On generalizations of the Conway-Gordon theorems

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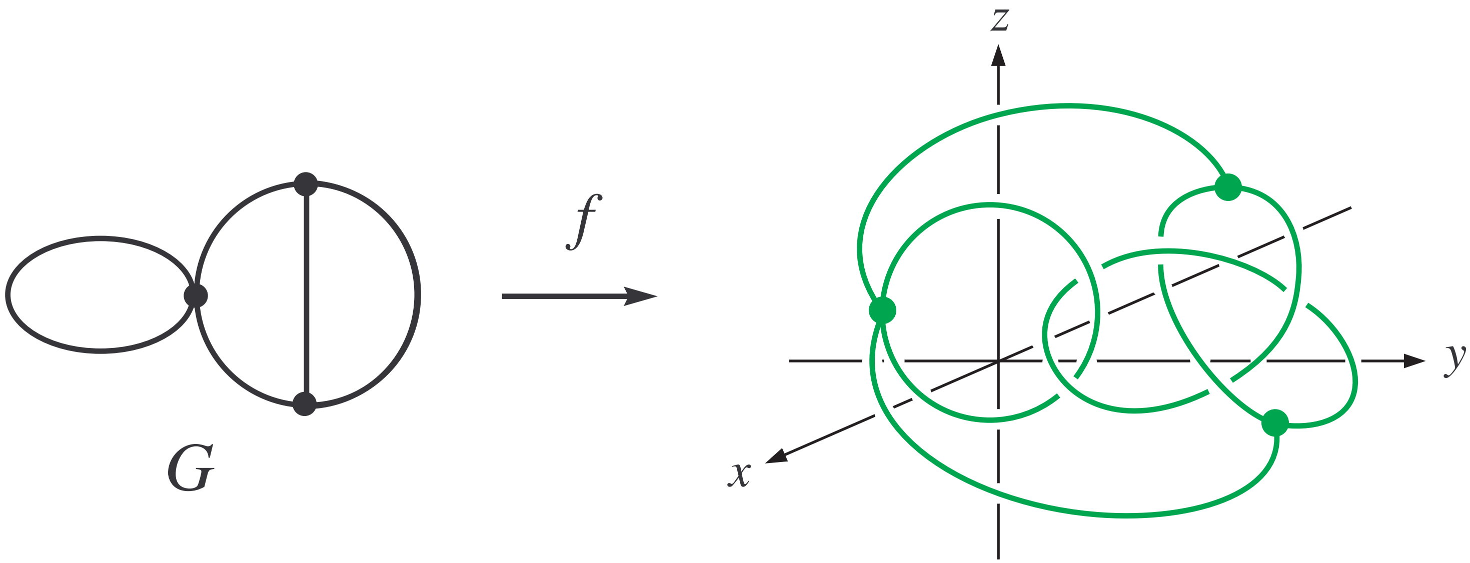
§1. Conway-Gordon theorems

Spatial graph = The image of a spatial embedding f of a finite graph G into $\mathbb{R}^3$



For a (disjoint union of) *cycle(s)* λ of G, $f(\lambda)$ is called a *constituent knot (link)* of the spatial graph.

$$\mathrm{SE}(G) \overset{\text{def.}}{=} \{\text{embedding } f : G \to \mathbb{R}^3\}$$

$$\Gamma_k(G) \overset{\text{def.}}{=} \{k\text{-cycles of } G\}$$

$$\Gamma_{k,l}(G) \overset{\text{def.}}{=} \{\text{a disjoint pair of } k\text{-cycle and } l\text{-cycle of G}\}$$

