Writhe Polynomials and Shell Moves for Virtual Knots and Links

Joint Work with
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Main Theorems

K: an oriented virtual knot

$J_n(K)$: the <u>n-writhe</u> of K $(n \neq 0)$

$W_K(t) = \sum_{n \neq 0} J_n(K) \cdot (t^n - 1)$: the <u>writhe polynomial</u> of K

$\overset{\text{shell}}{\sim}$: the equivalence relation generated by <u>shell moves</u>

<u>Thm 1</u> ['19 NNS]

K, K' : oriented virtual knots

$W_K(t) = W_{K'}(t) \Longleftrightarrow K \overset{\text{shell}}{\sim} K'$

$L = K_1 \cup K_2$: an oriented 2-comp. virtual link

$\lambda(L) = Lk(K_1, K_2) - Lk(K_2, K_1)$: the virtual linking number

$J_n(K_1; L)$: the n-writhe of K_1 in L $(n \neq 0, -\lambda)$

$J_n(K_2; L)$: the n-writhe of K_2 in L $(n \neq 0, \lambda)$

$F(L) \in \left(\mathbb{Z}[t, t^{-1}]/(t^{\lambda} - 1)\right)^2 / \doteq$: the linking class of L

Thm. 2 [NNS] $\lambda(L) = \lambda(L') = 0$

$L \overset{\text{shell}}{\sim} L' \Leftrightarrow$ (i) $J_n(K_1; L) = J_n(K_1'; L')$ for $\forall n \neq 0, 1$

(ii) $J_n(K_2; L) = J_n(K_2'; L')$ for $\forall n \neq 0, 1$

(iii) $F(L) = F(L')$

Thm. 3 [NNS] $\lambda(L)=\lambda(L')=1$

$L \overset{\text{shell}}{\sim} L' \Leftrightarrow$ (i) $J_n(K_1;L)=J_n(K_1';L')$ for $\forall n\neq 0,-1,1$

(ii) $J_n(K_2;L)=J_n(K_2';L')$ for $\forall n\neq 0,1,2$

(iii) $F(L)=F(L')$

Thm. 4 [NNS] $\lambda(L)=\lambda(L')\geq 2$

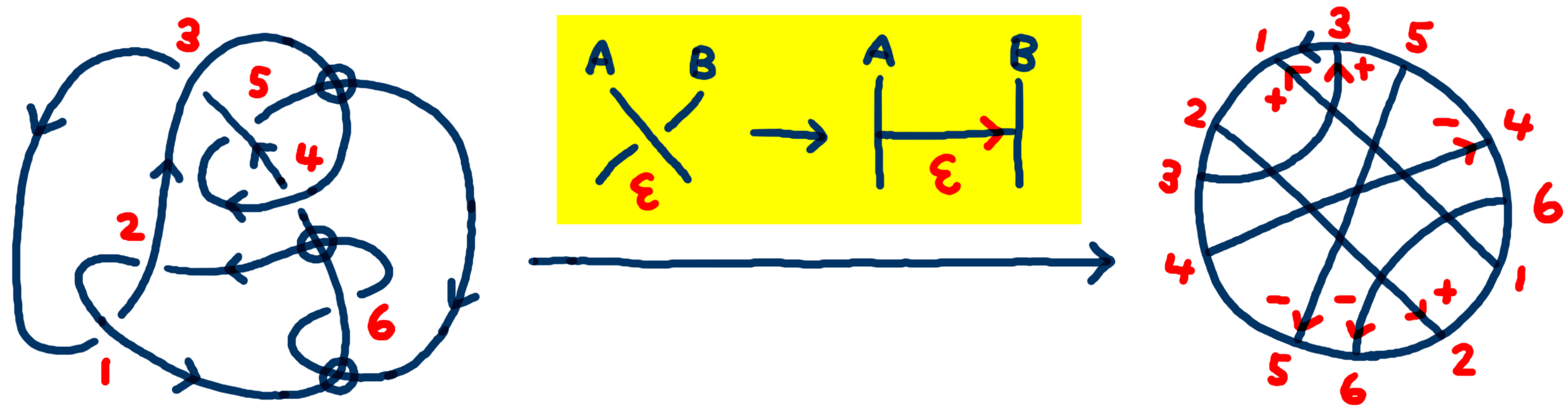
$L \overset{\text{shell}}{\sim} L' \Leftrightarrow$ (i) $J_n(K_1;L)=J_n(K_1';L')$ for $\forall n\neq 0,-\lambda,1,-\lambda+1$

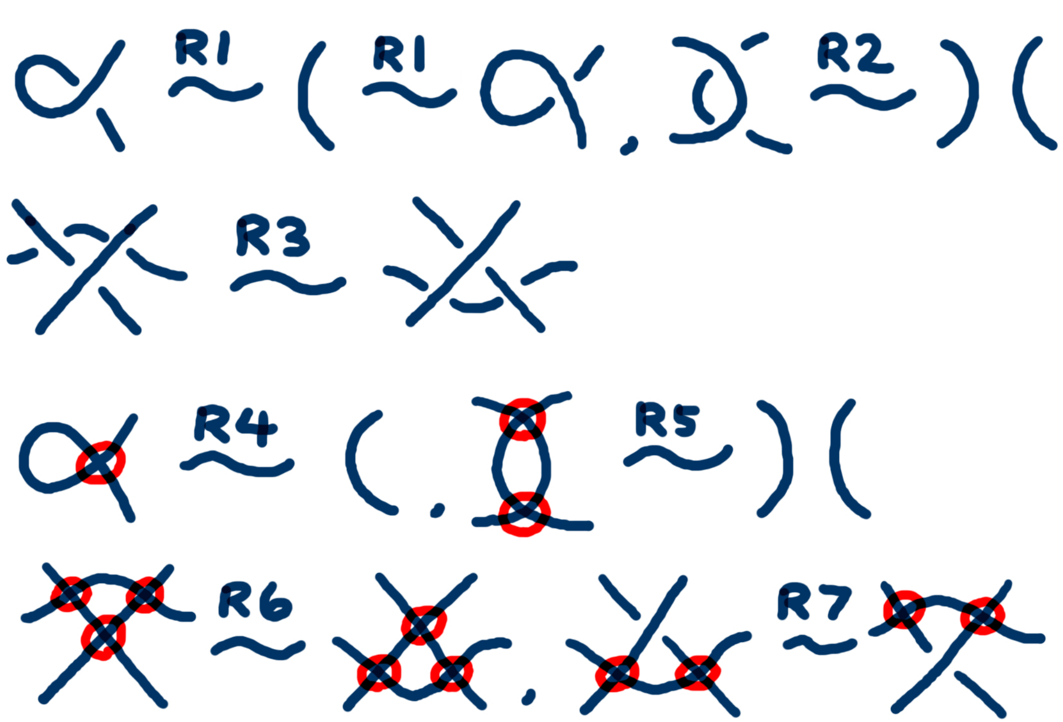
(ii) $J_n(K_2;L)=J_n(K_2';L')$ for $\forall n\neq 0,\lambda,1,\lambda+1$

