Hidden symmetries of hyperbolic links

Han Yoshida

National Institute of Technology, Nara College

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1. Introduction

Definition

A hyp. mfd. M admits a hidden symmetry
$\Leftrightarrow \exists$ an isometry between finite sheeted covers of M which is not a lift of an isometry of M.

We will define hidden symmetry by using "commensurator" and "normalizer" later.

Conjecture 1 (Neumann-Reid)

The figure-eight knot and the two dodecahedral knots are the only hyperbolic knots in S^3 admitting hidden symmetries.

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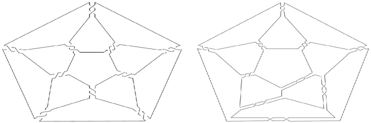

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One cusped-hyp. mfds. s781, v1241, v1859, v2037, v2274, v2573, v2731, v2875 admit hidden symmetries. (Notation as in the SnapPea census.)

Partial Answers

Th 1 (A. Reid, G. S. Walsh)

Two bridge knots ($\neq$ figure-eight) do not admit hidden symmetries.

Commensurability classes of 2-bridge knot complements. Algebr. Geom. Topol., 8(2):1031–1057, 2008

Th 2 (M. Macasieb, T. W. Mattman)

$(-2, 3, n)$ *pretzel knot* ($n \in \mathbb{N}$) *does not admit hidden symmety.*

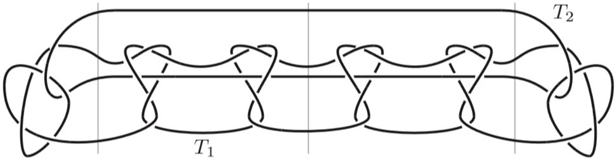
Commensurability classes of $(-2; 3; n)$ pretzel knot complements. Algebr. Geom. Topol., 8(3):1833–1853, 2008

N. Dunfield checked this conjecture is true for the knots up to 16 crossings.

2-comp. link and hidden symmetry

Th 3 (E. Chesebro, J. DeBlois)

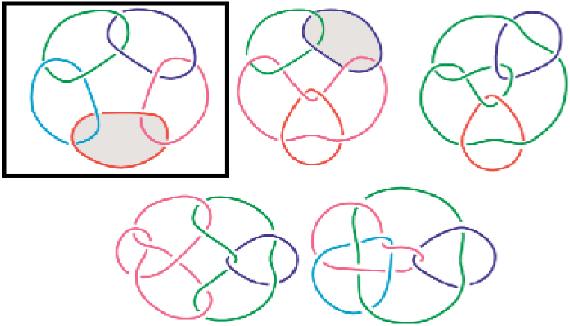
The 2-component links as in figure admit hidden symmetry.



"Hidden symmetries via hidden extensions"
https://arxiv.org/pdf/1501.00726.pdf

3-comp. and 4-comp. link and hidden symmetries

O. Goodman, D. Heard and C Hodgson showed the four links as in figure admit hidden symmetries by using computer.
Commensurators of cusped hyperbolic manifolds, Experiment. Math. Volume 17, Issue 3 (2008) 283-306.

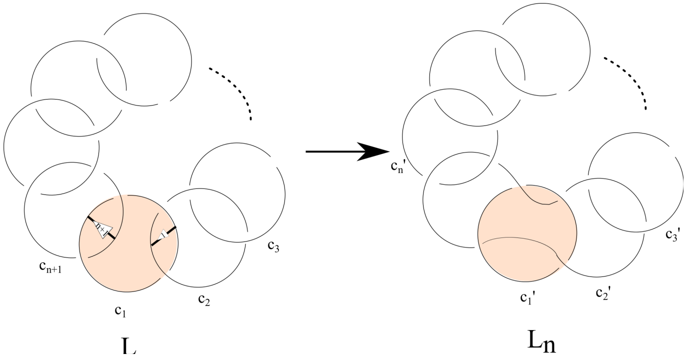


Main Theorem

We generalize the result of O. Goodman, D. Heard and C Hodgson.