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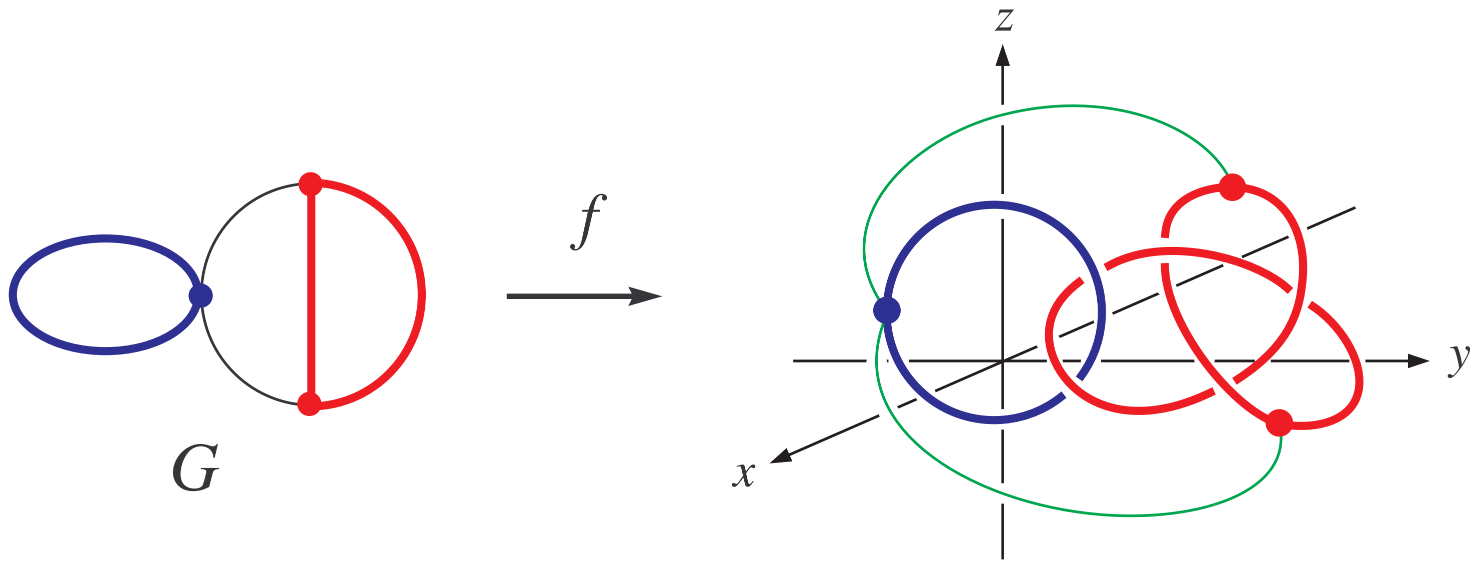

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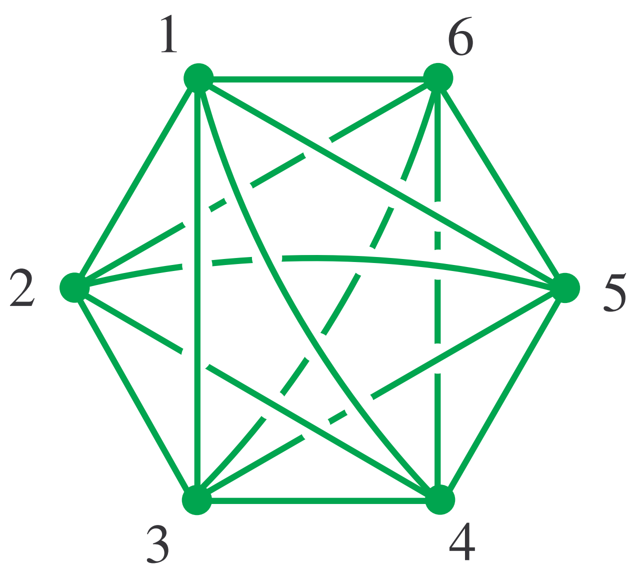
K_n : *complete graph* on n vertices

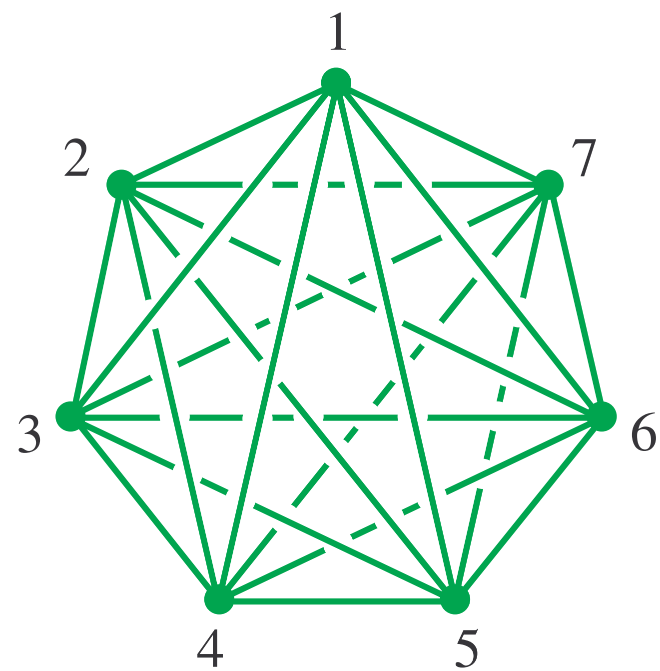
Theorem 1.1. [Conway-Gordon '83]

(1) $\forall f \in \mathrm{SE}\,(K_6)$, $\displaystyle\sum_{\lambda \in \Gamma_{3,3}(K_6)} \mathrm{lk}(f(\lambda)) \equiv 1 \pmod 2$.

(2) $\forall f \in \mathrm{SE}\,(K_7)$, $\displaystyle\sum_{\gamma \in \Gamma_7(K_7)} a_2\,(f\,(\gamma)) \equiv 1 \pmod 2$.

Here, lk: *linking number*, a_2: 2nd coefficient of $\nabla(z)$.





$\therefore\ \forall f\,(K_6) \supset$ nonsplittable link, $\forall f\,(K_7) \supset$ nontrivial knot.

