# Divisibility of Lee's class and its relation with Rasmussen's invariant

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# Introduction : History

- [Kho00] M. Khovanov, “A categorification of the Jones polynomial”.
  - **Khovanov homology** - a bigraded link homology theory, constructed combinatorially from a planar link diagram.
  - Its graded Euler characteristic gives the Jones polynomial.

- [Lee05] E. S. Lee, “An endomorphism of the Khovanov invariant”.
  - **Lee homology** - a variant of Khovanov homology, originally introduced to prove the “Kight move conjecture” for the $\mathbb{Q}$-Khovanov homology of alternating knots.

- [Ras10] J. Rasmussen, “Khovanov homology and the slice genus”.
  - $s$**-invariant** - an integer valued knot invariant obtained from Lee homology.
  - $s$ gives a combinatorial proof for the Milnor conjecture.

# Introduction : Khovanov homology

**Khovanov homology** $H_{Kh}$ is a bigraded link homology theory, constructed combinatorially from a planar link diagram.

Theorem ([Kho00, Theorem 1])

*For any diagram $D$ of an oriented link $L$, the isomorphism class of $H^{\cdot\cdot}_{Kh}(D; R)$ (as a bigraded $R$-module) is an invariant of $L$.*

Proposition ([Kho00, Proposition 9])

*The graded Euler characteristic of $H^{\cdot\cdot}_{Kh}(L; \mathbb{Q})$ gives the (unnormalized) Jones polynomial of $L$:*

$$\sum_{i,j}(-1)^i q^j \dim_{\mathbb{Q}}(H^{ij}_{Kh}(L; \mathbb{Q})) \;=\; (q + q^{-1})V(L)|_{\sqrt{t}=-q}.$$

# Introduction : Khovanov homology

## Example (1)

$0_1$

| j \ i | 0 |
|---|---|
| 1 | 1 |
| -1 | 1 |

$3_1$

| j \ i | -3 | -2 | -1 | 0 |
|---|---|---|---|---|
| -1 | | | | 1 |
| -3 | | | | 1 |
| -5 | | 1 | | |
| -7 | | | | |
| -9 | 1 | | | |

$4_1$

| j \ i | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| 5 | | | | | 1 |
| 3 | | | | | |
| 1 | | | 1 | 1 | |
| -1 | | 1 | 1 | | |
| -3 | | | | | |
| -5 | 1 | | | | |

Figure 1: $H_{Kh}(K;\mathbb{Q})$ for $K = 0_1, 3_1, 4_1$

# Introduction : Khovanov homology

## Example (2)

$9_{42}$

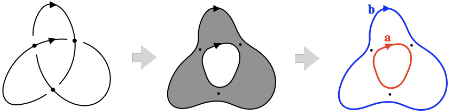
| | -4 | -3 | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|---|---|
| 7 | | | | | | | 1 |
| 5 | | | | | | | |
| 3 | | | | | 1 | 1 | |
| 1 | | | | 1 | 1 | | |
| -1 | | | | 1 | 1 | | |
| -3 | | 1 | 1 | | | | |
| -5 | | | | | | | |
| -7 | 1 | | | | | | |

Figure 2: $H_{Kh}(K;\mathbb{Q})$ for $K = 9_{42}$

# Introduction : Lee homology and Lee's classes

**Lee homology** $H_{Lee}$ is a variant of Khovanov homology. Although the construction is similar, when $R = \mathbb{Q}$ the structure of $H_{Lee}$ is strikingly simple.

For a knot diagram $D$, there are two distinct classes $[\alpha], [\beta]$ constructed combinatorially from $D$.




## Theorem ([Lee05, Theorem 4.2])

*When $R = \mathbb{Q}$, the two classes $[\alpha], [\beta]$ form a basis of $H_{Lee}(D; \mathbb{Q})$.*

## Remark

For a link diagram $D$ with $\ell$ components, there are $2^\ell$ distinct classes $[\alpha(D, o)]$ one for each alternative orientation $o$ of $D$. These form a basis of $H_{Lee}(D; \mathbb{Q})$.

