Classification of small ribbon 2-knots

Taizo KANENOBU

Osaka City University

13:15–13:45, 13 May 2020
Intelligence of Low-dimensional Topology
13–15 May 2020, RIMS, Kyoto University

Outline

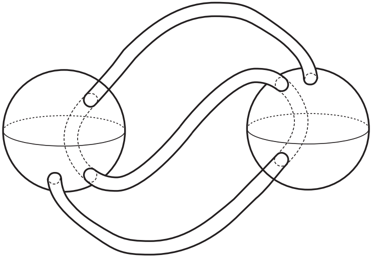
We consider classification of ribbon 2-knots with small ribbon crossing numbers.
More precisely, we classify the ribbon 2-knots in Yasuda's table of ribbon 2-knots with up to 4 ribbon crossing number.
We show the difference by:

- The Alexander polynomial.
- The fundamental group of the branched cyclic covering space of S^4.
- The trace set, which is obtained from the representations of the knot group to $\mathrm{SL}(2, \boldsymbol{C})$.
- The twisted Alexander polynomial.

Ribbon 2-knot of m-fusion

A *ribbon 2-knot* is an embedded 2-sphere in S^4 obtained by adding m 1-handles to a trivial 2-link with $(m+1)$ components for some m, which we call an m-fusion presentation of a ribbon 2-knot.

- A projection image of a ribbon 2-knot of 1-fusion. $(m = 1)$



Ribbon handlebody (ribbon 2-disk)

A ribbon handlebody is a 2-dimensional handlebody in $\mathbf{R}^3$ consisting of:

- $(m+1)$ 0-handles $\mathcal{D} = D_0 \cup D_1 \cup \cdots \cup D_m$, and
- m 1-handles $B = B_1 \cup \cdots \cup B_m$ for some m,

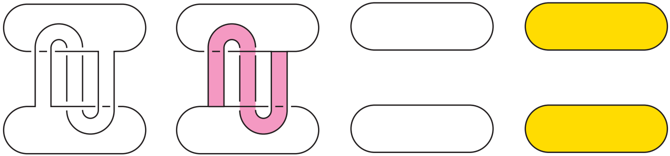
which has only ribbon singularities; the preimage of each ribbon singularity consists of an arc in the interior of a 0-handle and a cocore of a 1-handle.

Example

$m = 1.$ $\quad \mathcal{D} = D_0 \cup D_1 \quad \mathcal{B} = B_1 \quad \mathcal{D} \cup \mathcal{B}$

Ribbon handlebody $\longrightarrow$ Ribbon 2-knot

Given a ribbon handlebody $\mathcal{D} \cup \mathcal{B}$, we define the associated 2-knot in $\boldsymbol{R}^4 = \boldsymbol{R}^3 \times \boldsymbol{R}$ as the ribbon 2-knot that bounds the immersed 3-disk $\mathcal{D} \times [-2, 2] \cup \mathcal{B} \times [-1, 1]$:



$|t| < 1$ $\quad t = \pm 1$ $\quad 1 < |t| < 2$ $\quad t = \pm 2$

Conversely, for any ribbon 2-knot K, there exists a ribbon handlebody whose associated ribbon 2-knot is K.
Therefore, we may represent a ribbon 2-knot by a ribbon handlebody.

Ribbon handlebody presentation

For a ribbon handlebody $H = \mathcal{D} \cup \mathcal{B}$, where
$\mathcal{D} = D_0 \cup D_1 \cup D_2 \cup \cdots \cup D_m$: 0-handles,
$\mathcal{B} = B_1 \cup B_2 \cup \cdots \cup B_m$: 1-handles,

we define a ribbon handlebody presentation $[X|R]$ as follows:

- $X = \{x_0, x_1, \ldots, x_m\}$,
  where each letter x_q corresponds to the 0-handle D_q.
- $R = \{\rho_1, \rho_2, \ldots, \rho_m\}$,
  where each relation $\rho_q : x_{\iota_q}^{w_q} = x_{\tau_q}$ (or $x_{\tau_q} = x_{\iota_q}^{w_q}$) corresponds to the 1-handle B_q that joins D_{ι_q} to D_{τ_q} passing through 0-handles according to the word w_q:

$$w_q = x_{\lambda(q,1)}^{\epsilon(q,1)} x_{\lambda(q,2)}^{\epsilon(q,2)} \cdots x_{\lambda(q,\ell_q)}^{\epsilon(q,\ell_q)} \in F[x_0, x_1, \ldots, x_m],$$

$\epsilon(q,1), \epsilon(q,2), \ldots, \epsilon(q,\ell_q) = \pm 1.$

Knot group of a ribbon 2-knot

K: a ribbon 2-knot presented by a ribbon handlebody presentation $[X|R]$,

$$X = \{x_0, x_1, \ldots, x_m\}, \quad R = \{\rho_1, \ldots, \rho_m\}, \quad \rho_q : x_{\iota_q}^{w_q} = x_{\tau_q}.$$

Then

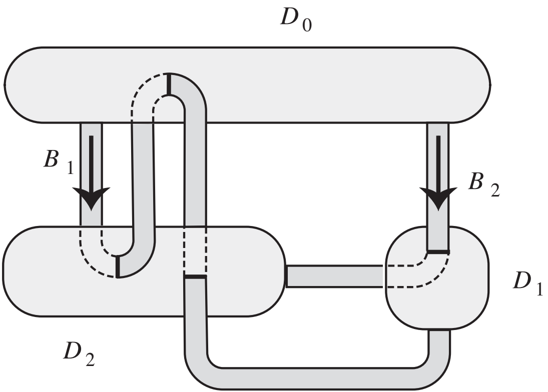
$$\pi_1(\boldsymbol{R}^4 - K) = \langle X|\tilde{R}\rangle,$$

$\tilde{R} = \{\tilde{\rho}_1, \ldots, \tilde{\rho}_m\}, \quad \tilde{\rho}_q : w_q^{-1} x_{\iota_q} w_q = x_{\tau_q}.$

A ribbon 2-knot K is represented by a ribbon handlebody with m ribbon singularities.
The *ribbon crossing number* of a ribbon 2-knot K is the least number of m possible for K.

Example of a ribbon handlebody presentation

Y21



$$\left[\, x_0,\ x_1,\ x_2 \;\middle|\; \rho_1 : x_0^{x_2 x_0 x_2} = x_1,\ \ \rho_2 : x_0^{x_1^{-1}} = x_2 \,\right]$$

$$\left\langle\, x_0,\ x_1,\ x_2 \;\middle|\; \tilde{\rho}_1 : (x_2 x_0 x_2)^{-1} x_0 (x_2 x_0 x_2) = x_1,\ \ \tilde{\rho}_2 : (x_1) x_0 (x_1^{-1}) = x_2 \,\right\rangle$$

Enumeration of ribbon 2-knots by T. Yasuda

- T. Yasuda, Crossing and base numbers of ribbon 2-knots, JKTR **10** (2001) 999–1003.
  Ribbon 2-knots with ribbon crossing number ≤ 3.
- T. Yasuda, Ribbon 2-knots of ribbon crossing number four, JKTR **27** (2018) 1850058 (20 pages).
  Ribbon 2-knots with ribbon crossing number 4.

Ribbon crossing number	0	2	3	4
Number of ribbon 2-knots	1	3	13	≤ 111
(Each chiral pair is counted as one knot)	1	2	7	≤ 60

- Alexander polynomial

Classification of ribbon 2-knots presented by virtual arcs with up to 4 classical crossings

- Any ribbon 2-knot is presented by an oriented virtual arc diagram due to Shin Satoh.
- If a ribbon 2-knot is presented by a virtual arc with n classical crossings, then its ribbon crossing number is $\leq n$.
- We have enumerated ribbon 2-knots presented by virtual arc diagrams with up to 4 classical crossings.