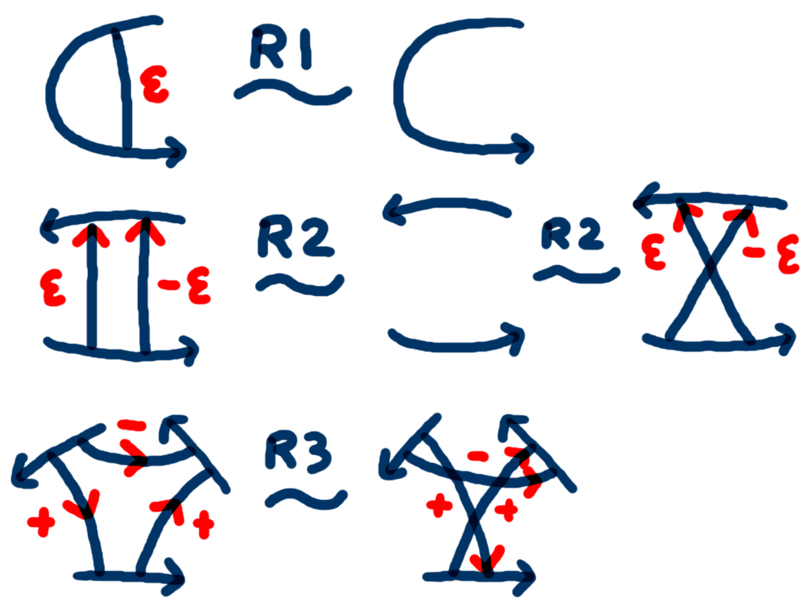
(iii) $F(L)=F(L')$

(iv) $J_1(K_1;L)+J_{-\lambda+1}(K_1;L)+J_1(K_2;L)+J_{\lambda+1}(K_2;L)$
$=J_1(K_1';L')+J_{-\lambda+1}(K_1';L')+J_1(K_2';L')+J_{\lambda+1}(K_2';L')$

Virtual knot diagrams v.s. Gauss diagrams

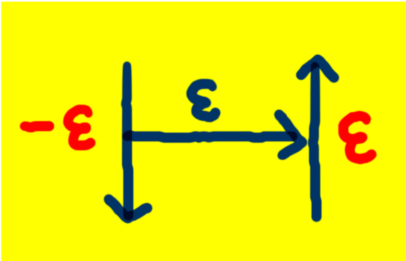


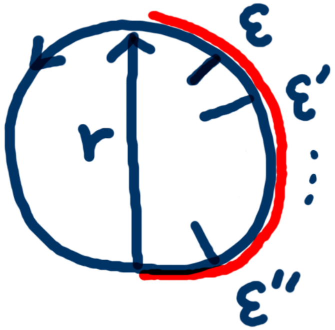




→ virtual knots

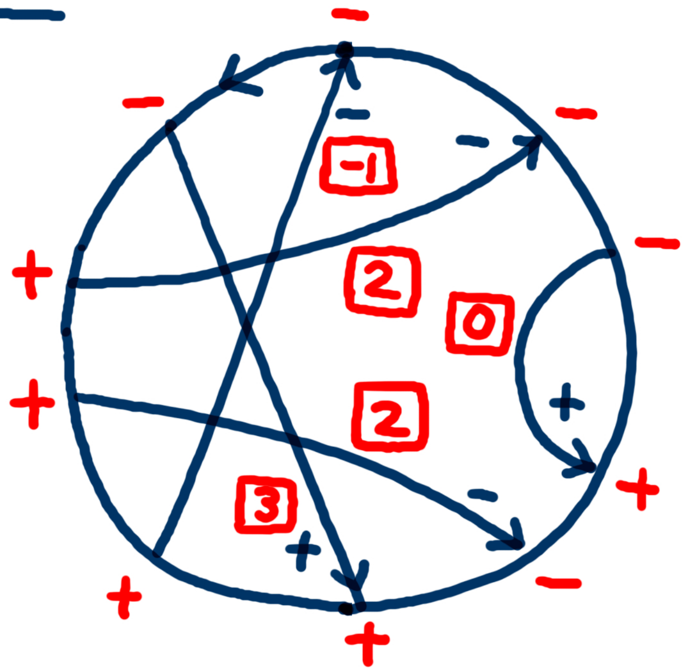
The index of a chord





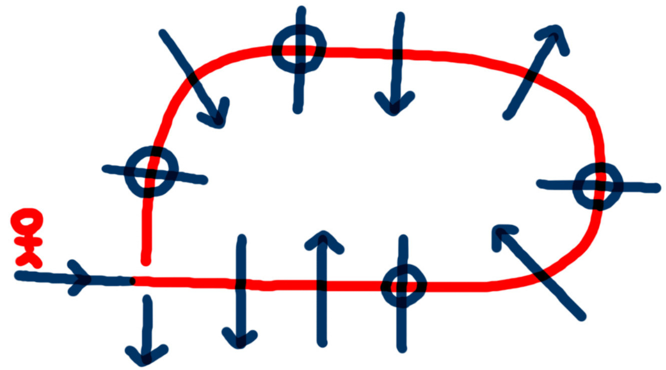
The **index** of γ is the sum of signs on ——, denoted by $\mathrm{Ind}(\gamma)$

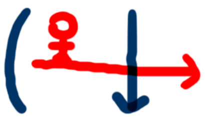
Ex.



Rem.