# Introduction : Rasmussen's $s$-invariant

Rasmussen introduced in [Ras10] an integer-valued knot invariant, called the $s$-**invariant**, based on $\mathbb{Q}$-Lee homology.

$H_{Lee}$ is not bigraded (unlike $H_{Kh}$), but admits a filtration by q-degree. For a knot $K$, the $s$-invariant is defined by

$$s(K) := \frac{q_{\max} + q_{\min}}{2},$$

where $q_{\max}$ (resp. $q_{\min}$) denotes the maximum (resp. minimum) q-degree of $H_{Lee}(K;\mathbb{Q})$.

Many properties of $s$ are proved from the fact that $[\alpha], [\beta]$ are invariant (up to unit) under the Reidemeister moves. Hence Rasmussen called them the "**canonical generators**" of $H_{Lee}(K;\mathbb{Q})$.

# Introduction : Properties of $s$

### Theorem ([Ras10, Theorem 2])

*$s$ defines a homomorphism from the knot concordance group in $S^3$ to $2\mathbb{Z}$:*

$$s\colon Conc(S^3) \to 2\mathbb{Z}.$$

### Theorem ([Ras10, Theorem 1])

*$s$ gives a lower bound of the slice genus:*

$$|s(K)| \leq 2g_*(K).$$

### Theorem ([Ras10, Theorem 4])

*If $K$ is a positive knot, then*

$$s(K) = 2g_*(K) = 2g(K).$$

# Introduction : Properties of $s$

With the above three properties of $s$, one obtains:

## Corollary (The Milnor Conjecture, [Mil68])

*The (smooth) slice genus and the unknotting number of the $(p, q)$ torus knot are both equal to $(p-1)(q-1)/2$.*

## Remark

The Milnor Conjecture was first proved by Kronheimer and Mrowka in [KM93] using gauge theory, but Rasmussen's result was notable since it provided a purely combinatorial proof.

# Our observations

Now we consider Lee homology over $\mathbb{Z}$. Let $D$ be a knot diagram, and denote

$$H_{Lee}(D;\mathbb{Z})_f = H_{Lee}(D;\mathbb{Z})/Tor.$$

The two classes $[\alpha], [\beta]$ can be defined over $\mathbb{Z}$, but they do not form a basis of $H_{Lee}(D;\mathbb{Z})_f \cong \mathbb{Z}^2$.

We created a computer program[1] that calculates the components of $[\alpha], [\beta]$ with respect to some basis of $H_{Lee}(D;\mathbb{Z})_f$. It turned out, that for any prime knot diagram of crossing number up to 11, only **2-powers** appear in those components.

## Question

*Where does the 2-powers come from, and what information can we extract from the 2-divisibility of* $[\alpha], [\beta]$*?*

[1] `https://github.com/taketo1024/SwiftyMath`

# Computational results

```
3_1
b⊗a(111) = [2, -2]
a⊗b(111) = [-2, -2]

4_1
a⊗b⊗a(0011) = [-2, -2]
b⊗a⊗b(0011) = [2, -2]

5_1
b⊗a(11111) = [2, -2]
a⊗b(11111) = [-2, -2]

5_2
b⊗a⊗b⊗a(11111) = [-8, 8]
a⊗b⊗a⊗b(11111) = [-8, -8]

6_1
a⊗b⊗a⊗b⊗a(110011) = [8, 8]
b⊗a⊗b⊗a⊗b(110011) = [8, -8]

6_2
a⊗b⊗a(110011) = [2, 2]
b⊗a⊗b(110011) = [2, -2]

6_3
a⊗b⊗b(001110) = [-2, 2]
b⊗a⊗a(001110) = [2, 2]

7_1
b⊗a(1111111) = [2, -2]
a⊗b(1111111) = [-2, -2]

7_2
b⊗a⊗b⊗a⊗b⊗a(1111111) = [-32, 32]
a⊗b⊗a⊗b⊗a⊗b(1111111) = [32, 32]

7_3
b⊗a⊗b⊗a(0000000) = [-1, 1]
a⊗b⊗a⊗b(0000000) = [1, 1]

7_4
b⊗a⊗b⊗a⊗b⊗a(0000000) = [-1, 1]
a⊗b⊗a⊗b⊗a⊗b(0000000) = [1, 1]

7_5
b⊗a⊗b⊗a(1111111) = [-8, 8]
a⊗b⊗a⊗b(1111111) = [-8, -8]

7_6
a⊗b⊗b⊗b(1110011) = [4, -4]
b⊗a⊗a⊗a(1110011) = [4, 4]

7_7
a⊗b⊗a⊗a(1100100) = [2, -2]
b⊗a⊗b⊗b(1100100) = [-2, -2]
```