Main Theorem

n-componet link as in Figure is non-arithmetic and admits a hidden symmetry. $(n \geq 4)$

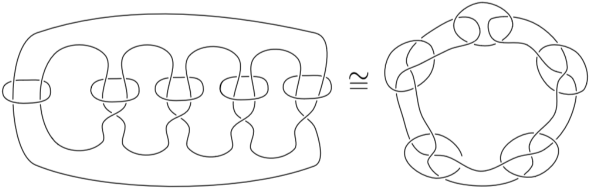


We prove this by using a tessellation of $\mathbb{H}^3$.

n-component links and hidden symmetry

Th 4 (J. S. Meyer, C. Millichap, R. Trapp)

n-componet link as in Figure is non-arithmetic and admits hidden symmetry. $(n \geq 6)$



https://arxiv.org/pdf/1811.00679
They showed this by using totally geodesic surfaces.

2.Definitions

Definition

$\Gamma_1,\ \Gamma_2 < \mathrm{Isom}(\mathbb{H}^3)$ *are commensurable*
$\Leftrightarrow |\Gamma_1 : \Gamma_1 \cap \Gamma_2| < \infty$ *and* $|\Gamma_2 : \Gamma_1 \cap \Gamma_2| < \infty$
Γ_1 *and* Γ_2 *are commensurable in the wide sense*
$\Leftrightarrow \exists h \in \mathrm{Isom}(\mathbb{H}^3)$ *such that* Γ_1 *is commensurable with* $h^{-1}\Gamma_2 h$.
$M_1 = \mathbb{H}^3/\Gamma_1$ *and* $M_2 = \mathbb{H}^3/\Gamma_2$ *are commensurable*
$\Leftrightarrow \Gamma_1$ *and* Γ_2 *are commensurable in the wide sense.*
$\Leftrightarrow M_1$ *and* M_2 *have a common finite sheeted cover.*

Rem: Commensurability is an equivalence relation.

Definition

For a Kleinian group Γ, the commensurator of Γ is defined by

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{Isom}(\mathbb{H}^3) : \mathrm{g}\Gamma\mathrm{g}^{-1} \text{ and } \Gamma \text{ are commensurable.}\}$$
$$= \{g \in \mathrm{Isom}(\mathbb{H}^3) : |\mathrm{g}\Gamma\mathrm{g}^{-1} : \mathrm{g}\Gamma\mathrm{g}^{-1} \cap \Gamma| < \infty,\ |\Gamma : \mathrm{g}\Gamma\mathrm{g}^{-1} \cap \Gamma| < \infty \}.$$

Rem:

- $\Gamma < \mathrm{Comm}(\Gamma)$
- Γ_1 and Γ_2 are commensurable. $\Rightarrow \mathrm{Comm}(\Gamma_1) = \mathrm{Comm}(\Gamma_2)$.

Th 5 (Margulis)

$\mathrm{Comm}(\Gamma)$ *is discrete* $\Leftrightarrow$ Γ *is "non-arithmetic".*

Rem: For a non-arithmetic group Γ,
$\mathrm{Comm}(\Gamma)$ contains every member of the commensurability class "in finite index".
i.e. $\mathbb{H}^3/\mathrm{Comm}(\Gamma)$ is the minimum orbifold(or manifold) in the commensurability class of $\mathbb{H}^3/\Gamma$.

Definition

A Kleinian group Γ is called arithmetic if it is commensurable with the group norm 1 elements of an order of quaternion algebra A ramified at all real places over a number field k with exactly one complex place.

Ex: The figure-eight knot complement and the Whitehead link complement are arithmetic.
Rem : arithmetic hyp. mfds and non-arithmetic hyp. mfds are incommenurable.
K : knot, $S^3 - K$ is arithmetic $\Rightarrow$ K is figure-eight knot.