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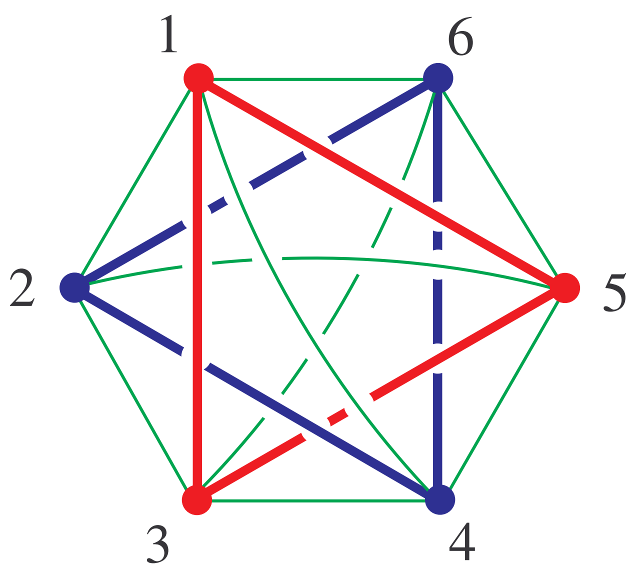

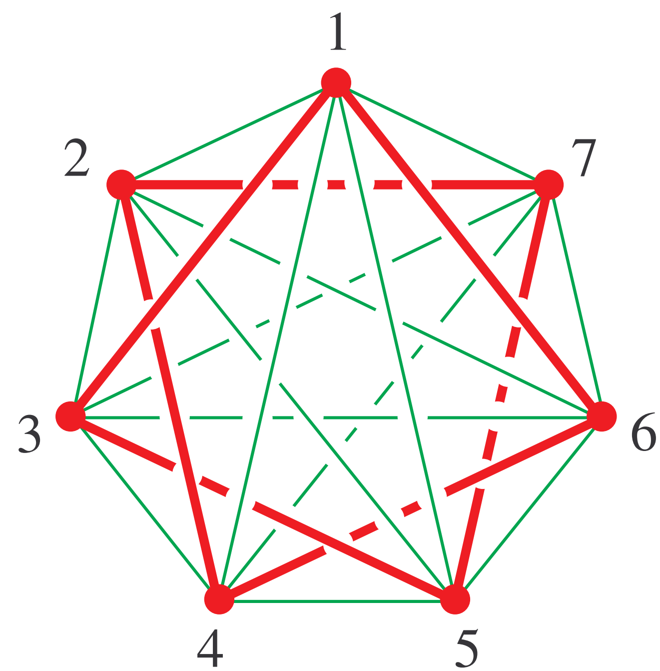

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There have been little results about a generalization of the Conway-Gordon type congruences for K_n ($n \geq 8$):

Theorem 1.2.

(1) [Foisy '08] + [Hirano '10]

$$\forall f \in \mathrm{SE}(K_8), \quad \sum_{\gamma \in \Gamma_8(K_8)} a_2(f(\gamma)) \equiv 3 \pmod 6.$$

(2) [Hirano '10] For $n \geq 9$,

$$\forall f \in \mathrm{SE}(K_n), \quad \sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) \equiv 0 \pmod 2.$$

(3) [Kazakov-Korablev '14] For $n \geq 7$,

$$\forall f \in \mathrm{SE}(K_n), \quad \sum_{p+q=n} \sum_{\lambda \in \Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda)) \equiv 0 \pmod 2.$$

Integral lifts of the Conway-Gordon theorems are known:

Theorem 1.3. [Nikkuni '09]

(1) $\forall f \in \mathrm{SE}(K_6)$,

$$\sum_{\gamma \in \Gamma_6(K_6)} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_5(K_6)} a_2(f(\gamma)) = \frac{1}{2} \sum_{\lambda \in \Gamma_{3,3}(K_6)} \mathrm{lk}(f(\lambda))^2 - \frac{1}{2}.$$

(2) $\forall f \in \mathrm{SE}(K_7)$,

$$\sum_{\gamma \in \Gamma_7(K_7)} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(K_7)} a_2(f(\gamma))$$
$$= \frac{1}{7}\Bigl(2 \sum_{\lambda \in \Gamma_{3,4}(K_7)} \mathrm{lk}(f(\lambda))^2 + 3 \sum_{\lambda \in \Gamma_{3,3}(K_7)} \mathrm{lk}(f(\lambda))^2\Bigr) - 6.$$

Remark. **Thm 1.3** $\xrightarrow{\text{mod } 2}$ Conway-Gordon theorems

Our purposes are to give integral lifts of **Theorem 1.2** for K_n with arbitrary $n \geq 6$ and investigate the behavior of the "Hamiltonian" constituent knots and links.