(i)



$\Rightarrow \mathrm{Ind}(\gamma) = \#\big($  $\big) - \#\big($  $\big)$

(ii) D : classical $\Rightarrow \forall \gamma,\ \mathrm{Ind}(\gamma) = 0$

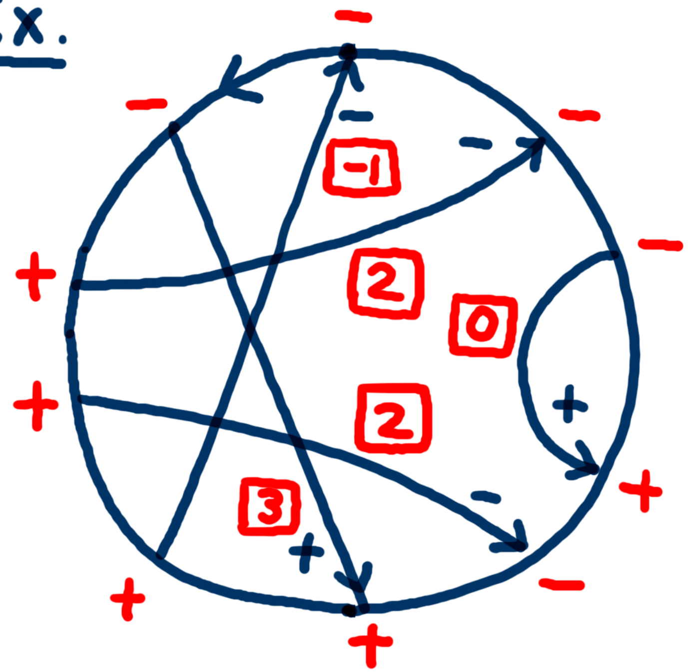
The n-writhe and the writhe polynomial

$$J_n(G) = \sum_{\mathrm{Ind}(r)=n} \varepsilon(r) \rightsquigarrow J_n(K) : \text{the } n\text{-writhe of } K \ (n \neq 0)$$

$$W_G(t) = \sum_{n \neq 0} J_n(G)(t^n - 1) \rightsquigarrow W_K(t) : \text{the writhe poly. of } K$$

['14 Cheng, '13 Kauffman, '14 Taniguchi-S]

Ex.

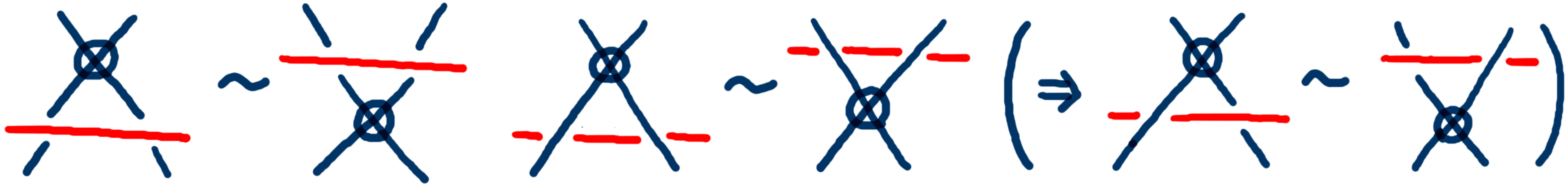


$$J_n(K) = \begin{cases} 1 & (n=3) \\ -2 & (n=2) \\ -1 & (n=-1) \\ 0 & (n \neq 3, 2, -1, 0) \end{cases}$$

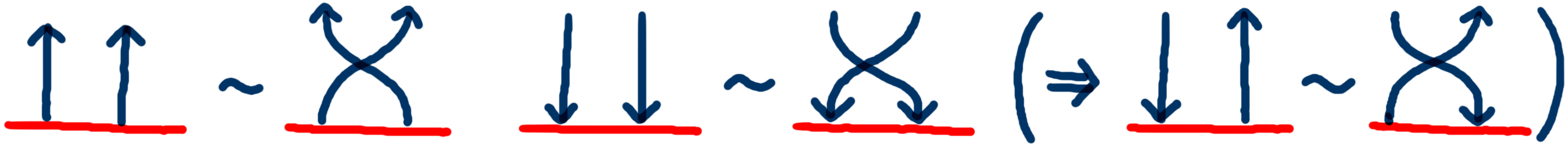
$$W_K(t) = 1\cdot(t^3-1) - 2(t^2-1) - 1\cdot(t^{-1}-1)$$
$$= t^3 - 2t^2 + 2 - t^{-1}$$

Forbidden moves

(virtual knot diagrams)

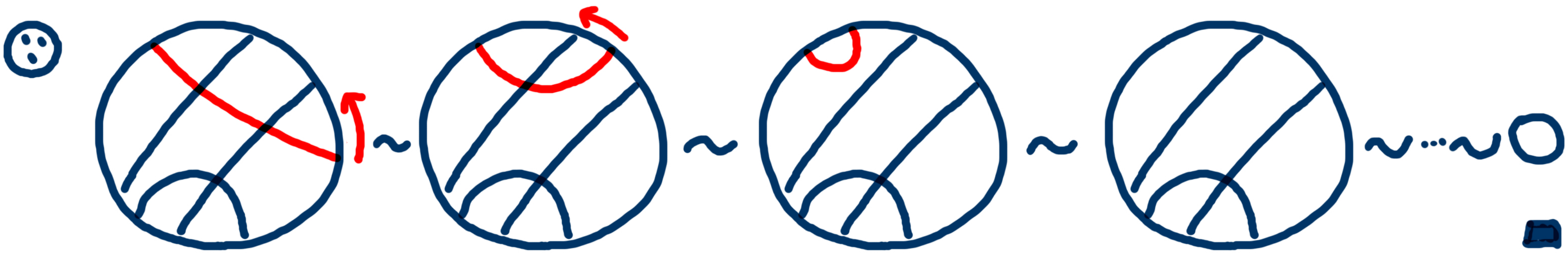


(Gauss diagrams)

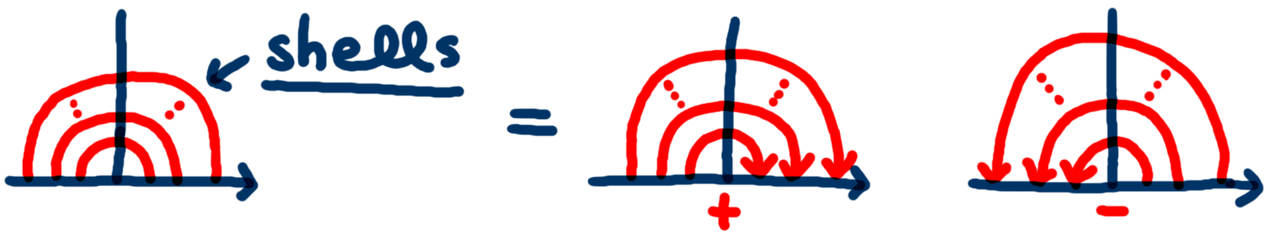


Thm. [’01 Kanenobu, Nelson]

The forbidden move is an unknotting operation for virtual knots.