Definition

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Definition

For a Kleinian group Γ, the normalizer of Γ is defined by

$$N(\Gamma) = \{g \in \mathrm{Isom}(\mathbb{H}^3) : \mathrm{g}\Gamma\mathrm{g}^{-1} = \Gamma\}.$$

Note:

$\cdot$ $\Gamma < N(\Gamma) < \mathrm{Comm}(\Gamma)$

$\cdot$ $N(\Gamma)/\Gamma \cong \mathrm{Isom}(\mathbb{H}^3/\Gamma)$

$\cdot$ $M = \mathbb{H}^3/\Gamma$: a cusped hyp. mfd with cusps $c_1, \cdots, c_n$

$V_i = \{x \in \partial\mathbb{H}^3 | x$ corresponds to the cusp c_i $\}$

$\gamma \in N(\mathbb{H}^3/\Gamma)$

$\Rightarrow \gamma(V_i) = V_{\sigma(i)}$ $(i = 1, \cdots, n)$ for some $\sigma \in S_n$.

Definition

$N(\Gamma) \neq \mathrm{Comm}(\Gamma) \Rightarrow$ *We say "Γ admits a hidden symmetry".*

Rem:

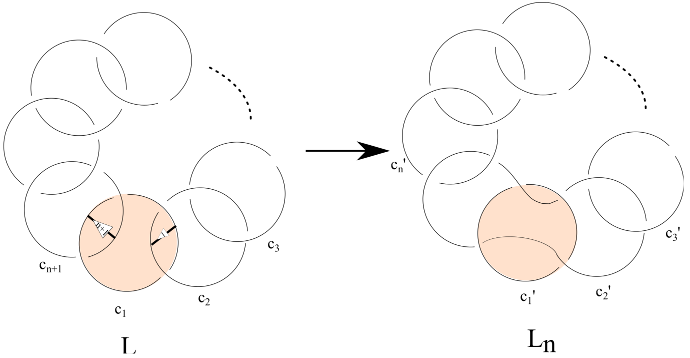
Γ is arithmetic. $\Rightarrow$ Comm(Γ) is not discrete. $\Rightarrow$ Γ admits a hidden symmetry.

figure-eight knot complement admits a hidden symmetry.

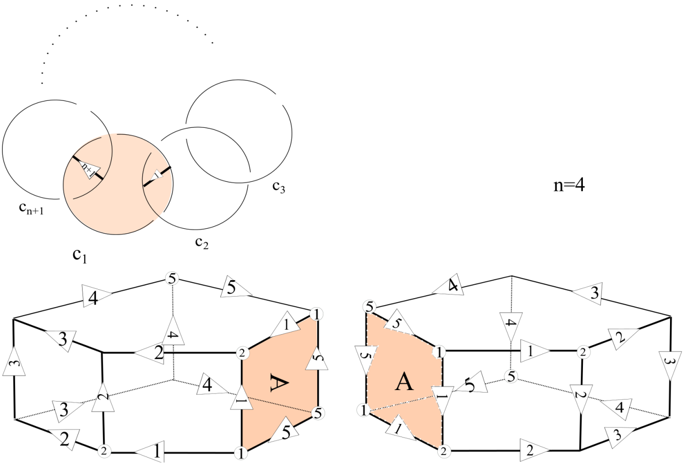
3. Proof of Main Theorem

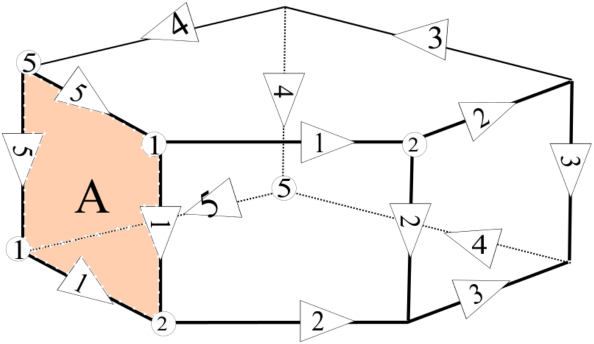
Main Theorem

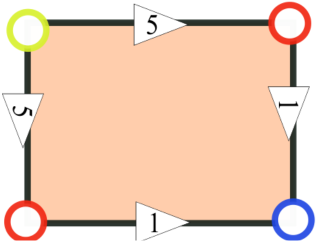
n-component link as in Figure is non-arithmetic and admits hidden symmetry. ($n \geq 4$)

