Braid group actions on the n-adic integers

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Overview

1. Polynomials and braids
2. Braid group actions
3. Braid sequences
4. Normal subgroups
5. Real algebraic links

1 Polynomials and braids

2 Braid group actions

3 Braid sequences

4 Normal subgroups

5 Real algebraic links

Polynomials and braids

Consider the space $\tilde{X}_n$ of monic complex polynomials f of fixed degree n and with n distinct roots (x_i s.t. $f(x_i) = 0$).
$f \mapsto \{x_1, x_2, \ldots, x_n\}$ gives a homeomorphism

$$\tilde{X}_n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : x_i \neq x_j \text{ if } i \neq j\}/S_n.$$

and an isomorphism $\pi_1(\tilde{X}_n) = \mathbb{B}_n$.

Critical values

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Or we could consider the space of monic complex polynomials of degree n, with $n-1$ distinct critical values (v_j s.t. $f(c_j)=v_j$ and $f'(c_j)=0$) and constant term equal to 0.

$f \mapsto \{v_1, v_2, \dots, v_{n-1}\}$ is not a homeomorphism, but it induces a homomorphism from the fundamental group of this subspace of the space of polynomials to $\mathbb{B}_{n-1}$.

Something new

In all of these spaces there is some way to associate an unordered tuple of distinct complex numbers to a point in the space (i.e., a polynomial). Hence a way to associate braids to loops in the space of polynomials.

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Consider the space X_n of monic complex polynomials of degree n with constant term equal to 0, with n distinct roots AND $n-1$ distinct (non-zero) critical values.

Since the critical values of a polynomial $f \in X_n$ are non-zero, the space of possible sets of critical values is

$$V_n = \{(v_1, v_2, \ldots, v_{n-1}) \in (\mathbb{C}\backslash\{0\})^{n-1} : v_i \neq v_j \text{ if } i \neq j\}/S_{n-1},$$
$$\pi_1(V_n) = \mathbb{B}^{aff}_{n-1}. \tag{1}$$

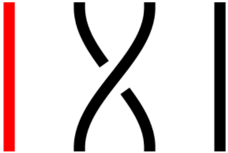
Affine braids

Definition

The affine braid group $\mathbb{B}_{n-1}^{aff}$ is generated by $x, \sigma_2, \sigma_3, \ldots, \sigma_{n-1}$ subject to the relations

$$\begin{aligned}
\sigma_2 x \sigma_2 x &= x \sigma_2 x \sigma_2 & \\
\sigma_i \sigma_j &= \sigma_j \sigma_i & \text{if } |i-j|>1 \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{if } i=2,3,\ldots,n-2 \\
\sigma_i x &= x \sigma_i & \text{if } i>2.
\end{aligned} \tag{2}$$

a) 

b) 

Affine braids

There are now two ways in which a loop $f_t \subset X_n$ corresponds to a n-strand braid, namely

$$A: \qquad (x_1(t), x_2(t), \ldots, x_n(t)), \tag{3}$$

$$B: \qquad (v_1(t), v_2(t), \ldots, v_{n-1}(t), 0). \tag{4}$$

Question: What is the relation between A and B?

1. Polynomials and braids
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A covering map

Theorem (Beardon-Carne-Ng)

The map $\theta_n : X_n \to V_n$ that sends a polynomial to its set of critical values is a covering map of degree n^{n-1}.

In particular, we have the homotopy lifting property. Therefore the braid type of the braid that is formed by the roots (A) depends only on the braid type of the braid formed by the critical values (B), not on the particular parametrisation.