§2. Generalizations of the Conway-Gordon theorems

Theorem 2.1. [Morishita-Nikkuni '19]

For $n \geq 6$, $\forall f \in \mathrm{SE}\,(K_n)$,

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2\,(f(\gamma)) - (n-5)! \sum_{\gamma \in \Gamma_5(K_n)} a_2\,(f(\gamma))$$
$$= \frac{(n-5)!}{2}\Bigg(\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\,(f(\lambda))^2 - \binom{n-1}{5} \Bigg).$$

$n = 6$: $\displaystyle \sum_{6} a_2 - \sum_{5} a_2 = \frac{1}{2}\sum_{3,3} \mathrm{lk}^2 - \frac{1}{2}.$ **(Thm. 1.3** (1)**)**

$n = 7$: $\displaystyle \sum_{7} a_2 - 2\sum_{5} a_2 = \sum_{3,3} \mathrm{lk}^2 - 6.$ **(Thm. 1.3** (2)**)**

$n = 8$: $\displaystyle \sum_{8} a_2 - 6\sum_{5} a_2 = 3\sum_{3,3} \mathrm{lk}^2 - 63.$

Note: $\nexists \lambda \in \Gamma_{3,3}(K_n)$ s.t. λ is shared by two distinct subgraphs of K_n isomorphic to K_6.

$$\overset{\text{Thm. 1.1 (1)}}{\Longrightarrow} \quad \forall f \in \mathrm{SE}(K_n), \quad \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\,(f(\lambda))^2 \geq \binom{n}{6}.$$

Corollary 2.2.

For $n \geq 6$, $\forall f \in \mathrm{SE}\,(K_n)$,

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2\,(f(\gamma)) - (n-5)! \sum_{\gamma \in \Gamma_5(K_n)} a_2\,(f(\gamma)) \geq \frac{(n-5)(n-6)(n-1)!}{2 \cdot 6!}.$$

Remark. [Otsuki '96] For $n \geq 6$, $\exists f_{\mathrm{b}} \in \mathrm{SE}(K_n)$ s.t. $f_{\mathrm{b}}\,(K_n) \supset$ exactly $\binom{n}{6}$ triangle-triangle Hopf links.
(f_{b}: *canonical book presentation* of K_n [Endo-Otsuki '94])

Thus the lower bound of **Cor. 2.2** is sharp.

For $\forall f, g \in \mathrm{SE}(K_n)$, by **Thm. 1.1** (1), we also have

$$\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\,(f(\lambda))^2 \equiv \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\,(g(\lambda))^2 \equiv \binom{n}{6} \pmod 2.$$

Then by **Thm. 2.1**, we have

$$\begin{aligned}
&\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_n(K_n)} a_2(g(\gamma)) \\
&\equiv \frac{(n-5)!}{2}\Bigg(\underbrace{\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\,(f(\lambda))^2 - \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\,(g(\lambda))^2}_{\text{even}}\Bigg) \\
&\equiv 0 \pmod{(n-5)!}.
\end{aligned}$$

Since $\displaystyle\sum_{\gamma \in \Gamma_n(K_n)} a_2(f_{\mathrm{b}}(\gamma)) = \frac{(n-5)!}{2}\left(\binom{n}{6} - \binom{n-1}{5}\right)$, we have

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) \equiv \frac{(n-5)!}{2}\left(\binom{n}{6} - \binom{n-1}{5}\right) \pmod{(n-5)!}.$$

Note. $\binom{n}{6} \equiv 1 \pmod 2 \iff n \equiv 6, 7 \pmod 8$,

$$\binom{n-1}{5} \equiv 1 \pmod 2 \iff n \equiv 0, 6 \pmod 8.$$

Corollary 2.3. For $n \geq 7$, $\forall f \in \mathrm{SE}(K_n)$, we have the following congruence modulo $(n-5)!$:

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) \equiv \begin{cases} -\dfrac{(n-5)!}{2}\dbinom{n-1}{5} & (n \equiv 0 \pmod 8) \\ 0 & (n \not\equiv 0, 7 \pmod 8) \\ \dfrac{(n-5)!}{2}\dbinom{n}{6} & (n \equiv 7 \pmod 8). \end{cases}$$

$n = 7$: $\sum_7 a_2 \equiv 7 \equiv 1 \pmod 2$. (**Thm. 1.1** (1))

$n = 8$: $\sum_8 a_2 \equiv -63 \equiv 3 \pmod 6$. (**Thm. 1.2** (1))

$n = 9$: $\sum_9 a_2 \equiv 0 \pmod{24}$.