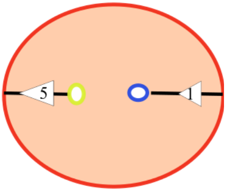


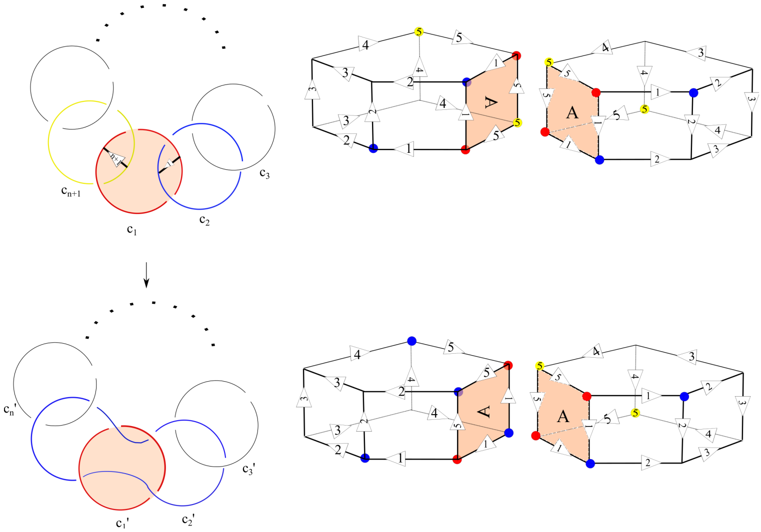
First, we will show that $S^3 - L_n$ is non-arithmetic.
W. Neumann and A. Reid showed that $S^3 - L$ is non-arithmetic.
W. Thurston showed $S^3 - L$ is obtained by glueing two ideal drums as in Figure.
The colored two punctured disc corresponds to the colored ideal quadrilaterals as in Figure.




A







c_{n+1}
c_1
c_2
c_3
A
c_n'
c_1'
c_2'
c_3'
A

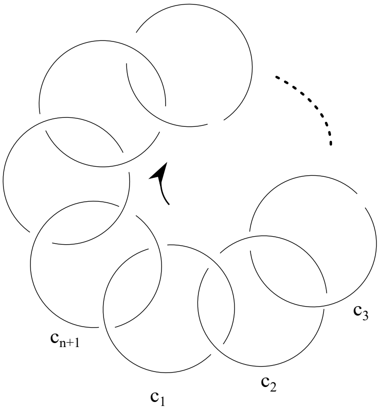
Γ (resp. Γ_n) : a Kleinian group such that $S^3 - L = \mathbb{H}^3/\Gamma$ (resp. $S^3 - L_n = \mathbb{H}^3/\Gamma_n$).
Lift the ideal polyhedral decompositions of $S^3 - L$ and $S^3 - L_n$.
We can get the same ideal polyhedral tessellation of $\mathbb{H}^3$.
Denote it by T. The symmetry group of T is discrete. Γ and Γ_n preserve the tessellation T.
Thus Γ and Γ_n are finite index subgroups of the symmetry group of T.
As commensurability is an equivalence relation, Γ_n is commensurable with non-arithmetic group Γ.
Hence, Γ_n is non-arithmetic and $\mathrm{Comm}(\Gamma) = \mathrm{Comm}(\Gamma_n)$.

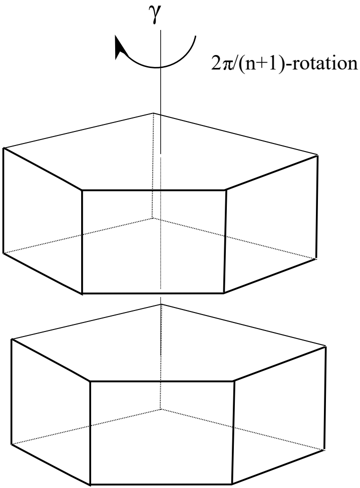
We will show that $S^3 - L_n$ admits a hidden symmetry.

P: an ideal drum in $\mathbb{H}^3$ which is a lift of this ideal drum.