Shell moves (for Gauss diagrams)

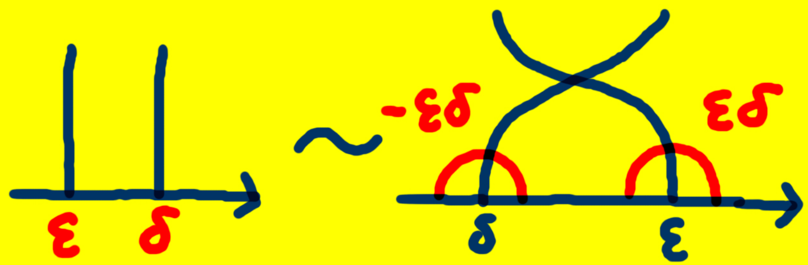


Rem. γ: a shell
$\Rightarrow \mathrm{Ind}(\gamma) = 1$

Shell moves

(S1)

(S2)



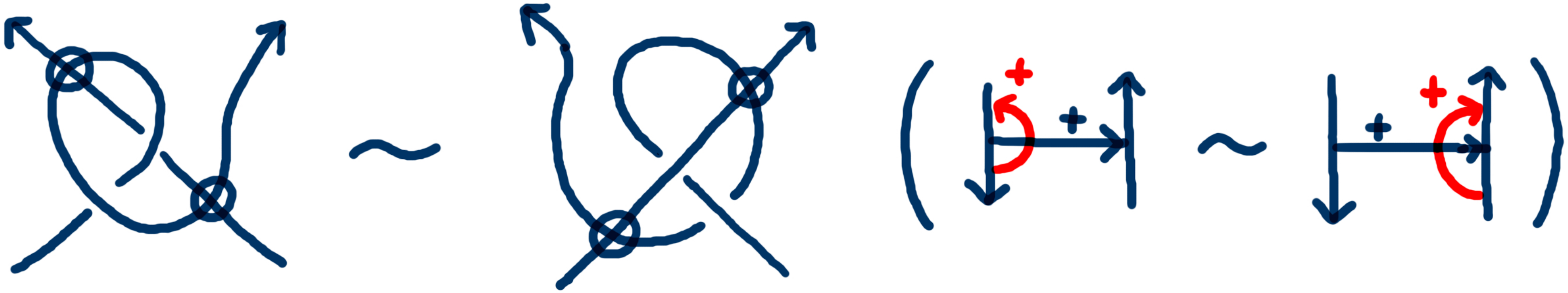
$$G \overset{\text{shell}}{\sim} G' \overset{\text{def}}{\iff} G = G_0, G_1, \cdots, G_n = G'$$

s.t. $G_i \to G_{i+1}$: R1, R2, R3, S1, S2

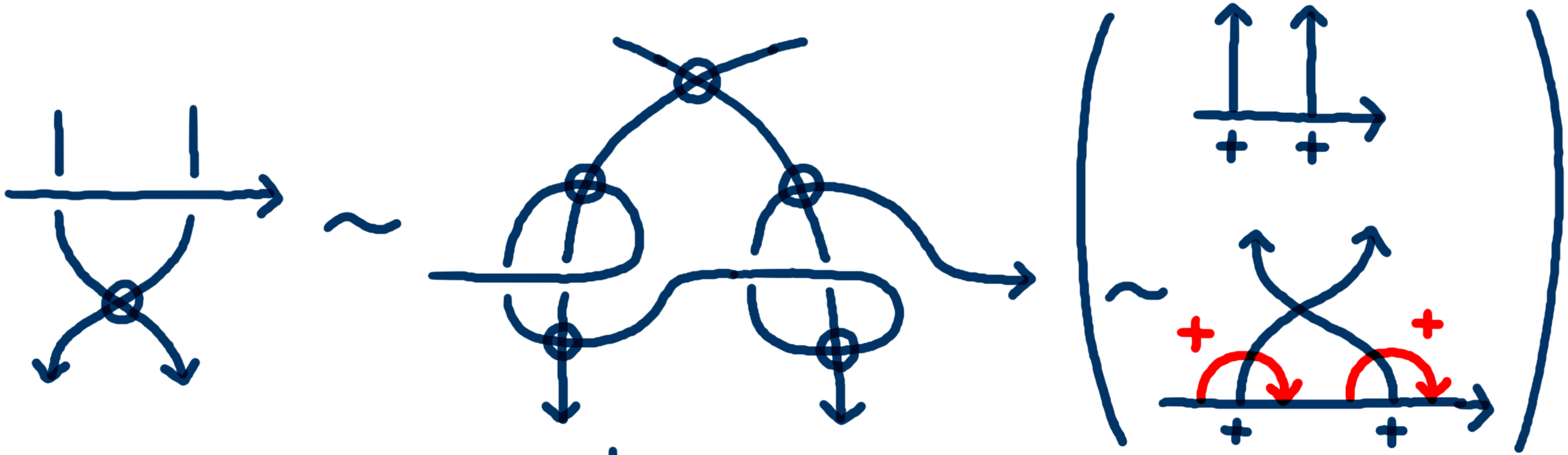
$$K \overset{\text{shell}}{\sim} K' \overset{\text{def}}{\iff} \exists\, G, G' \text{ for } K, K' \text{ s.t. } G \overset{\text{shell}}{\sim} G'$$

Shell moves (for virtual knot diagrams)