An infinite tower

If $f \in X_n$, then one of its roots is zero. We can embed X_n into V_n by sending a polynomial to its $n-1$ non-zero roots. We define $X_n^1 = X_n$ and $X_n^{j+1} = \theta_n^{-1}(X_n^j)$.

$$\begin{array}{ccccccc} & & & & X_n & \xrightarrow{\theta_n} & V_n \\ & & & & \downarrow & & \\ & & X_n^2 := \theta_n^{-1}(X_n) & \xrightarrow{\theta_n} & V_n & & \\ & & \downarrow & & & & \\ X_n^3 := \theta_n^{-1}(X_n^2) & \xrightarrow{\theta_n} & V_n & & & & \end{array} \tag{5}$$

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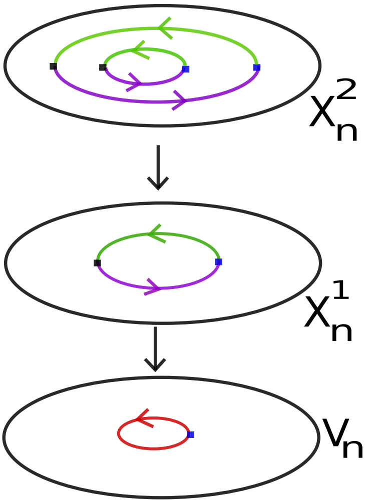
$$\dots X_n^{j+1} \to X_n^j \to \dots \to X_n^2 \to X_n \to V_n \tag{6}$$

Action of $\pi_1(V_n) = \mathbb{B}_{n-1}^{aff}$ on the fibers.

The fiber in X_n^j consists of $(n^{n-1})^j$ points, i.e. we have homomorphisms $\mathbb{B}_{n-1}^{aff} \to S_{(n^{n-1})^j}$. The actions are compatible, meaning that for any loop $\gamma \in V_n$ with basepoint $v \in V_n$ we have $\theta_n(x.\gamma) = \theta_n(x).\gamma$ for all $x \in (\theta_n^{-1})^j(v)$.

We obtain an action on the fiber in $\varprojlim_j X_n^j = \varprojlim_j \mathbb{Z}/(n^{n-1})^j\mathbb{Z}$.

Figure



The *n*-adic integers

Definition

The set of *n*-adic integers $\mathbb{Z}_n$ is the inverse limit $\varprojlim_{j} \mathbb{Z}/n^j\mathbb{Z}$.

In other words, an *n*-adic integer is a sequence of numbers $(a_1, a_2, a_3, \ldots)$ such that $a_j \in \mathbb{Z}/n^j\mathbb{Z}$ with $a_{j+1} \equiv a_j \bmod n^j$.
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E. g. $(0, 2, 2, 10, \ldots) \in \mathbb{Z}_2$
$\mathbb{Z}_n$ is a topological group and metric space.
$ord_n(a) = \min\{k \geq 1 : a_i = 0 \text{ for all } i < k\}$, $|a|_n = n^{-ord_n(a)}$.
Remarks: $\mathbb{Z}_{\prod_i^\ell p_i^{n_i}} \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \ldots \times \mathbb{Z}_{p_\ell}$

The fiber in $\varprojlim_j X_n^j$ over any basepoint in V_n is bijective to $\mathbb{Z}_{n^{n-1}} \cong \mathbb{Z}_n$.

Braid group actions

Definition

We say a point $v = (v_1, v_2, \dots, v_{n-1}) \in V_n$ has 0 in the jth position if exactly $j-1$ of the v_i have negative real part.

We pick n basepoints $w_j \in V_n$, $j = 1, 2, \dots, n$ such that w_j has 0 in the jth position.

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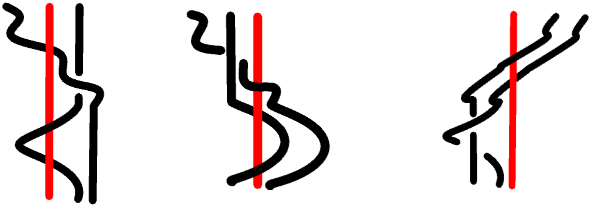
Given a braid word B on n strands, we consider n parametrisations of B as paths in V_n, each one starting at one of the w_j and ending at w_k. By construction $k=\pi(B)(j)$, where $\pi:\mathbb{B}_n\to S_n$ is the permutation representation.