Figure 3: Computational results[2]

[2] https://git.io/fphro

# Overview of our results (1/2)

We consider the question in a more generalized setting. There exists a family of Khovanov-type homology theories $\{H_c(-; R)\}_{c \in R}$ over a commutative ring $R$ parameterized by $c \in R$.

For each $c \in R$, Lee's classes $[\alpha], [\beta]$ of a knot diagram $D$ can also be defined in $H_c(D; R)$.

If $R$ is an integral domain and $c$ is non-zero, non-invertible, then we can define the $c$-**divisibility** of $[\alpha]$ (modulo torsions) by the exponent of its $c$-power factor. We denote it by $k_c(D)$.

By inspecting the variance of $k_c$ under the Reidemeister moves, we prove that

$$\bar{s}_c(K) := 2k_c(D) + w(D) - r(D) + 1$$

is a knot invariant, where $w$ is the writhe, and $r$ is the number of Seifert circles.

# Overview of our results 2/2

Again by computations, we saw that values of $\bar{s}_2(-;\mathbb{Z})$ coincide with values of $s$ for all prime knots of crossing number up to 11.

### Question

*Are all $\bar{s}_c$ equal to $s$?*

The following theorems support the affirmative answer.

### Theorem (S.)

*Each $\bar{s}_c$ possesses properties common to $s$. In particular, the each $\bar{s}_c$ can be used to reprove the Milnor conjecture.*

### Theorem (S.)

*If $(R, c) = (\mathbb{Q}[h], h)$, then the knot invariant $s'_h$ coincides with $s$*

$$s(K) = s'_h(K; \mathbb{Q}[h]).$$

# Conventions

In this talk, all knots and links are assumed to be **oriented**. For simplicity, we mainly focus on **knots**, but many of the results can be generalized to links.

1. Introduction
2. Preliminary : Khovanov homology theory
3. Generalizing Lee's classes
4. $k_c(D)$ and $\bar{s}_c(K)$
5. Behavior of $\bar{s}_c(K)$ under cobordisms
6. Coincidence with $s$
7. Future prospects

# Frobenius algebra (1/2)

Let $R$ be a commutative ring with unity. A **Frobenius algebra** over $R$ is a quintuple $(A, m, \iota, \Delta, \varepsilon)$ satisfying:

1. $(A, m, \iota)$ is an associative $R$-algebra with multiplication $m : A \otimes A \to A$ and unit $\iota : R \to A$,
2. $(A, \Delta, \varepsilon)$ is a coassociative $R$-coalgebra with comultiplication $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to R$, and
3. the Frobenius relation holds:

$$\Delta \circ m = (id \otimes m) \circ (\Delta \otimes id) = (m \otimes id) \circ (id \otimes \Delta).$$

# Frobenius algebra (2/2)

A commutative Frobenius algebra $A$ gives a 1+1 TQFT

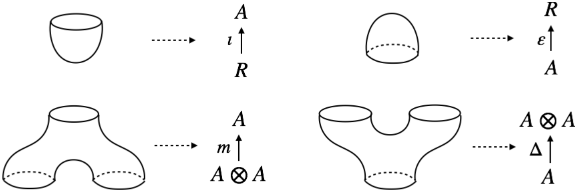
$$\mathcal{F}_A : Cob_2 \longrightarrow Mod_R,$$

by mapping:

- Objects:

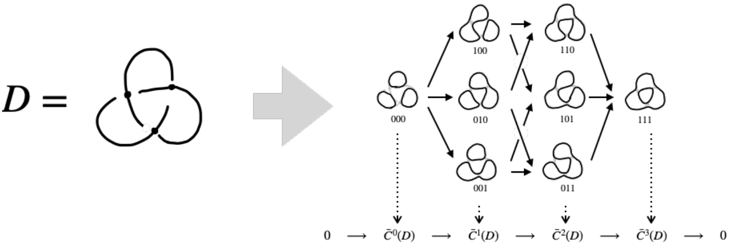
$$\underbrace{\bigcirc \sqcup \cdots \sqcup \bigcirc}_{r} \longrightarrow \underbrace{A \otimes \cdots \otimes A}_{r}$$

- Morphisms:



# Construction of the chain complex

Let $D$ be a link diagram with $n$ crossings. The $2^n$ resolutions of the crossings yields a commutative cubic diagram in $Cob_2$.




By applying $\mathcal{F}_A$ we obtain a commutative cubic diagram in $Mod_R$.

Then we turn this cube skew commutative by appropriately adjusting the signs of the edge maps.

Finally we fold the cube and obtain a chain complex $C_A(D)$ and its homology $H_A(D)$.

# Khovanov homology and its variants

Khovanov's original theory is given by $A = R[X]/(X^2)$. Other variant theories are given by:

- $A = R[X]/(X^2 - 1) \quad \rightarrow$ Lee's theory
- $A = R[X]/(X^2 - hX) \rightarrow$ Bar-Natan's theory

Khovanov unified these theories in [Kho06] by considering the following special Frobenius algebra with $h, t \in R$:

$$A_{h,t} = R[X]/(X^2 - hX - t).$$

Denote the corresponding chain complex by $C_{h,t}(D; R)$ and its homology by $H_{h,t}(D; R)$. The isomorphism class of $H_{h,t}(D; R)$ is invariant under Reidemeister moves, thus gives a link invariant.

# Generalizing Lee's classes (1/2)

In order to generalize Lee's classes $[\alpha], [\beta]$ in $H_{h,t}(D; R)$, we assume $(R, h, t)$ satisfies the following condition:

### Condition

*$X^2 - hX - t$ factors into linear polynomials in $R[X]$.*

This is equivalent to:

### Condition

*There exists $c \in R$ such that $h^2 + 4t = c^2$ and $(h \pm c)/2 \in R$.*

With this condition, fix one square root $c = \sqrt{h^2 + 4t}$, and let $X^2 - hX - t = (X - u)(X - v)$ with $c = v - u$. Define

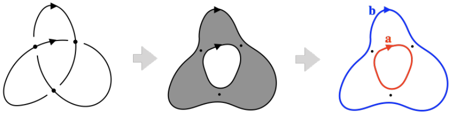
$$\mathbf{a} = X - u, \quad \mathbf{b} = X - v \ \in \ A.$$

# Generalizing Lee's classes (2/2)

With **a** and **b**, the multiplication and comultiplication on $A$ diagonalizes as:

$$m(\mathbf{a} \otimes \mathbf{a}) = c\mathbf{a}, \qquad \Delta(\mathbf{a}) = \mathbf{a} \otimes \mathbf{a},$$
$$m(\mathbf{a} \otimes \mathbf{b}) = 0, \qquad \Delta(\mathbf{b}) = \mathbf{b} \otimes \mathbf{b}$$
$$m(\mathbf{b} \otimes \mathbf{a}) = 0$$
$$m(\mathbf{b} \otimes \mathbf{b}) = -c\mathbf{b}$$

We define the cycles $\alpha, \beta \in C_{h,t}(D; R)$ by the orientation preserving resolution of $D$.