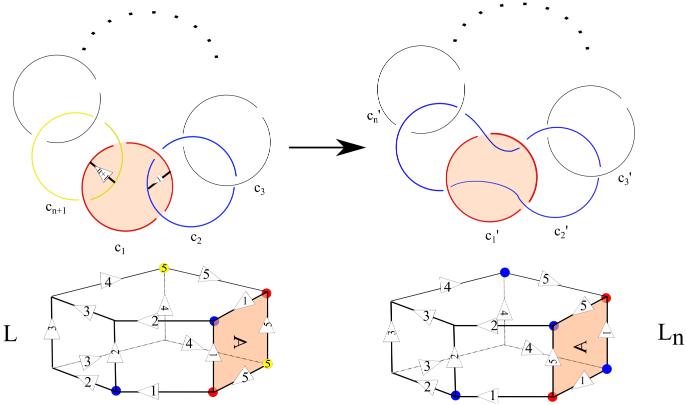
The symmetry that rotates the chain L clockwise, taking each link into the next.

This corresponds to $2\pi/(n+1)$ rotation as in Figure

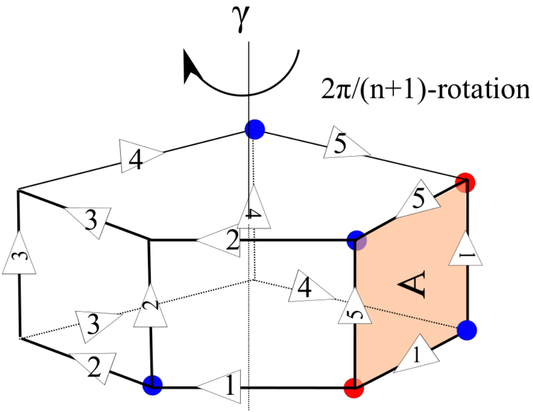




Denote it by γ. $\gamma \in N(\Gamma) < \mathrm{Comm}(\Gamma) = \mathrm{Comm}(\Gamma_\mathrm{n})$.
$c_1, \cdots, c_{n+1}$ (resp. $c'_1, \cdots, c'_n$) : cusps of $S^3 - L$ (resp. $S^3 - L_n$) as in Figure .
The cusp c_i corresponds to two ideal vertices of P.
By cuttng and re-glueing along the colored twice punctured disk, c_2 and c_{n+1} correspond to the cusp c'_2.

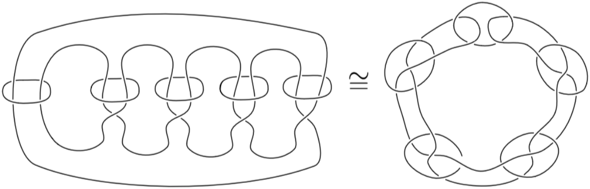


We can see that $\gamma(V_1) \neq V_i \ (i = 1, \cdots, n-1)$
$(V_i = \{x \in \partial\mathbb{H}^3 | x \text{ corresponds to the cusp } c_i\})$
$\gamma \notin N(\Gamma_n)$. We have $N(\Gamma_n) \neq \mathrm{Comm}(\Gamma_n)$. Hence Γ_n admits a hidden symmetry.



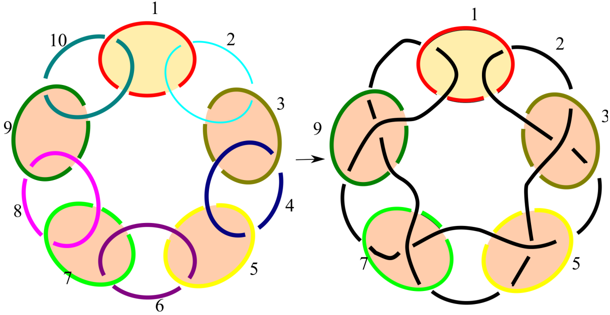
Th 6 (J. S. Meyer, C. Millichap, R. Trapp)

n-componet link as in Figure is non-arithmetic and admits hidden symmetry. ($n \geq 6$)