Braid group actions



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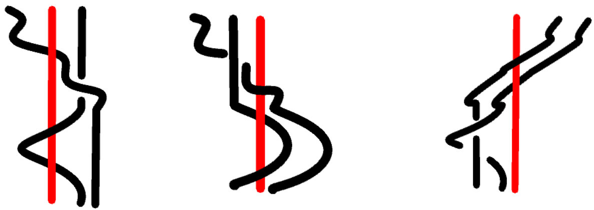
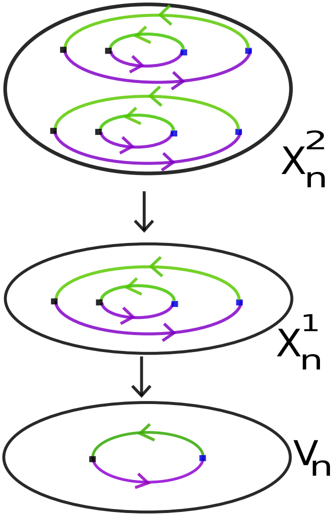


The lifts of these n paths in V_n permute the points in the fibers over $\{w_1, w_2, \ldots, w_n\}$ in any X_n^j. Again these permutations are compatible with each other and we obtain an action φ_n of $\mathbb{B}_n$ on $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n^{n-1}} \cong \mathbb{Z}_n$.

Braid group actions



Transitivity and continuity

$\mathbb{Z}_n$ is uncountable, while $\mathbb{B}_n$ is countable, so none of the defined actions can be transitive.

Proposition

X_n^j is path-connected for all n, j.

Corollary

Restricting the action φ_n to the fiber in X_n^j for a fixed j results in an action on $n \times (n^{n-1})^j$ points. This action is transitive for all j.

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Proposition

The action on $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n^{n-1}}$ is by isometries ($|x.\gamma - y.\gamma| = |x - y|$ for all $x, y \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n^{n-1}}$, $\gamma \in \mathbb{B}_n$). The action on $\mathbb{Z}_n$ is continuous.

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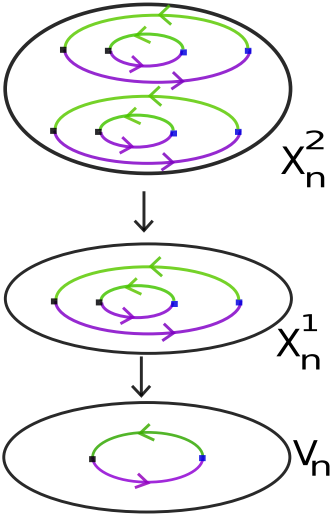
Braid sequences



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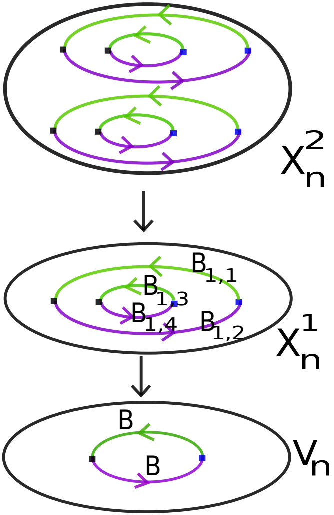

Braid sequences

Let z_i, $i = 1, 2, \ldots, n^n$, denote the points in $\theta_n^{-1}(\{w_1, w_2, \ldots, w_n\})$ in X_n. The lifted paths in X_n correspond to braids on n strands (by sending a polynomial to its set of roots). Denote the braid that corresponds to the lifted path that starts at z_i by $B_{1,i}$. Then $B_{1,i}$ are invariants of the original braid B (corresponding to n paths in V_n).

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The whole sequence $(B, \{B_{1,i}\}_{i=1,2,\ldots,n^n}, \{B_{2,i}\}_{i=1,2,\ldots,n^{2n-1}}, \ldots)$ is an invariant of B.