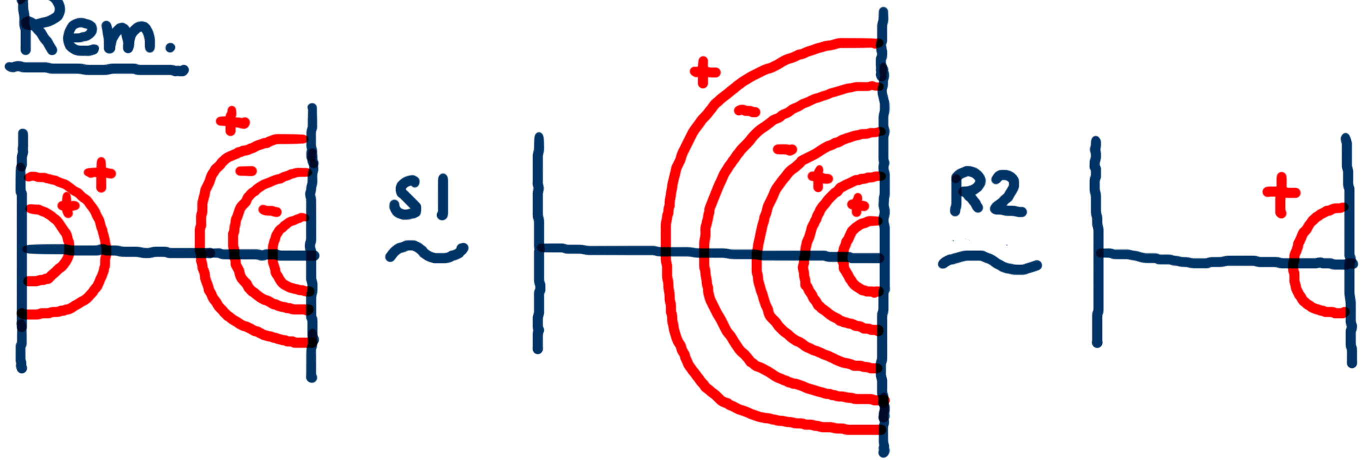
(S1)



(S2)



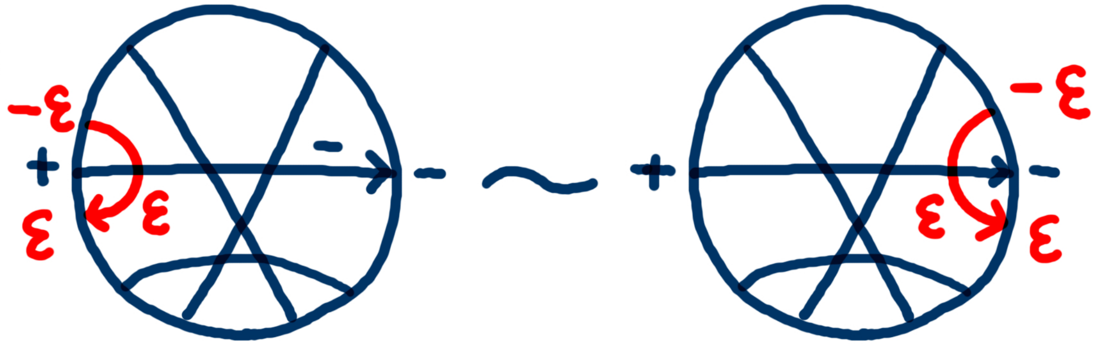
Rem.



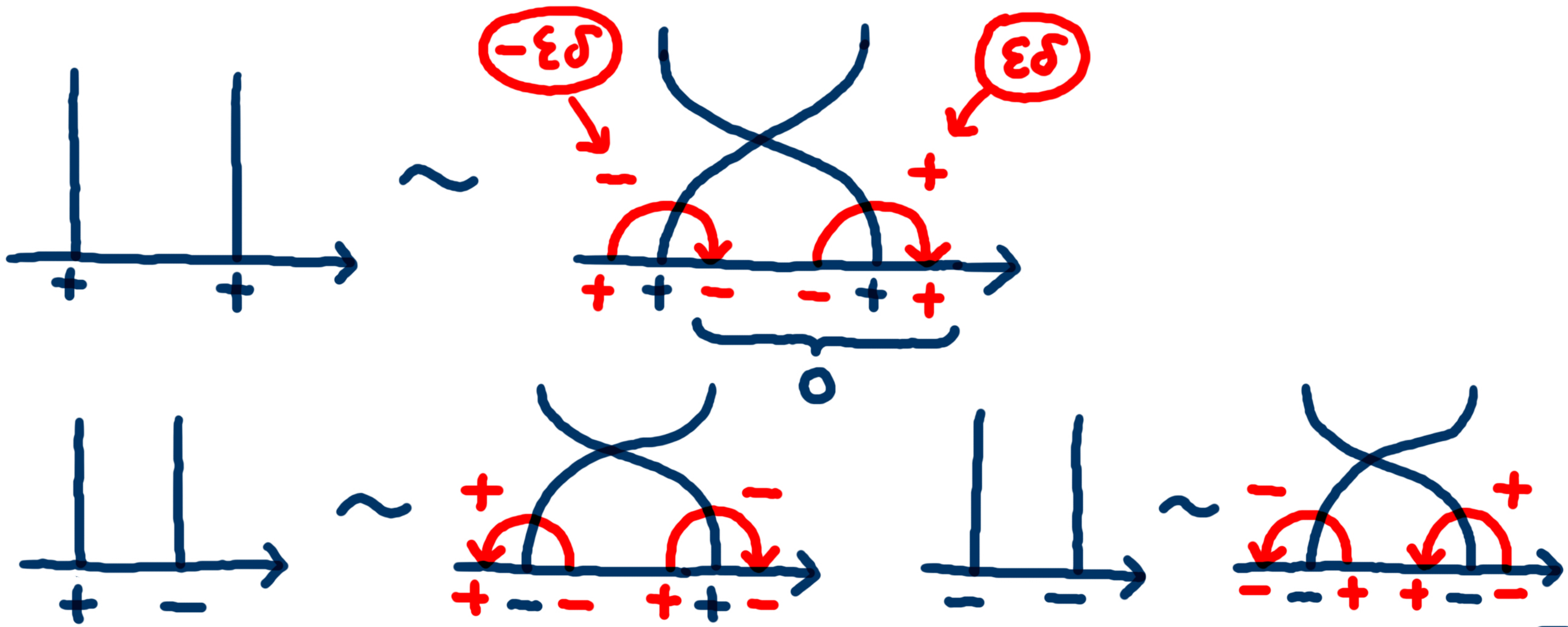
Invariance of the n-writhe under shell moves