For “Hamiltonian” 2-component constituent links, we also have the following formula:

Theorem 2.4. [Morishita-Nikkuni ’19]

(1) For $n = p + q$ $(p, q \geq 3)$, $\forall f \in \mathrm{SE}\,(K_n)$,

$$\sum_{\lambda \in \Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 = \begin{cases} (n-6)! \displaystyle\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 & (p = q) \\ 2(n-6)! \displaystyle\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 & (p \neq q). \end{cases}$$

(2) For $n \geq 6$, $\forall f \in \mathrm{SE}\,(K_n)$,

$$\sum_{p+q=n} \sum_{\lambda \in \Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 = (n-5)! \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2.$$

$n = 7$: $\displaystyle\sum_{3,4} \mathrm{lk}^2 = 2 \sum_{3,3} \mathrm{lk}^2.$

$n = 8$: $\displaystyle\sum_{3,5} \mathrm{lk}^2 = 2 \sum_{4,4} \mathrm{lk}^2 = 4 \sum_{3,3} \mathrm{lk}^2.$

Since $\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 \geq \binom{n}{6}$, we have the following:

Corollary 2.5.

(1) For $n = p + q$ $(p, q \geq 3)$, $\forall f \in \mathrm{SE}\,(K_n)$,

$$\sum_{\lambda\in\Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 \geq \begin{cases} n!/6! & (p = q) \\ 2 \cdot n!/6! & (p \neq q) \end{cases}.$$

(2) For $n \geq 6$, $\forall f \in \mathrm{SE}\,(K_n)$,

$$\sum_{p+q=n} \sum_{\lambda\in\Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 \geq (n-5) \cdot \frac{n!}{6!}.$$

$n = 7$: $\sum_{3,4} \mathrm{lk}^2 \geq 2 \cdot 7 = 14$. [Fleming-Mellor '09]

$n = 8$: $\sum_{3,5} \mathrm{lk}^2 \geq 2 \cdot 8 \cdot 7 = 112$, $\sum_{4,4} \mathrm{lk}^2 \geq 8 \cdot 7 = 56$.

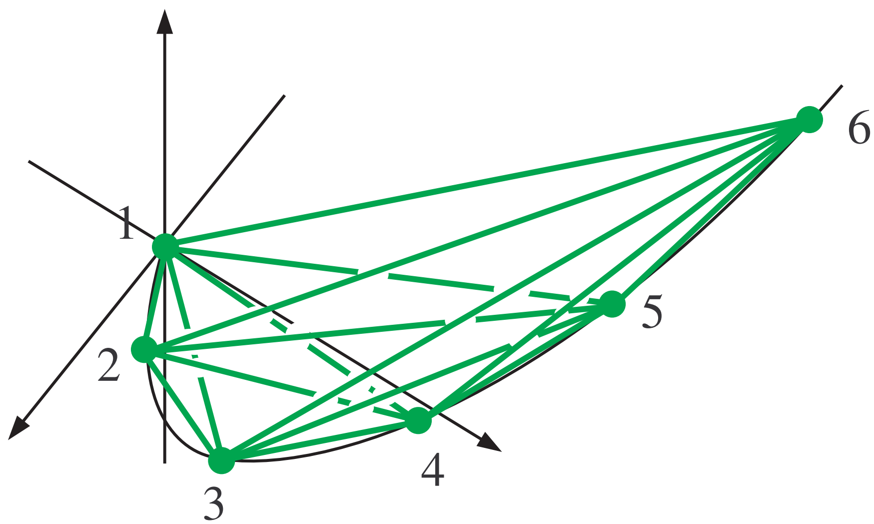
Remark. The lower bounds in **Cor. 2.5** are sharp.

§3. Rectilinear spatial complete graphs

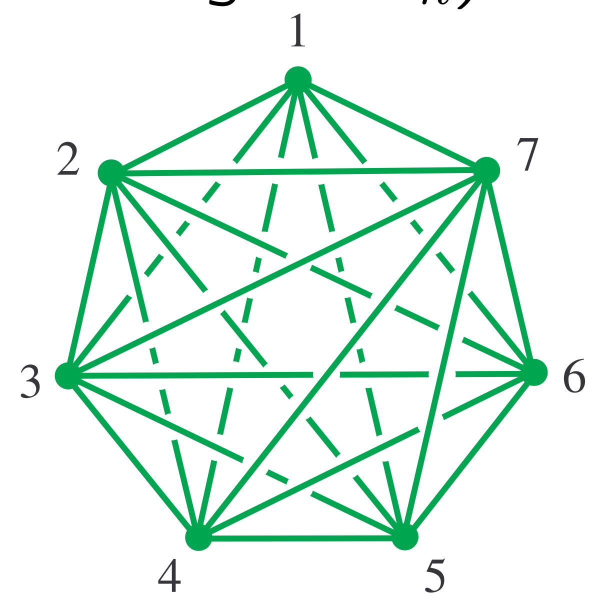
A spatial embedding f_r of K_n is *rectilinear*
$\overset{\text{def.}}{\Leftrightarrow}$ $\forall$ edge e of K_n, $f_r(e)$ is a straight line segment in $\mathbb{R}^3$

$$\mathrm{RSE}\,(K_n) \overset{\text{def.}}{=} \left\{\text{rectilinear embedding } f_r : K_n \to \mathbb{R}^3\right\}$$

Example. (*Standard* rectilinear embedding of K_n)



$f_r(K_6)$



$f_r(K_7)$

Take n vertices on the curve (t, t^2, t^3) and connect every pair of distinct vertices by a straight line segment.