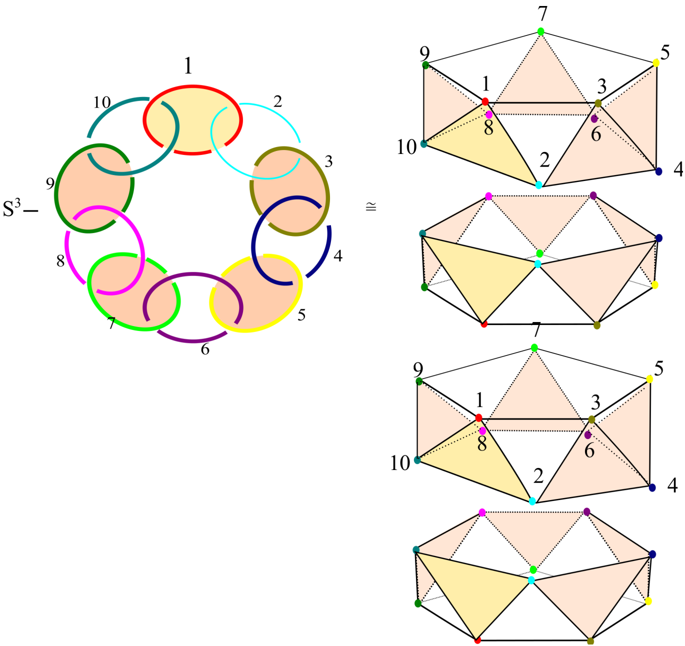


https://arxiv.org/pdf/1811.00679

We can prove this theorem in the same fashion of Main Theorem.



L → L_n


S^3 −
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≅

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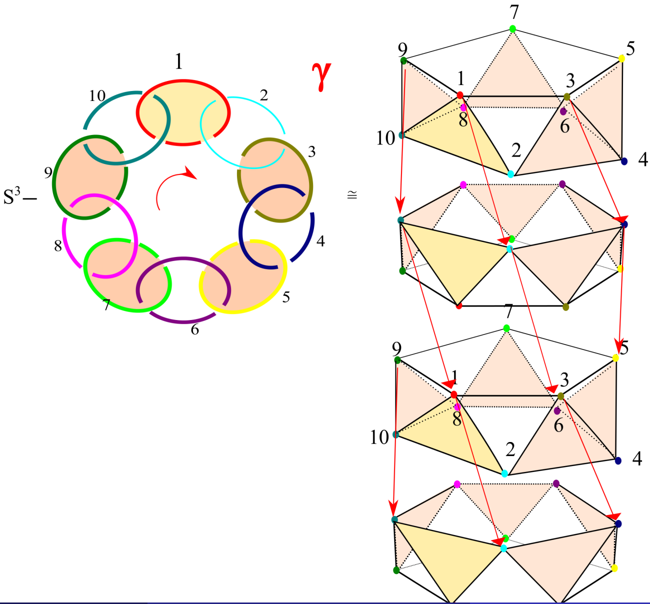

S³ −
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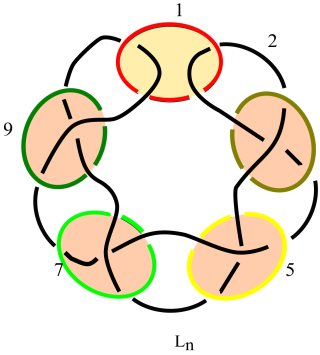
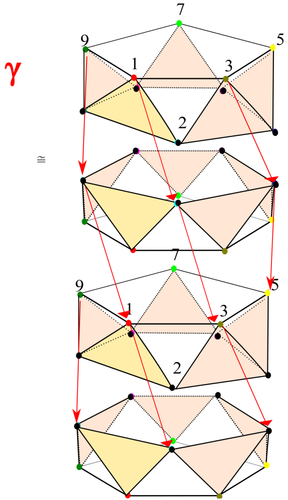




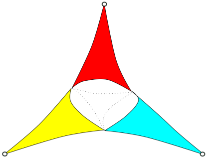
$S^3 - L$ is commensurable to $S^3 - L_n$, $\mathrm{Comm}(\Gamma) = \mathrm{Comm}(\Gamma_n)$,
$\gamma \in \mathrm{Comm}(\Gamma) = \mathrm{Comm}(\Gamma_n)$.
$\gamma \notin N(\Gamma_n)$