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The whole sequence $(B, \{B_{1,i}\}_{i=1,2,\ldots,n^n}, \{B_{2,i}\}_{i=1,2,\ldots,n^{2n-1}}, \ldots)$ is an invariant of B.
Say we have a braid invariant $I_n : \mathbb{B}_n \to X$ valued in some set X. Then $(I_n(B), \{I_n(B_{1,i})\}, \{I_n(B_{2,i})\}, \ldots)$ is an invariant of B, presumably a lot stronger than $I_n(B)$, but not much harder to compute.

Braid sequences

Algebraically, the lifting process corresponds to a homomorphism

$$\mathbb{B}_n \to \mathbb{B}_n^{n\times(n^{n-1})^j} \rtimes S_{n\times(n^{n-1})^j}. \tag{7}$$

The action is constructed from the projection to $S_{n\times(n^{n-1})^j}$ and the braid sequences come from the projection to $\mathbb{B}_n^{n\times(n^{n-1})^j}$.

Braid sequences

Conjugate braids lift to conjugate braids in the sense that if A and B are conjugate braids, then for every j there is a $\rho_j \in S_{n\times(n^{n-1})^j}$ such that $A_{j,i}$ is conjugate to $B_{j,\rho_j(i)}$.

We can therefore apply invariants of conjugacy classes of braids to the sequences $(B, \{B_{1,i}\}, \ldots)$ and obtain stronger invariants of conjugacy classes.

Question: How do the braid sequences change under (de)stabilization moves? Can we use them to define link invariants?

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Normal subgroups

By construction the actions φ_n correspond to sequences of homomorphisms $h_j : \mathbb{B}_n \to S_{n\times(n^{n-1})^j}$. Let $H_j := \ker(h_j)$. Then $H_j \trianglerighteq H_{j+1} \ldots$ is a descending series of normal subgroups of $\mathbb{B}_n$.

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Proposition

The sequence H_j does not stabilize, i.e., there is no N such that $H_n = H_N$ for all $n \geq N$.

Question: Is $\bigcap_{j=1}^{\infty} H_j = \{e\}$? If and only if φ_n is faithful.

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Real algebraic links

Definition

Consider a polynomial map $f : \mathbb{R}^4 \to \mathbb{R}^2$ such that

- $f(0) = 0$, $\nabla f(0) = 0$,
- There is a nbhd U of $0 \in \mathbb{R}^4$ such that 0 is the only point in U, where ∇f does not have full rank.

Then we call 0 an *isolated singularity* of f.

Definition

If 0 is an isolated singularity of $f : \mathbb{R}^4 \to \mathbb{R}^2$, the intersection $f^{-1}(0) \cap S^3_\rho$ is a link L for all small enough radii ρ. We call L the *link of the singularity*. A link L is *real algebraic* if there is a polynomial $f : \mathbb{R}^4 \to \mathbb{R}^2$ with an isolated singularity and L is the link of that singularity.

Real algebraic links

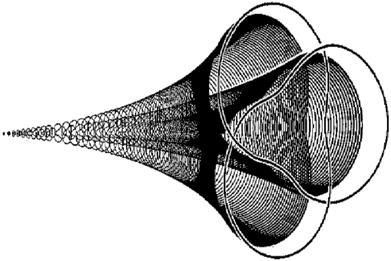

Figure: José Seade: On the topology of isolated singularities in analytic spaces

Real algebraic links

Theorem (Milnor)

L real algebraic $\implies$ L fibered.

Real algebraic links

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Conjecture (Benedetti, Shiota)

L real algebraic $\iff$ L fibered.

