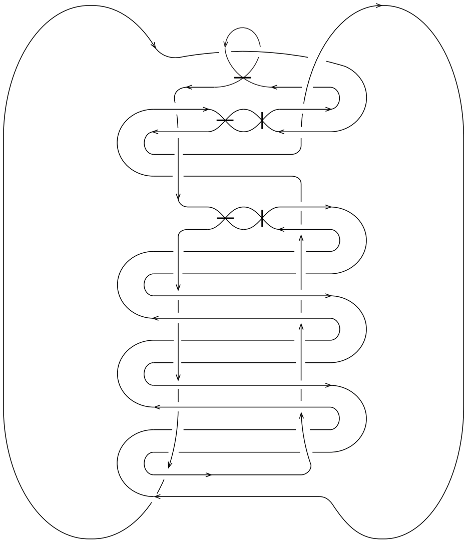
I answer the following question.

Question

For any integer $n \geq 1$, is there an immersed 2-knot with n double point singularities every of which is essential?

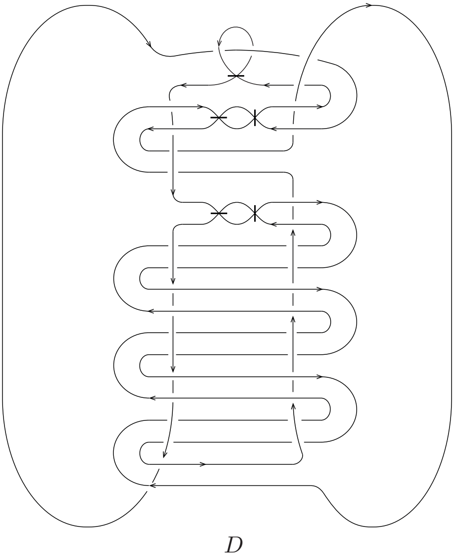
Yes. There are infinitely many immersed 2-knots with n double point singularities every of which is essential.

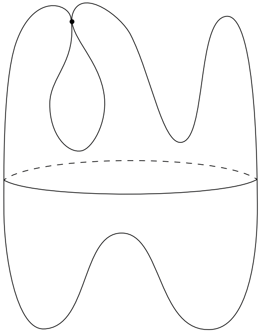
Example



D

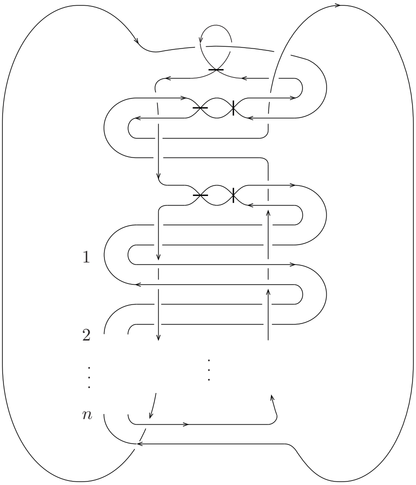
Example



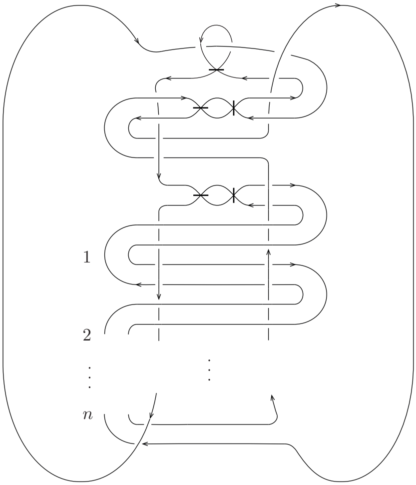


The knot group is $< x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}$ $, x_{14}, x_{15} | x_1 = x_2^{-1} x_3 x_2,\ x_2 = x_3^{-1} x_5 x_3,\ x_1 = x_3^{-1} x_4 x_3,\ x_2 = x_1^{-1} x_3 x_1,$ $x_6 = x_2^{-1} x_1 x_2,\ x_6 = x_1^{-1} x_7 x_1,\ x_1 = x_7^{-1} x_8 x_7,\ x_2 = x_7^{-1} x_9 x_7,$ $x_{10} = x_2^{-1} x_7 x_2,\ x_{10} = x_1^{-1} x_{11} x_1,\ x_1 = x_{11}^{-1} x_{12} x_{11},\ x_2 = x_{11}^{-1} x_{13} x_{11},$ $x_{14} = x_2^{-1} x_{11} x_2,\ x_{14} = x_1^{-1} x_2 x_1,\ x_1 = x_2^{-1} x_{15} x_2 >$.

The first elementary ideal $\varepsilon(D)$ is $< 1 - 2t, 4 - 3t >$ and it is equivalent to the ideal $< 2t - 1, 5 >$. Since $< 2t - 1, 5 >$ is not equivalent to the ideal $< t - 2, 5 >$, it is non-symmetric. ($\because \mathbb{Z}_5[t, t^{-1}]$ is a principal ideal domain.)



D_n



D'_n

We have $\varepsilon(D_n) =< 2t-1, n+2 >$, $\varepsilon(D'_n) =< 2t-1, n-1 >$.

Denote the first Alexander module $H_1(\tilde{E}(K))$ of a 2-knot K by $H(K)$. Let

$$DH(K) = \{x \in H(K) |\ \exists \{\lambda_i\}_{1 \le i \le m} : \text{coprime } (m \ge 2) \text{ with } \lambda_i x = 0,\ \forall i\},$$

called the annihilator Λ-submodule. The following lemma is used in our argument.

Lemma

If K is a 2-knot such that the dual Λ-module $DH(K)^* = \hom(DH(K), \mathbb{Q}/\mathbb{Z})$ is Λ-isomorphic to $DH(K)$, then the first elementary ideal $\varepsilon(K)$ is symmetric.

Lemma (Kawauchi-K.)

The following statements are equivalent:

1. The ideal $< 2t-1, m >$ is symmetric.
2. An integer m is $\pm 2^r$ or $\pm 2^r 3$ for any integer $r \geq 0$.

Lemma (Kawauchi-K.)

There are infinitely many immersed 2-knots with one essential double point singularity.

Sketch of Proof. Let K_n and K'_n be immersed 2-knots represented by D_n and D'_n, resp. Suppose that $K_n = K\#S(\pm)$, where K is a 2-knot and $S(\pm) = S(+)$ or $S(-)$. Then the ideal $\varepsilon(K_n) =< 2t-1, n+2 >$ is symmetric. There is a contradiction if n isn't $2^{r+2}-2$ nor $2^r 3 - 2$ ($r \geq 0$). Hence K_n is an immersed 2-knot with essential singularity except that n is $2^{r+2}-2$ or $2^r 3 - 2$ ($r \geq 0$). So is K'_n except that n is 1, $2^r + 1$ or $2^r 3 + 1$ ($r \geq 0$).

Theorem (Kawauchi-K.)

Let $K = nK_m^*$ be the connected sum of n copies of an immersed 2-knot K_m^* with one essential double point singularity whose first elementary ideal is $< 2t-1, m >$ for any integer $m \geq 5$ without factors 2 and 3. Then K gives infinitely many immersed 2-knots with n double point singularities every of which is essential.

Thank you