  T. Kanenobu and S. Komatsu, Enumeration of ribbon 2-knots presented by virtual arcs with up to four crossings, JKTR **26** (2017) 1750042.
- We have classified these ribbon 2-knots.

  T. Kanenobu and T. Sumi, Classification of ribbon 2-knots presented by virtual arcs with up to four crossings JKTR **28** (2019) 1950067 (18 pages).

Classification of ribbon 2-knots with up to 4 crossings

Numbers of the ribbon 2-knots presented by virtual arcs.

Number of classical crossings of a virtual arc	0	2	3	4
Number of ribbon 2-knots	1	3	9	91/92
(Each chiral pair is counted as one knot)	1	2	5	49/50

Today's results:

Ribbon crossing number	0	2	3	4
Number of ribbon 2-knots	1	3	13	111/112
(Each chiral pair is counted as one knot)	1	2	7	60/61

Enumeration: There is one more ribbon 2-knot with 4 ribbon crossings which is not included in Yasuda's table.

Classification: The 112 ribbon 2-knots with 4 ribbon crossings are mutually non-isotopic except for one pair.

Chirality: The chirality of these knots are confirmed.

Classification of ribbon 2-knots with up to 4 crossings

These ribbon 2-knots are classified into two types:

Type 1 Ribbon 2-knots of 1-fusion.

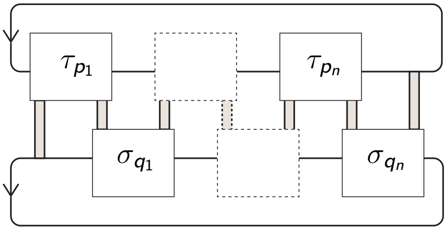
- The knot group is presented by 2 generators and 1 relation.
- π_1(2-fold branched covering space)$= \boldsymbol{Z}_d$, $d = \det K = |\Delta_K(-1)|$.

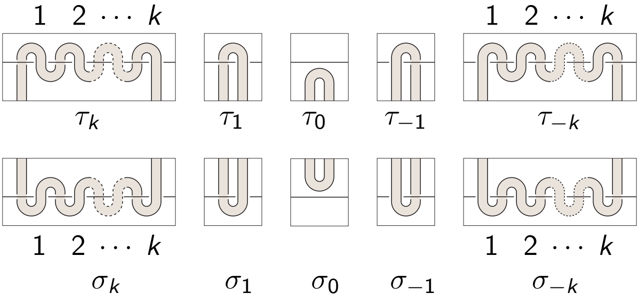
Type 2 Composition of two ribbon 2-knots of 1-fusion.

- π_1(2-fold branched covering space)$= \boldsymbol{Z}_3 * \boldsymbol{Z}_3$.

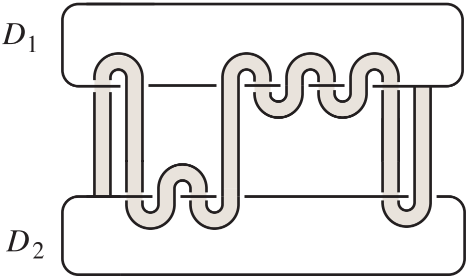
Ribbon crossing number	0	2	3	4
Number of ribbon 2-knots of Type 1	1	3	13	105/106
(Each chiral pair is counted as one knot)	1	2	7	56/57
Number of ribbon 2-knots of Type 2	0	0	0	6
(Each chiral pair is counted as one knot)	0	0	0	4

Ribbon 2-knot of 1-fusion $R(p_1, q_1, \ldots, p_n, q_n)$, $p_i, q_i \in \mathbb{Z}$.