## Remark

For a link diagram $D$ with $\ell$ components, there are $2^\ell$ distinct cycles $\alpha(D, o)$ one for each alternative orientation $o$ of $D$.

# Reduction of parameters

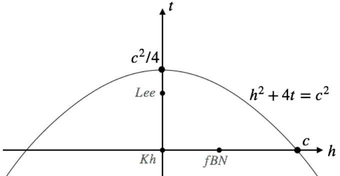
In our setting, we may reduce the parameters $(h, t)$ to a single parameter $c$.

## Proposition

*For another $(h', t')$ such that $c = \sqrt{h'^2 + 4t'}$, the corresponding groups $H_{h,t}(D; R)$ and $H_{h',t'}(D; R)$ are naturally isomorphic, and under the isomorphism the Lee classes correspond one-to-one.* □

Thus we denote the isomorphism class by $H_c(D; R)$ and regard $[\alpha], [\beta] \in H_c(D; R)$.

The figure depicts the $(h, t)$-parameter space, where each point $(h, t)$ corresponds to $H_{h,t}(D; R)$ and the parabola $h^2 + 4t = c^2$ corresponds to the isomorphism class $H_c(D; R)$.



# Generalizing Lee's theorem

The following proposition generalizes Lee's theorem for $\mathbb{Q}$-Lee homology ($\mathbb{Q}$-Lee homology corresponds to $(R, c) = (\mathbb{Q}, 2)$).

## Proposition

*If c is invertible in $R$, then $\{[\alpha], [\beta]\}$ form a basis of $H_c(D; R)$.*

## Proof.

$A$ is free over $R$ with basis $\{1, X\}$. Now $\{\mathbf{a}, \mathbf{b}\}$ also form a basis of $A$, since the transformation matrix $\begin{pmatrix} -u & -v \\ 1 & 1 \end{pmatrix}$ has determinant $v - u = c$.

By the *admissible colorings decomposition* of $C_c(D; R)$ (proposed by Wehrli in [Weh08]), one can show that the subcomplex generated by $\alpha$ and $\beta$ becomes $H_c(D; R)$, whereas the remaining part is acyclic. □

## Remark

For a link diagram $D$, the $2^\ell$ classes $\{[\alpha(D, o)]\}_o$ form a basis of $H_c(D; R)$.

# Correspondence under Reidemeister moves (1/2)

Next, the following proposition generalizes the "invariance of $[\alpha]$ and $[\beta]$ (up to unit) in $\mathbb{Q}$-Lee theory".

## Proposition

*Suppose $D, D'$ are two diagrams related by a single Reidemeister move. Under the corresponding isomorphism:*

$$\rho : H_c(D; R) \to H_c(D'; R)$$

*there exists some $j \in \{0, \pm 1\}$ and $\varepsilon, \varepsilon' \in \{\pm 1\}$ such that the $\alpha, \beta$-classes of $D$ and $D'$ are related as:*

$$[\alpha'] = \varepsilon c^j \cdot \rho[\alpha],$$
$$[\beta'] = \varepsilon' c^j \cdot \rho[\beta].$$

*(Here $c$ is not necessarily invertible, so when $j < 0$ the equation $z = c^j w$ is to be understood as $c^{-j} z = w$.)*

# Correspondence under Reidemeister moves (2/2)

## Proposition (continued)

*Moreover the exponent $j$ is given by*

$$j = \frac{\Delta r - \Delta w}{2}$$

*where $r$ denotes the number of Seifert circles, $w$ denotes the writhe, and the prefixed $\Delta$ is the difference of the corresponding numbers for $D$ and $D'$.*

## Proof.

The isomorphism $\rho$ is given explicitly, and the proof is done by checking all possible patterns of $[\alpha]$ and those images under $\rho$. □

Note that $[\alpha], [\beta]$ are invariant (up to unit) iff c is invertible.

## Remark

Similar statement holds for link diagrams.