$s(L) =$ min. # of edges in a polygon which represents L
: *stick number* of a link (knot) L

Proposition 3.1.

(1) L is a nontrivial knot $\Longrightarrow s(L) \geq 6$.

(2) $s(L) = 6 \iff L \cong 3_1$, 0_1^2 or 2_1^2.

(3) $s(L) = 7 \iff L \cong 4_1$ or 4_1^2.

In particular, for $f_r \in \mathrm{RSE}(K_n)$ $(n \geq 6)$,

$$\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f_r(\lambda))^2 = \#\text{ of triangle-triangle Hopf links}$$

Theorem 3.2. [Morishita-Nikkuni '19]

For $n \geq 6$, $\forall f_r \in \mathrm{RSE}\,(K_n)$,

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2\,(f_r(\gamma)) = \frac{(n-5)!}{2}\left(\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\,(f_r(\lambda))^2 - \binom{n-1}{5}\right).$$

Proposition 3.3. [Hughes '06] [Huh-Jeon '06] [N '09]

$\forall f_{\mathrm{r}} \in \mathrm{RSE}\,(K_6)$, $f_{\mathrm{r}}(K_6) \supset$ at most 3 Hopf links.

$$\Longrightarrow \ \forall f_{\mathrm{r}} \in \mathrm{RSE}(K_n),\ \binom{n}{6} \leq \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\,(f_{\mathrm{r}}(\lambda))^2 \leq 3\binom{n}{6}.$$

Corollary 3.4. For $n \geq 6$, $\forall f_{\mathrm{r}} \in \mathrm{RSE}\,(K_n)$,

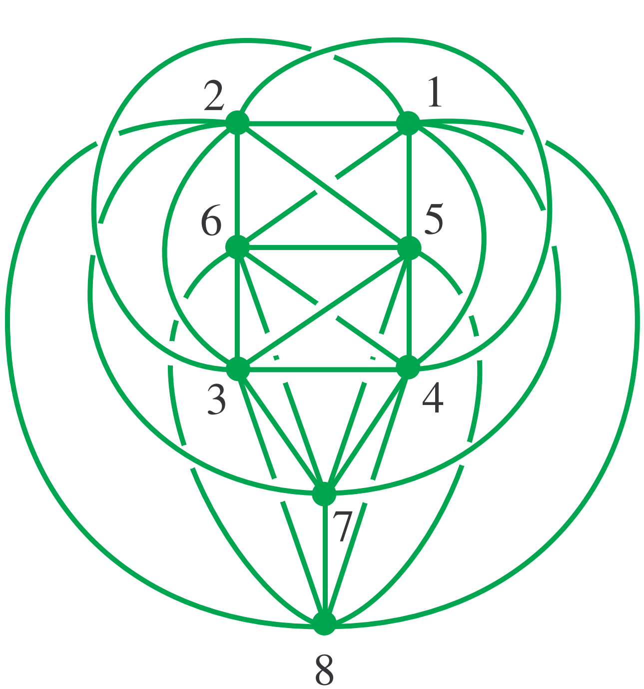
$$\frac{(n-5)(n-6)(n-1)!}{2 \cdot 6!} \leq \sum_{\gamma \in \Gamma_n(K_n)} a_2\,(f_{\mathrm{r}}(\gamma)) \leq \frac{3(n-2)(n-5)(n-1)!}{2 \cdot 6!}.$$

$n = 6$: $0 \leq \sum_{6} a_2 \leq 1$. ($\Longrightarrow$ $\exists$ at most one trefoil knot)

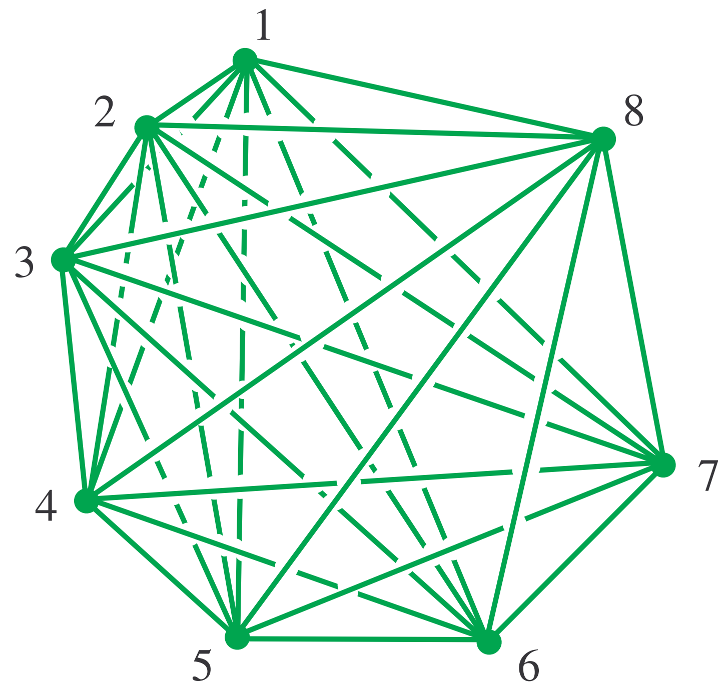
$n = 7$: $1 \leq \sum_{7} a_2 \leq 15$, $\sum_{7} a_2 \equiv 1 \pmod 2$. ($\Longrightarrow$ $\exists$ trefoil)