Example: $R(1, 2, -3, 1)$



$$\left[\, x_1,\ x_2 \;\middle|\; \rho_1 : x_2^{x_1 x_2^2 x_1^{-3} x_2} = x_1 \,\right]$$

$$\left\langle\, x_1,\ x_2 \;\middle|\; \tilde{\rho}_1 : (x_1 x_2^2 x_1^{-3} x_2)^{-1} x_2 (x_1 x_2^2 x_1^{-3} x_2) = x_1 \,\right\rangle$$

Knot group and Alexander polynomial of $R(p_1, q_1, \ldots, p_n, q_n)$

- Knot group:

$$G(p_1,q_1,\ldots,p_n,q_n) = \pi_1(\mathbf{R}^4 - R(p_1,q_1,\ldots,p_n,q_n)) \\ = \langle\, x, y \mid y = wxw^{-1} \,\rangle, \qquad w = x^{p_1}y^{q_1}\cdots x^{p_n}y^{q_n}.$$

- Alexander polynomial:

$$\Delta(t) = 1 - t^{p_1} + t^{p_1+q_1} - t^{p_1+q_1+p_2} + t^{p_1+q_1+p_2+q_2} - \cdots \\ - t^{p_1+q_1+\cdots+q_{n-1}+p_n} + t^{p_1+q_1+\cdots+q_{n-1}+p_n+q_n}.$$

- We normalize the Alexander polynomial $\Delta(t) \in \mathbf{Z}[t^{\pm 1}]$ so that $\Delta(1) = 1$ and $(d/dt)\Delta(1) = 0$.
  We abbreviate $\Delta(t)$ as follows: for $c_i \in \mathbf{Z}$

$$(c_{-m}\ \ c_{-m+1}\ \ \ldots\ \ c_{-1}\ \ [c_0]\ \ c_1\ \ c_2\ \ \ldots\ \ c_n) = \sum_{i=-m}^{n} c_i t^i.$$

Example: Ribbon 2-knots with $\Delta(t) =$([0] 0 4 -4 1)

$$Y47 = \left[x_1, x_2, x_3 \,\middle|\, x_1^{x_2 x_1^{-1}} = x_2,\ x_1^{x_3 x_1^{-1}} = x_3 \right]$$

$$Y50 = \left[x_1, x_2, x_3 \,\middle|\, x_1^{x_2 x_1^{-1}} = x_2,\ x_1^{x_3 x_2^{-1}} = x_3 \right], \quad x_1 = x_3^{x_2 x_3^{-1}}$$

$$\xrightarrow{x_1 \to x_3 x_2^{-1} x_3 x_2 x_3^{-1}} \left[x_1, x_2, x_3 \,\middle|\, x_1^{x_2 x_3 x_2^{-1} x_3^{-1} x_2 x_3^{-1}} = x_2,\ x_1 = x_3^{x_2 x_3^{-1}} \right]$$

$$\xrightarrow{x_1 \to x_3^{x_2 x_3^{-1}}} \left[x_2, x_3 \,\middle|\, x_3^{x_2 x_3^{-1} x_2 x_3 x_2^{-1} x_3^{-1} x_2 x_3^{-1}} = x_2 \right]$$

Y47	$R(1,-1)\#R(1,-1)$	Type 2
Y50	$R(1,-1,1,1,-1,-1,1,-1)$	Type 1

We can distinguish by $\pi_1(\Sigma_2)$, Σ_2 is the 2-fold covering of S^4 branched over the knot.