# Summary

- We defined a family of Khovanov-type link homology theories $\{H_c(-; R)\}_{c \in R}$, where Knovanov's theory corresponds to $c = 0$ and Lee's theory corresponds to $c = 2$.
- For each $c \in R$, we generalized Lee's classes $[\alpha], [\beta]$ of a knot diagram $D$ in $H_c(D; R)$.
- $[\alpha], [\beta]$ form a basis of $H_c(D; R)$ and are invariant (up to unit) under the Reidemeister moves iff $c$ is invertible.

Thus the situation is completely analogous to $\mathbb{Q}$-Lee theory when $c$ is invertible. Our main concern is when $c$ is not invertible.

# $c$-divisibility of the $\alpha$-class

Let $R$ be an integral domain, and $c \in R$ be a non-zero non-invertible element in $R$. Denote

$$H_c(D;R)_f = H_c(D;R)/Tor.$$

By abuse of notation, we denote the images of $[\alpha], [\beta]$ in $H_c(D;R)_f$ by the same symbols.

## Definition

For any knot diagram $D$, define the $c$-**divisibility** of $[\alpha]$ by:

$$k_c(D) := \max\{k \geq 0 \mid [\alpha] \in c^k H_c(D;R)_f \}.$$

Note that there is a filtration:

$$H_c(D;R)_f \supset cH_c(D;R)_f \supset \cdots \supset c^k H_c(D;R)_f \supset \cdots$$

so $k_c(D)$ is the maximal filtration level that contains $[\alpha]$.

# Basic properties of $k_c(D)$

Proposition

$$0 \leq k_c(D) \leq n^{-}(D)$$

.

In particular if $D$ is positive, then $k_c(D) = 0$. We can regard $k_c$ as the measure of the "non-positivity" of the diagram.

Proposition

1. $k_c(D) = k_c(-D)$.
2. $k_c(D) + k_c(D') \leq k_c(D \sqcup D')$.
3. $k_c(D \# D') \leq k_c(D \sqcup D') \leq k_c(D \# D') + 1$.

# Variance of $k_c$ under Reidemeister moves

## Proposition

*Let $D, D'$ be two diagrams of the same knot. Then*

$$\Delta k_c = \frac{\Delta r - \Delta w}{2},$$

*where the prefixed $\Delta$ is the difference of the corresponding numbers for $D$ and $D'$.* □

## Theorem (S.)

*For any knot $K$,*

$$\bar{s}_c(K) := 2k_c(D) - r(D) + w(D) + 1$$

*is an invariant of $K$.* □

## Remark

$\bar{s}_c$ can also be defined for links.

# Basic properties of $\bar{s}_c(K)$

**Proposition**

$$\bar{s}_c(K) \in 2\mathbb{Z}.$$

**Proposition**

1. $\bar{s}_c(L) = \bar{s}_c(-L)$.
2. $\bar{s}_c(L \sqcup L') \geq \bar{s}_c(L) + \bar{s}_c(L') - 1$.
3. $\bar{s}_c(L \# L') = \bar{s}_c(L \sqcup L') \pm 1$.

# Behavior under cobordisms (1/2)

The important properties of $s$ are obtained by inspecting its behavior under cobordisms between knots. By tracing the arguments given in [Ras10], we obtain a similar proposition for $\bar{s}_c$.

## Proposition (S.)

*If $S$ is an oriented connected cobordism between knots $K, K'$, then*

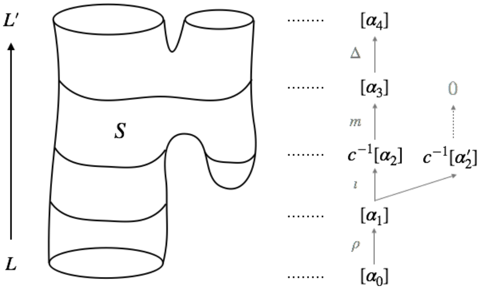
$$|\bar{s}_c(K') - \bar{s}_c(K)| \leq -\chi(S).$$

## Remark

Similar statement holds for links.