$n = 8$: $21 \leq \sum_{8} a_2 \leq 189$, $\sum_{8} a_2 \equiv 3 \pmod 6$.

Example. ($n = 8$) Each of the following spatial K_8 contains exactly 21 trefoils as nontrivial Hamiltonian knots:



[BBFHL '07]



[Alfonsín '08]

$$\forall\text{5-cycle knots are trivial} \overset{\text{Thm.2.1}}{\Longrightarrow} \sum_8 a_2 \geq 21.$$

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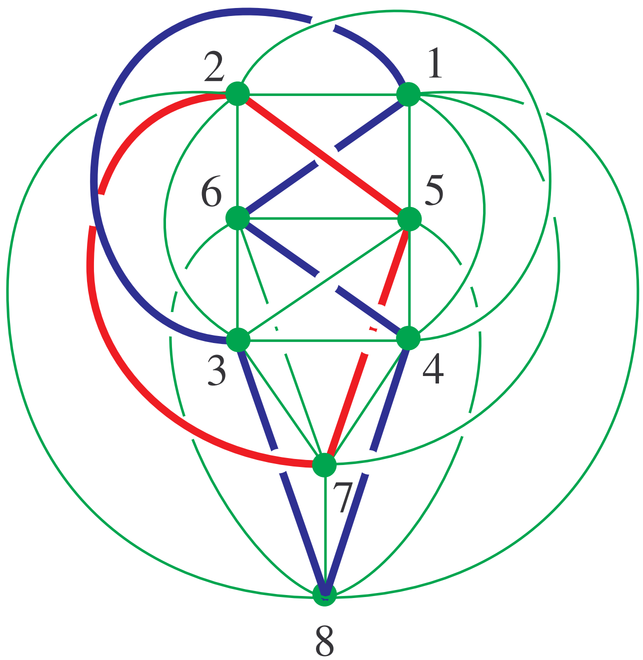

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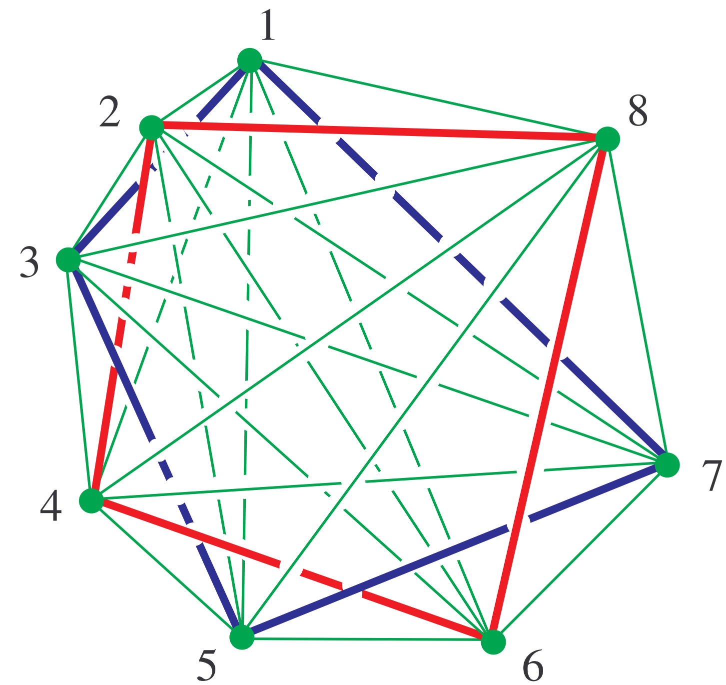

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Remark. The above mentioned spatial graphs of K_8 are NOT ambient isotopic. (Observe the link with $\mathrm{lk} = 2$)

Remark. The lower bound in **Cor. 3.4** is sharp.
(The standard recti. emb. of K_n realizes the lower bound)

On the other hand in $n = 7$ (our upper bound is 15):
According to a computer search in [Jeon et al. 2010],

$$\nexists f_{\mathsf{r}} \in \mathsf{RSE}(K_7) \text{ s.t } \sum_{\gamma \in \Gamma_7(K_7)} a_2(f_{\mathsf{r}}(\gamma)) = 13, 15.$$

$$\left(\Longleftrightarrow \sum_{\lambda \in \Gamma_{3,3}(K_7)} \mathsf{lk}(f_{\mathsf{r}}(\gamma))^2 = 19, 21 \right)$$

(announced in IWSG 2010)

Problem 3.5. Determine the sharp upper bound of $\sum_n a_2$ for all rect. emb. $f_{\mathsf{r}} \in \mathsf{RSE}(K_n)$ for each $n \geq 7$.