Y47: $\pi_1(\Sigma_2) = \boldsymbol{Z}_3 * \boldsymbol{Z}_3, \quad H_1(\Sigma_2; \boldsymbol{Z}) = \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3$

Y50: $\pi_1(\Sigma_2) = H_1(\Sigma_2; \boldsymbol{Z}) = \boldsymbol{Z}_9$

Nonabelian representation to $\mathrm{SL}(2, \boldsymbol{C})$

Any nonabelian representation

$$G(p_1, q_1, \ldots, p_n, q_n) = \langle\, x,\, y \mid y = wxw^{-1} \,\rangle \to \mathrm{SL}(2, \boldsymbol{C}),$$

$w = x^{p_1} y^{q_1} \cdots x^{p_n} y^{q_n}$, is conjugate to a representation $\rho : G \to \mathrm{SL}(2, \boldsymbol{C})$ given by

$$\rho(x) = X = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}, \quad \rho(y) = Y = \begin{pmatrix} s & 0 \\ u & s^{-1} \end{pmatrix}, \qquad (1)$$

for some $s,\ u \in \boldsymbol{C}$ with $s \neq 0$ and $(s, u) \neq (\pm 1, 0)$; such a representation ρ is parametrized by $(s + s^{-1}, u)$.

R. Riley, *Nonabelian representations of 2-bridge knot groups*, Quart. J. Math. Oxford Ser. (2) **35** (1984) 191–208.

Lemma

A nonabelian representation ρ in Eq. (1) is reducible if and only if either $u = -(s - s^{-1})^2$ or $u = 0$.

For the group $G = G(p_1, q_1, \dots, p_n, q_n) = \langle\, x, y \mid y = wxw^{-1} \,\rangle$, $w = x^{p_1} y^{q_1} \cdots x^{p_n} y^{q_n}$, we define:

$$\mathrm{tr}\, G = \{\, s + s^{-1} \mid \rho : G \to \mathrm{SL}(2, \boldsymbol{C}) \text{ is an irreducible representation given by Eq. (1)} \},$$

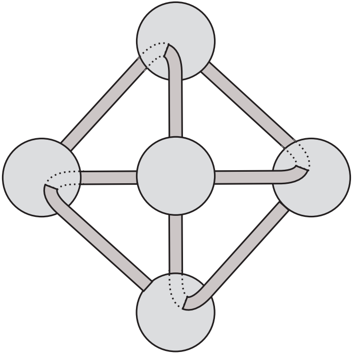
which we consider a multiset, i.e., we allow multiple instances for each of its elements. We call this the *trace set* for the group G.

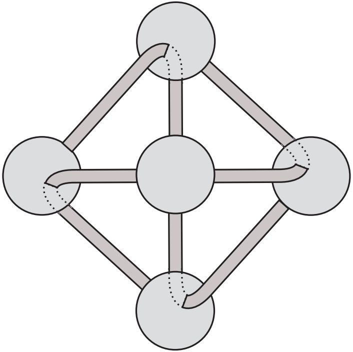
Proposition

The trace set is an ambient isotopy invariant for a ribbon 2-knot of 1-fusion.

Ribbon 2-knots with $\Delta(t) = (-1\ 4\ [-5]\ 4\ -1)$

Y109 and Y112 (the new knot)





They are both positive amphicheiral.

Y109	$R(1, 1, -1, -1, 1, -1, -1, 1, 1, 1, -1, 1, 1, -1)$
Y112	$R(1, -1, -1, 1, 1, 1, -1, 1, 1, -1, -1, -1, 1, 1)$

We can detect the difference of Y109 and Y112 by:

(1) (T. Sumi) The twisted Alexander polynomials associated to the representations to $\mathrm{SL}(2,2)$:

Y109: $\Delta(t) = 1 + t^6$; $\quad x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Y112: $\Delta(t) = 1 + t^2 + t^4 + t^6$; $\quad x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

(2) (T. Sumi) The numbers of the irreducible representations to $\mathrm{SL}(2,7)$.

(3) The trace sets of the irreducible representations to $\mathrm{SL}(2,\boldsymbol{C})$.