Known real algebraic links:

- All algebraic links,
- 4_1 (Perron, Rudolph),
- $K\#K$ if K is a fibered knot (Looijenga).
- The closure of B^2, where B is homogeneous. (B)

Real algebraic links from lifted affine braids

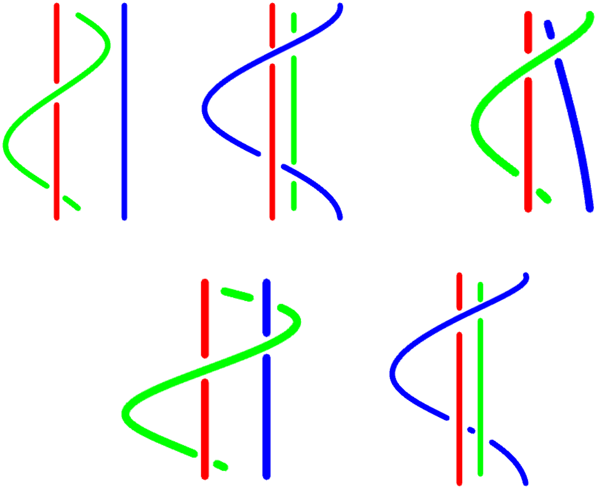
Proposition (B)

Let f_t be a loop in $\tilde{X}_n$ such that its critical values $(v_1(t), v_2(t), \ldots, v_{n-1}(t))$ are distinct, non-zero for all t and such that $\frac{\partial \arg v_j}{\partial t}(h) \neq 0$ for all $j = 1, 2, \ldots, n-1$ and $h \in [0,1]$. Let B denote the braid that is formed by the roots of f_t. Then the closure of B^2 is real algebraic.

Therefore, to construct real algebraic links we look for affine braids such that

- it can be parametrised by $(0, v_1(t), v_2(t), \ldots, v_{n-1}(t))$ as in the proposition and
- one of its lifts in $X_n \subset \tilde{X}_n$ is a loop.

Solar systems



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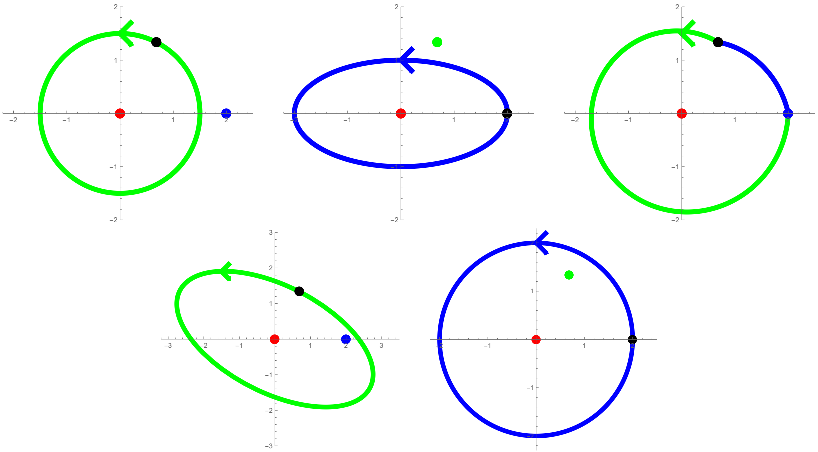

Real algebraic links

Corollary (B)

Let $\varepsilon \in \{\pm 1\}$ *and let* $B = \prod_{j=1}^{\ell} w_{i_j}^{\varepsilon}$ *be a 3-strand braid with*

$$w_1 = \sigma_2, \qquad w_2 = \sigma_2^{-1}\sigma_1^2\sigma_2, \qquad w_3 = (\sigma_1\sigma_2\sigma_1)^2$$
$$w_4 = (\sigma_2\sigma_1\sigma_2^{-1}\sigma_1\sigma_2)^2, \quad w_5 = \sigma_2^{-1}\sigma_1\sigma_2^2\sigma_1, \tag{8}$$

and such that either there is a j *with* $i_j = 3$ *or there is only one residue class* k *mod 3 such that* $i_j \neq k$ *for all* j.
Then the closure of B^2 *is real algebraic.*