# Behaviour under cobordisms (2/2)

Proof sketch.




Decompose $S$ into elementary cobordisms such that each factor corresponds to a Reidemeister move or a Morse move. Inspect the successive images of the $\alpha$-class at each level. □

# Consequences

The previous proposition implies properties of $\bar{s}_c$ that are common to the $s$-invariant:

## Theorem (S.)

- *$\bar{s}_c$ is a knot concordance invariant in $S^3$.*
- *For any knot $K$,*
$$|\bar{s}_c(K)| \leq 2g_*(K).$$
- *If $K$ is a positive knot, then*
$$\bar{s}_c(K) = 2g_*(K) = 2g(K).$$

These properties suffice to reprove the Milnor conjecture.

# The refined canonical generators (1/2)

Now we focus on the case

$$(R, c) = (\mathbb{Q}[h], h),\ \deg h = -2$$

and prove that $\bar{s}_h(-; \mathbb{Q}[h])$ coincides with $s$.

Recall that in general $[\alpha], [\beta]$ do not form a basis of $H_c(D; R)_f$. However in the above case, we can "normalize" them to obtain a class $[\zeta]$ such that $\{[\zeta], X[\zeta]\}$ is a basis of $H_c(D; R)_f$.

Moreover they are invariant under the Reidemeister moves, so it is reasonable to call them the "**canonical generators**" of $H_c(D; R)_f$.

## Remark

$X$ denotes an action on $H_c(D; R)$ defined by merging a circled labeled $X$ to a neighborhood of a fixed point of $D$.

# The refined canonical generators (2/2)

## Proposition

*There is a unique class* $[\zeta] \in H_h(D;R)_f$ *such that*

- $[\zeta], X[\zeta]$ *form a basis of* $H_h(D;R)_f$*, and are invariant under the Reidemeister moves.*
- $[\alpha], [\beta]$ *can be described as*

$$\begin{aligned} [\alpha] &= \quad h^k(\quad (h/2)[\zeta] + X[\zeta]) \\ [\beta] &= (-h)^k(-(h/2)[\zeta] + X[\zeta]), \end{aligned}$$

*where* $k = k_h(D)$*.* □

## Remark (1)

Unlike $[\alpha]$ or $[\beta]$, the definition of $[\zeta]$ is non-constructive.

## Remark (2)

Currently this result is only obtained for knots.

# The homomorphism property of $\bar{s}_h$

From the description of $[\alpha], [\beta]$ by the class $[\zeta]$, we can prove:

### Proposition

*For* $(R, c) = (\mathbb{Q}[h], h)$*,*

- $k_h(D) + k_h(\overline{D}) = r(D) - 1.$
- $k_h(D \# D') = k_h(D) + k_h(D').$

*where* $\overline{D}$ *denotes the mirror image of* $D$*.*

And we obtain:

### Theorem (S.)

*For* $(R, c) = (\mathbb{Q}[h], h)$*, the invariant* $\bar{s}_h$ *defines a homomorphism*

$$\bar{s}_h \colon Conc(S^3) \to 2\mathbb{Z}.$$

# Coincidence with the $s$-invariant (1/4)

## Theorem (S.)

*For* $(R, c) = (\mathbb{Q}[h], h)$*, our* $\bar{s}_h(-; \mathbb{Q}[h])$ *coincides with* $s$*:*

$$s(K) = 2k_h(D) + w(D) - r(D) + 1.$$

## Remark (1)

There is a well known lower bound for $s$ [Shu07, Lemma 1.3]

$$s(K) \geq w(D) - r(D) + 1,$$

so $2k_h(D)$ gives the correction term of the inequality.

## Remark (2)

Currently this result is only obtained for knots.