Lem. $K \overset{\text{shell}}{\sim} K' \Rightarrow {}^{\forall} n \neq 0,\ J_n(K) = J_n(K')$

☺ (S1)

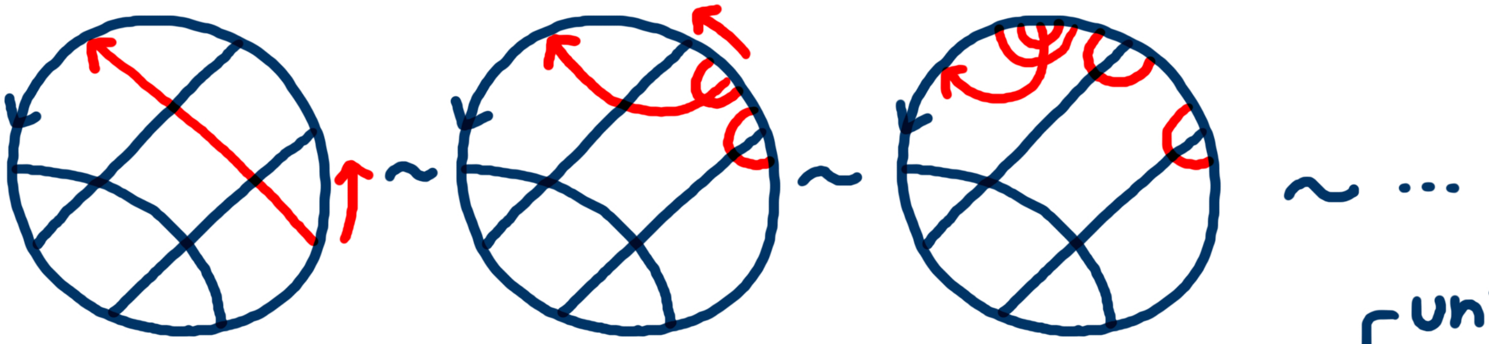


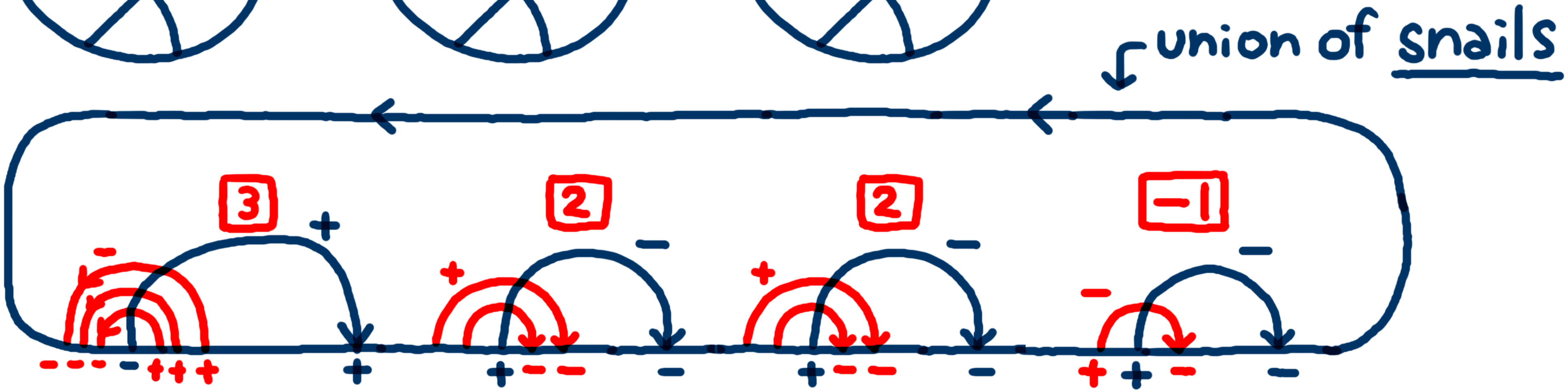
(S2)



∎

A representative under shell moves





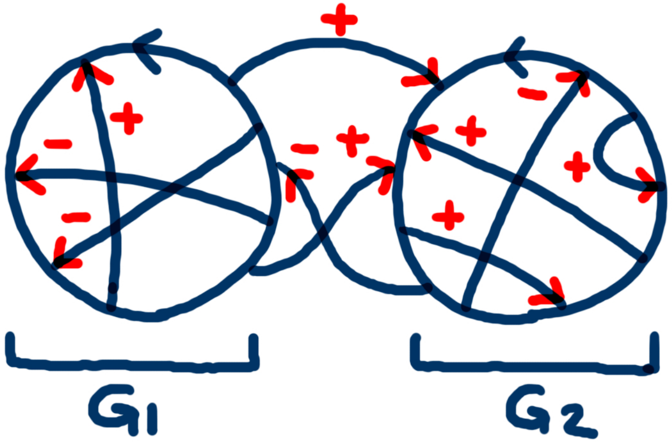
$$\Rightarrow W_K(t) = t^3 - 2t^2 + 2 - t^{-1}$$

Prop. $W_K(t) = \sum_{n \neq 0} a_n (t^n - 1) \Rightarrow K \sim$

Therefore, $W_K(t) = W_{K'}(t) \Rightarrow K \overset{\text{shell}}{\sim} K'$

The ori. 2-comp. virtual link case

G:



For $L = K_1 \cup K_2$, the following are invariants of L:

- $\boxed{Lk(K_i, K_j)} = \sum_{r : i \to j} \varepsilon(r)$

  $\rightsquigarrow \boxed{\lambda(L)} := Lk(K_1, K_2) - Lk(K_2, K_1)$

r: self-chord $\begin{cases} \mathrm{Ind}(r; G_i) \rightsquigarrow \\ \mathrm{Ind}(r; L) \rightsquigarrow \end{cases}$

- $J_n(K_1), J_n(K_2)$ $(n \neq 0)$
- $\boxed{J_n(K_1; L)}$ $(n \neq 0, -\lambda)$
- $\boxed{J_n(K_2; L)}$ $(n \neq 0, \lambda)$

Lem. The following are invariants under shell moves:

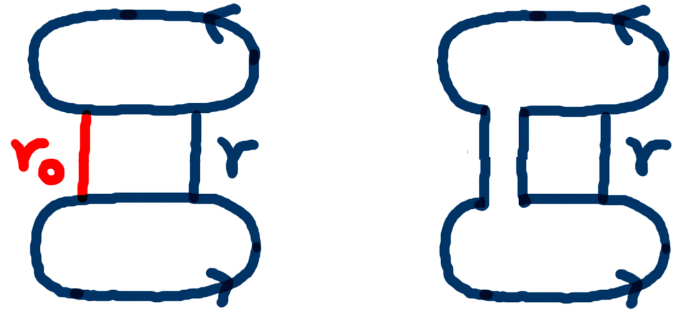
$Lk(K_1, K_2)$, $Lk(K_2, K_1)$, $\lambda = \lambda(L)$

$J_n(K_1; L)$ $(n \neq 0, -\lambda, 1, -\lambda+1)$, $J_n(K_2; L)$ $(n \neq 0, \lambda, 1, \lambda+1)$

$(\lambda \geq 2)$ $J_1(K_1, L) + J_{-\lambda+1}(K_1; L) + J_1(K_2; L) + J_{\lambda+1}(K_2; L)$

The relative index of a nonself-chord

$G \rightsquigarrow G(\gamma_0)$

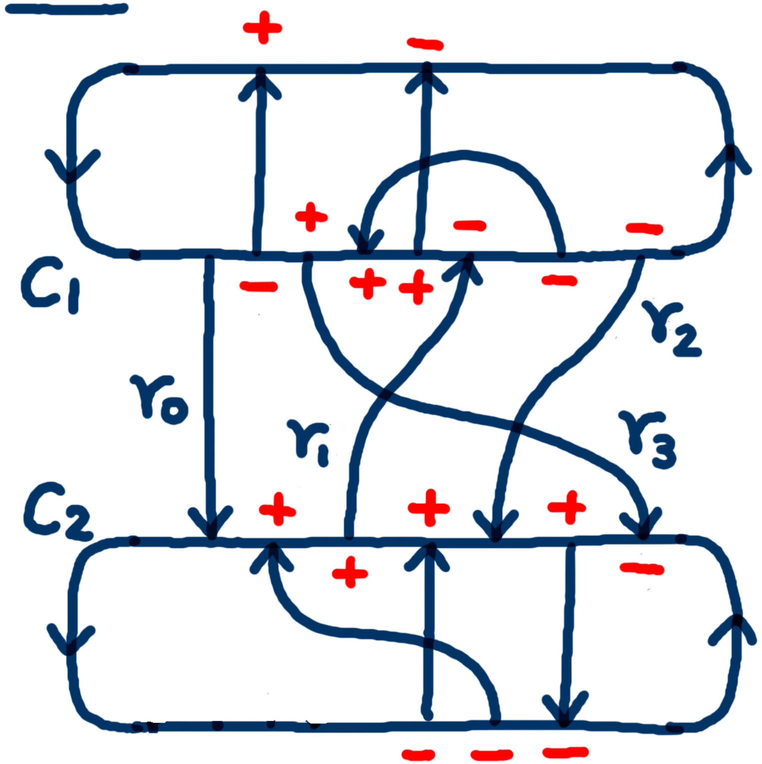


γ, γ_0 : nonself-chords

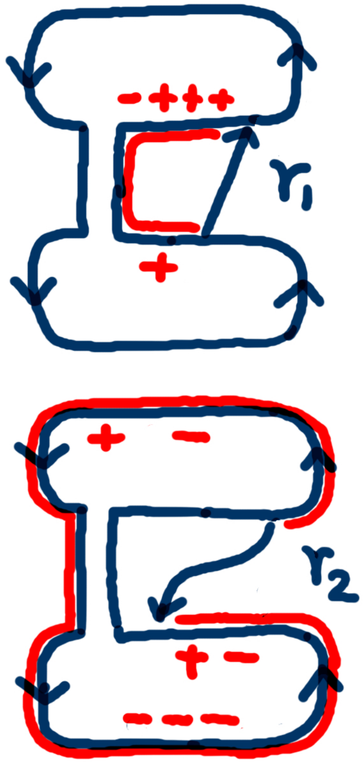
$\rightsquigarrow \boxed{\mathrm{Ind}(\gamma;\gamma_0) := \mathrm{Ind}(\gamma; G(\gamma_0))}$

$(\mathrm{Ind}(\gamma_0;\gamma_0) = 0)$

Ex.



$$\begin{cases} \mathrm{Ind}(\gamma_0;\gamma_0) = 0 \\ \mathrm{Ind}(\gamma_1;\gamma_0) = 3 \\ \mathrm{Ind}(\gamma_2;\gamma_0) = -3 \\ \mathrm{Ind}(\gamma_3;\gamma_0) = -4 \end{cases}$$



The linking class

$$J_n^{12}(G;r_0)=\sum_{\substack{r:1\to 2\\ \mathrm{Ind}(r;r_0)=n}}\varepsilon(r) \rightsquigarrow F_{12}(t;r_0)=\sum_n J_n^{12}(G;r_0)t^n$$

$$J_n^{21}(G;r_0)=\sum_{\substack{r:2\to 1\\ \mathrm{Ind}(r;r_0)=n}}\varepsilon(r) \rightsquigarrow F_{21}(t;r_0)=\sum_n J_n^{21}(G;r_0)t^n$$