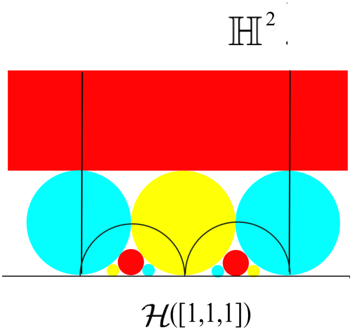
4. SnapPy and hidden symmetry

M : a cusped hyp. 3-mfd with cusps $c_1, \cdots, c_p$
$\pi : \mathbb{H}^3 \to M$ covering map
$\mathcal{W} = \{[w_1, \cdots, w_p] \in \mathbb{R}P^{p-1} | w_j > 0 (j = 1, \cdots, p)\}$.
c : a small positive number.

For $w = [w_1, \cdots, w_p]$, take the j-th cusp horoball nbhd. of C_j, so that $\mathrm{vol}(C_j) = cw_j$ $(j = 1, \cdots, p)$.
Let $\mathcal{H}(W) = \cup_{i=1,\cdots,p} \left(\pi^{-1}(C_j)\right)$.
The set of the ideal cell $\Delta \subset \mathbb{H}^3$ whose vertices are the centers of the nearest horoballs to some point $x \in \mathbb{H}^3$ in $\mathcal{H}(W)$ is called *the Epstein-Penner tilling* of M. We denote it by $\mathcal{T}(W)$.
(Dual of collision locus)

Example (2-dimensional hyperbolic manifold)





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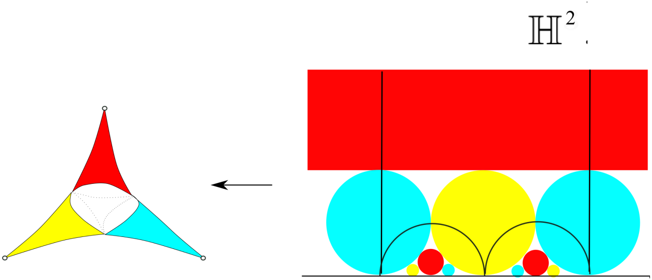
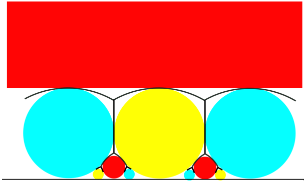


$\mathcal{H}([1,1,1])$



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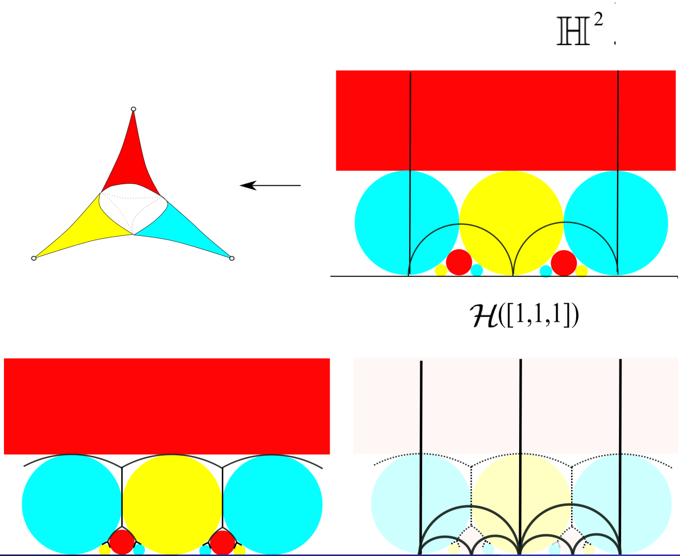
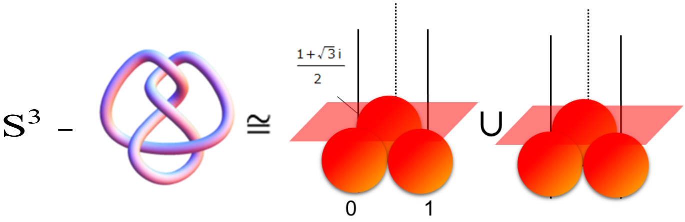
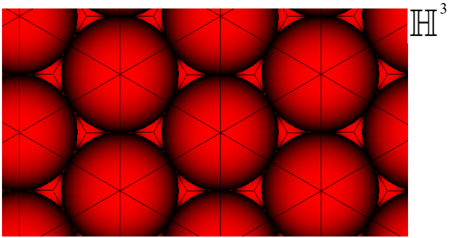