Problem 3.5 is equivalent to the following problem.

Problem 3.6. Determine the maximum number of triangle-triangle Hopf links in $f_{\mathsf{r}}(K_n)$ for each $n \geq 7$.

§4. Further applications

Theorem 4.1. $c(K)$: *(minimal) crossing number* of K

(1) [Calvo '01] For a knot K,

$$c(K) \leq \frac{(s(K) - 3)\,(s(K) - 4)}{2}.$$

(2) [Polyak-Viro '01] For a knot K,

$$a_2(K) \leq \frac{c(K)^2}{8}.$$

By **Thm. 4.1** (1) and (2), we have the following.

Lemma 4.2. For a polygonal knot K with $\leq n$ sticks,

$$a_2(K) \leq \left\lfloor \frac{(n-3)^2(n-4)^2}{32} \right\rfloor.$$

Theorem 4.3. [Morishita-Nikkuni '19] For $n \geq 7$, the minimum number of Hamiltonian knots with $a_2 > 0$ in every rectilinear spatial graph of K_n is at least

$$r_n = \left\lceil \frac{(n-5)(n-6)(n-1)!/(2 \cdot 6!)}{\lfloor (n-3)^2(n-4)^2/32 \rfloor} \right\rceil.$$

n	7	8	9	10	11	12	13	14	15
r_n	1	2	12	92	772	7187	73628	823680	10015889

Remark. For $f \in \mathrm{SE}(K_n)$ (not need to be rectilinear),

(1) [Hirano '10] $\exists$ at least 3 Hamiltonian knots with odd a_2 in $f(K_8)$.

(2) [BBFHL '07] For $n \geq 9$, $\exists$ at least $(n-1)!/7!$ nontrivial Hamiltonian knots with odd a_2 in $f(K_n)$.

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r_n	1	5	12	92	772	7187	73628	823680	10015889

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(1) For $n \geq 6$, $\forall f_{\mathrm{r}} \in \mathrm{RSE}(K_n)$,

$$\max_{\gamma \in \Gamma_n(K_n)} \{a_2(f_{\mathrm{r}}(\gamma))\} \geq \frac{(n-5)(n-6)}{6!}.$$

(2) For $n = p + q$ $(p, q \geq 3)$, $\forall f \in \mathrm{SE}(K_n)$,

$$\max_{\lambda \in \Gamma_{p,q}(K_n)} \{|\mathrm{lk}(f(\lambda))|\} \geq \begin{cases} \dfrac{\sqrt{10}\, n}{60} & (p = q) \\[2ex] \dfrac{\sqrt{10pq}}{30} & (p \neq q) \end{cases}.$$

(1): $\max_n \{a_2\} \cdot \overbrace{\sharp \Gamma_n(K_n)}^{\frac{(n-1)!}{2}} \geq \sum_n a_2 \geq \dfrac{(n-5)(n-6)(n-1)!}{2 \cdot 6!}.$

Corollary 4.5. If $n > \left(11 + \sqrt{2880m - 2879}\right)/2$, then
$\forall f_{\mathrm{r}} \in \mathrm{RSE}(K_n)$, $\exists \gamma \in \Gamma_n(K_n)$, s.t. $a_2(f_{\mathrm{r}}(\gamma)) \geq m$.

Remark. [Shirai-Taniyama '03]

(1) $\forall f \in \mathrm{SE}\left(K_{48 \cdot 2^k}\right)$, $\exists \gamma \in \Gamma$ s.t. $|a_2(f(\gamma))| \geq 2^{2k}$.

(2) $n \geq 96\sqrt{m} \Rightarrow \forall f \in \mathrm{SE}(K_n)$, $\exists \gamma \in \Gamma$ s.t. $|a_2(f(\gamma))| \geq m$.

m	1	2	3	4	5	6	7	8	9
n [S-T]	48	136	167	96	215	236	254	272	288
n [M-N]	7	33	44	52	60	66	72	77	82

$R(L) \overset{\text{def.}}{=} \min\{n \mid \forall f_{\mathrm{r}}(K_n) \supset L\}$: *Ramsey number* of L

For $m > 0$, $R(m) \overset{\text{def.}}{=} \min\{n \mid \forall f_{\mathrm{r}}(K_n) \supset \exists K \text{ s.t. } a_2(K) \geq m\}$

For a knot K with $a_2(K) > 0$, $R(a_2(K)) \leq R(K)$.