Rem. $F_{12}(1;r_0)=Lk(K_1,K_2)$, $F_{21}(1;r_0)=Lk(K_2,K_1)$

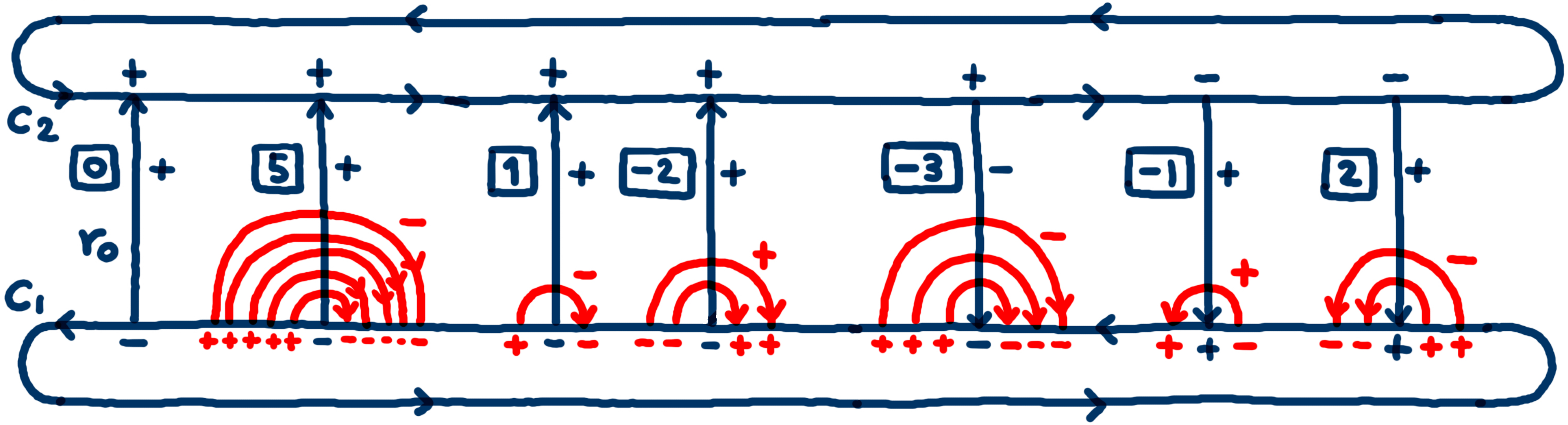
Thm. ['13 Cheng-Gao]

$$F(L)=[F_{12}(t),F_{21}(t)]\in\left(\mathbb{Z}[t,t^{-1}]/(t^\lambda-1)\right)^2/\doteq$$

is an invariant of L. Here,

$$(f_1(t),g_1(t))\doteq(f_2(t),g_2(t)) \iff \exists k\in\mathbb{Z} \text{ s.t. } \begin{cases} f_2(t)=t^k f_1(t)\\ g_2(t)=t^{-k}g_2(t)\end{cases}$$

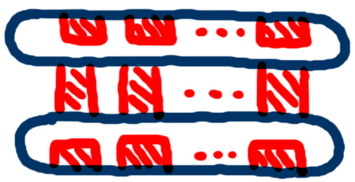
Ex.



$\lambda(L) = Lk(K_1, K_2) - Lk(K_2, K_1) = 4 - 1 = 3$

$$F(L) = [t^5 + t + 1 + t^{-2},\ t^2 + t^{-1} - t^{-3}] \in \left(\mathbb{Z}[t, t^{-1}]/(t^3 - 1)\right)^2 / \doteq$$
$$= [t^2 + 2t + 1,\ 2t^2 - 1] = [2t^2 + t + 1,\ -t^2 + 2t]$$

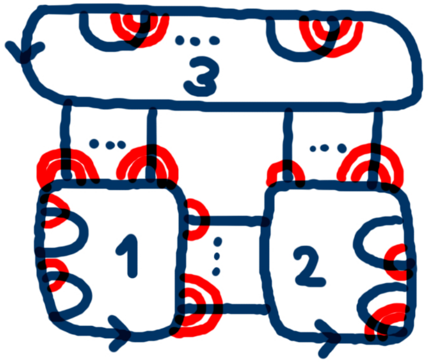
Proofs of Thm. 2-4

① Deform $^\forall G$ into  up to $\underset{\sim}{\text{shell}}$.

② Calculate the invariants for representatives. $\blacksquare$

Problems

① Study oriented 3-comp. virtual links up to shell moves.



Q. How can we define the index of a nonself-chord?

② Study the virtual knots up to Δ-moves.

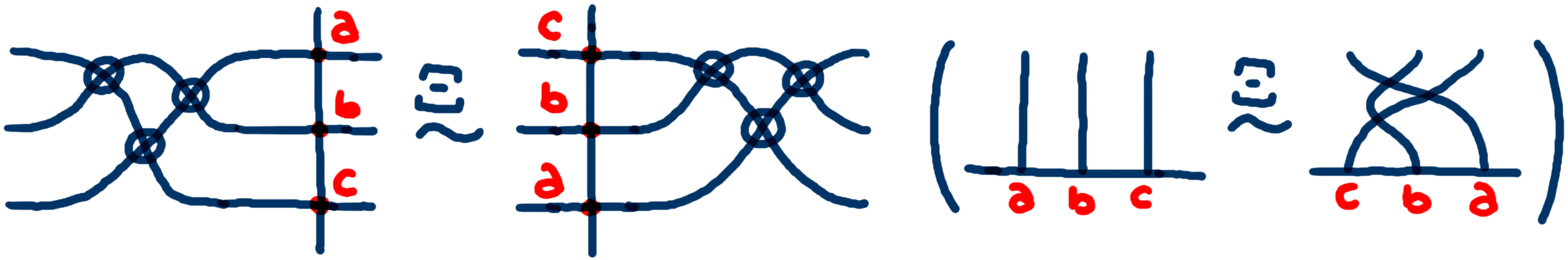
* A Δ-move is realized by crossing changes.

$$\left\{\begin{matrix}\text{virtual}\\\text{knots}\end{matrix}\right\}/\Delta \dashrightarrow \left\{\begin{matrix}\text{virtual}\\\text{knots}\end{matrix}\right\}/\text{c.c.} = \left\{\begin{matrix}\text{flat}\\\text{virtual}\\\text{knots}\end{matrix}\right\} \xrightarrow{\boxed{J_n(K)-J_{-n}(K)}} \mathbb{Z}$$

$$\left\{\begin{matrix}\text{virtual}\\\text{knots}\end{matrix}\right\}/\Delta \longrightarrow \mathbb{Z}, \quad \boxed{J(K)} = \sum_{n:\text{odd}} J_n(K)$$

Q. Which deformations are realized by Δ-moves?

③ Study oriented 2-comp. virtual links up to Ξ-moves.



Thm. [T-S] $J(K)=J(K') \Leftrightarrow K \overset{\Xi}{\sim} K'$

Q. Which invariants of 2-comp. virtual links correspond to the Ξ-move?

④ For each $n \neq 0$, which local moves correspond to $J_n(K)$?

⑤ Is there a "skein relation" for $W_K(t)$?