Example (3-dimensional hyperbolic manifold)



View from ∞



Proposition 1

Let $M = \mathbb{H}^3/\Gamma$ *be a cusped hyperbolic manifold.*

$$\mathrm{Comm}(\Gamma) > \mathrm{Symm}(\mathcal{T}(W))$$

where $\mathrm{Symm}(\mathcal{T}(W)) = \{\gamma \in \mathrm{Isom}(\mathbb{H}^3) | \gamma(\mathcal{T}(W)) = \mathcal{T}(W)\}$*.*

Proof. The group $\mathrm{Symm}(\mathcal{T}(W))$ is discrete and $\Gamma < \mathrm{Symm}(\mathcal{T}(W))$.
Thus $\mathrm{Symm}(\mathcal{T}(W))$ is commensurable with Γ.
As commensurator contains every member of the commensurability class,

$$\mathrm{Comm}(\Gamma) > \mathrm{Symm}(\mathcal{T}(W)).$$

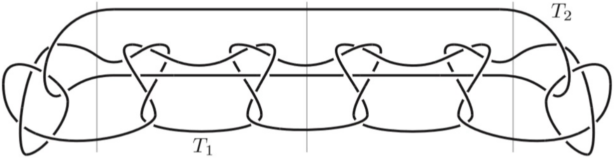
□

Rem:

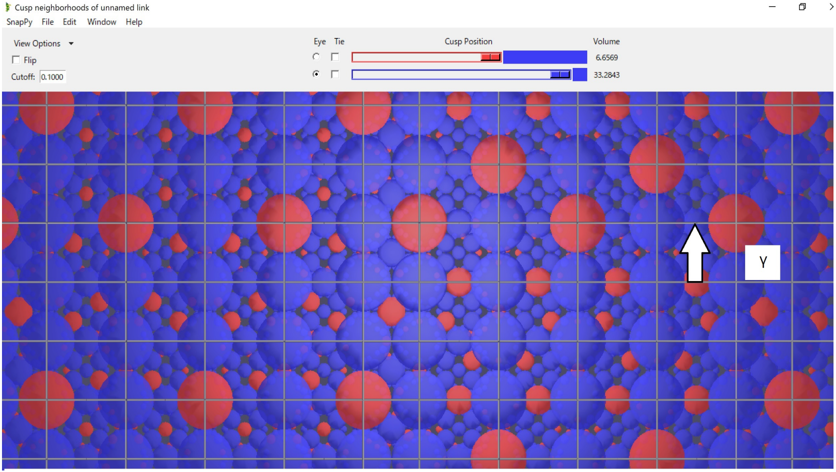
・For a non-arithmetic cusped hyperbolic manifold M, we can prove $\mathrm{Comm}(\Gamma) = \mathrm{Symm}(\mathcal{T}(W))$ for some $W \in \mathcal{W}$.

・This Proposition is proved by O. Goodman, D. Heard and C Hodgson. Commensurators of cusped hyperbolic manifolds, Experiment. Math. Volume 17, Issue 3 (2008) 283-306.

If we find $\gamma \in \mathrm{Symm}(\mathcal{T}(W))$ for some $W \in \mathcal{W}$ but $\gamma(V_1) \neq V_i$ $(i = 1, \cdots, p)$, $\gamma \in \mathrm{Comm}(\Gamma)$ and $\gamma \notin N(\Gamma)$.
Example)The links of Eric Chesebro, Jason DeBlois



(Perhaps) $\gamma \in \mathrm{Symm}(\mathcal{T}([1,5])) < \mathrm{Comm}(\Gamma)$.
$\gamma(V_1) \neq V_i$ $(i = 1, 2)$. Thus $\gamma \notin N(\Gamma)$



Thank you very much.
(ご清聴ありがとうございました.)