# Coincidence with the $s$-invariant (2/4)

## Proof.

Since both $s$ and $\bar{s}_h$ changes sign by mirroring, it suffices to prove

$$s(K) \geq \bar{s}_h(K).$$

Recall that $s(K)$ is defined by the (filtered) q-degree of $H_2(D; \mathbb{Q})$. On the other hand, q-degree on $H_h(D; \mathbb{Q}[h])$ gives a strict grading. There is a q-degree non-decreasing map

$$\pi : H_h(D; \mathbb{Q}[h]) \to H_2(D; \mathbb{Q})$$

induced from $\mathbb{Q}[h] \to \mathbb{Q},\ h \mapsto 2$.

Denote by $[\alpha_2], [\alpha_h]$ the $\alpha$-classes of $D$ in $H_2(D; F)$, $H_h(D; \mathbb{Q}[h])$ respectively. Then by definition $\pi[\alpha_h] = [\alpha_2]$.

# Coincidence with the $s$-invariant (3/4)

Proof continued.

Let $[\alpha_h] = h^k[\alpha'_h]$ with $k = k_h(D)$. Then from $\deg h = -2$,

$$\mathrm{qdeg}_h([\alpha'_h]) = \mathrm{qdeg}_h([\alpha_h]) + 2k = w(D) - r(D) + 2k,$$

so we have

$$\begin{aligned} s(K) &= \mathrm{qdeg}([\alpha_2]) + 1 \\ &= \mathrm{qdeg}(\pi[\alpha_h]) + 1 \\ &= \mathrm{qdeg}(\pi[\alpha'_h]) + 1 \\ &\geq \mathrm{qdeg}_h([\alpha'_h]) + 1 \\ &= w(D) - r(D) + 2k + 1 \\ &= \bar{s}_h(K; F[h]). \end{aligned}$$

□

# Coincidence with the $s$-invariant (4/4)

## Corollary

$$s(K) = \mathrm{qdeg}_h[\zeta] - 1.$$

## Remark

Khovanov gave an alternative definition of $s$ in [Kho06] by the q-degree of the generator of $H_{0,t}(D; \mathbb{Q}[t])$ where $\deg t = -4$. The equivalence of the two definitions can be proved using the above result.

## Question (1)

*Are all $\bar{s}_c$ equal to $s$?*

## Remark (1)

The $s$-invariant can be defined over any field $F$ of char $F \neq 2$. In fact we can prove that

$$s(-;F) = \bar{s}_h(-;F[h]).$$

It is an open question whether $s(-;F)$ for char $F \neq 2$ are all equal or not [LS14, Question 6.1]. If [Question 1] is solved affirmatively, then it follows that $s(-;F)$ are all equal.

## Remark (2)

In [LS14], an alternative definition of $s$ over any field $F$ (including char $F = 2$) is given. It is defined similarly to the original one, but is based on the filtered Bar-Natan homology. However C.Seed showed by direct computation that $K = K14n19265$ has $s(K;\mathbb{Q}) = 0$ but $s(K;\mathbb{F}_2) = -2$.

## Question (2)

*Can we construct* $[\zeta] \in H_c(D; R)$ *for any* $(R, c)$*?*

The existence of $[\zeta] \in H_h(D; \mathbb{Q}[h])$ was the key to prove $s = \bar{s}_h$.

If such class exists in general, then we can expect that Question (1) can also be solved. However the current proof for $(R, c) = (\mathbb{Q}[h], h)$ cannot be applied to the general case.

Maybe we can find a more geometric (or combinatorial) construction.

## Question (3)

*Does* $s = \bar{s}_c(-; \mathbb{Q}[h])$ *also hold for links?*

The definition of $s$ for links is given by Beliakova and Wehrli in [BW08]. Our $\bar{s}_c$ can also be defined for links, so the question makes sense.

Maybe we can construct the canonical generators $[\zeta_1], \cdots, [\zeta_{2^\ell}]$ of $H_h(D; \mathbb{Q}[h])$ such that the $\alpha$-classes $\{[\alpha(D, o)]\}_o$ can be described by them.

Thank you!

https://arxiv.org/abs/1812.10258

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