Trace sets of Y109 and Y112

Knot	Trace set
Y109	$\left\{ \begin{array}{l} \boldsymbol{C} - \{\pm\sqrt{3}\},\ \pm\sqrt{2},\ 0,0,0,0,0,0,0, \\ \boldsymbol{C} - \{\pm\sqrt{5}\},\ \boldsymbol{C} - \{\pm\sqrt{5}\},\ \pm 1, \\ (\delta + \epsilon\sqrt{13})/2,\ (\delta + \epsilon\sqrt{13})/2\ (\delta,\ \epsilon = \pm 1), \\ \pm\alpha_1,\ \pm\alpha_2,\ \pm\alpha_3,\ \pm\alpha_4 \end{array} \right\}$
Y112	$\left\{ \begin{array}{l} \boldsymbol{C} - \{\pm\sqrt{3}\},\ \pm\sqrt{2},\ 0,0,0,0,0,0,0, \\ \beta_1,\ \beta_2,\ \beta_3,\ \beta_4 \end{array} \right\}$

The complex numbers α_k, $k = 1, 2, 3, 4$, are the roots of the quartic equation $5 - 2x - 4x^2 + x^3 + x^4 = 0$;
$\alpha_1, \alpha_2 \fallingdotseq 1.25 \pm 0.27i$, $\alpha_3, \alpha_4 \fallingdotseq -1.75 \pm 0.17i$.

The complex numbers β_k, $k = 1, 2, 3, 4$, are the roots of the quartic equation $5 - 4x^2 + x^4 = 0$; $\beta_k \fallingdotseq \pm 1.46 \pm 0.34i$.

Example: Y69$\not\approx$ Y69!=Y80, $\Delta(t) = (-1\ 3\ [-3]\ 3\ -1)$

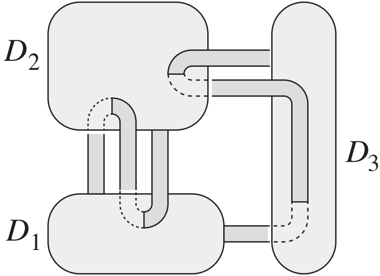
Y69$= R(1,-1,-1,-1,-1,1,1,1,-1,-1)$
The twisted Alexander polynomials of Y69 associated to the irreducible representations to $\mathrm{SL}(2,\boldsymbol{C})$ are not reciprocal, and so Y69 is not positive-amhicheiral, Y69$\not\approx$Y80.

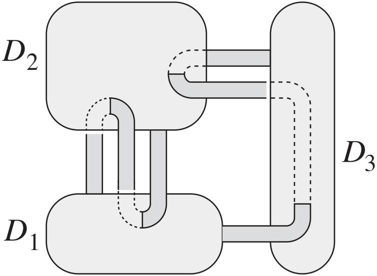
Knot	$(s+s^{-1},u)$	Twisted Alexander polynomial
Y69	$(0,\beta_k)$	$1+\alpha_k t^2+2t^4+t^6$

The numbers β_k, $k=1,\dots,5$, are the roots of the quintic equation $11-55x+77x^2-44x^3+11x^4-x^5=0$ with $0<\beta_2<1<\beta_1<2<\beta_5<3<\beta_3<7/2<\beta_4<4$.

The numbers α_k, $k=1,\dots,5$, are the roots of the quintic equation $1-30x-14x^2+29x^3-10x^4+x^5=0$ with $-1<\alpha_1<0<\alpha_2<1$, $2<\alpha_3<3<\alpha_4<4<\alpha_5<5$.

Open problem: Y43 vs. Y46, $\Delta(t) = (1\ \ -2\ [3]\ \ -2\ 1)$





They are both positive amphicheiral, and have isomorphic group:

$$\langle\, x_1, x_2, x_3 \mid x_1x_2x_1 = x_2x_1x_2,\ x_1(x_3x_2) = (x_3x_2)x_3 \,\rangle.$$

Problem 7.1 in: T. Kanenobu and T. Sumi, Classification of ribbon 2-knots presented by virtual arcs with up to four crossings JKTR **28** (2019) 1950067.

Thank you very much for your attention!