On quantum representation of knots via braided Hopf algebra

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Outline

1. Wirtinger presentation for closed braid
2. $\mathrm{SL}(2)$ representation space
3. Braided Hopf algebra
4. Representation space from a braided Hopf algebra
5. Braided $\mathrm{SL}(2)$

Problems

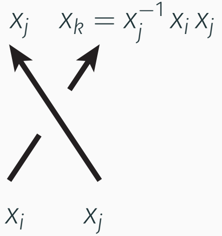
1. Wirtinger presentation for closed braid

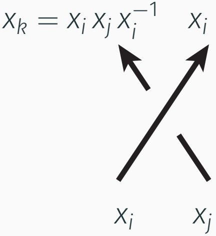
Wirtinger presentation

Let K be a knot in S^3 and D be its diagram. Then the fundamental group $\pi_1(S^3 \setminus K)$ of the complement of K has the following presentation.

$$\pi_1(S^3 \setminus K) = \langle x_1, x_2, \cdots, x_n \mid |r_1, r_2, \cdots, r_n \rangle$$

where n is the number of crossings of D, the generators $x_1, \cdots,$ x_n corresponds to the overpasses of D and r_i is the relation coming from the i-th crossing as follows.

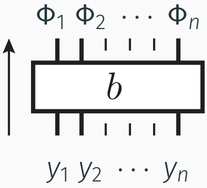




Presentation coming from a braid

Every knot can be expressed as a closed braid. For a knot K, let $b \in B_n$ be a braid whose closure is isotopic to K. Let $y_1, y_2, \cdots,$ y_n be elements of $\pi_1(S^3 \setminus K)$ corresponding to the overpasses at the bottom (and the top) of b. By applying the relations of the Wirtinger presentation at every crossings from bottom to top, we get $\Phi_1(y_1, \ldots, y_n), \cdots, \Phi_n(y_1, \cdots, y_n)$ at the top of b, and the Wirtinger presentation is equivalent to

$$\pi_1(S^3 \setminus K) = \\ \langle y_1, \cdots, y_n \mid y_1 = \Phi_1(y_1, \cdots, y_n), \cdots, y_n = \Phi_n(y_1, \cdots, y_n) \rangle .$$



2. $\mathrm{SL}(2)$ representation space

$\mathrm{SL}(2)$ representation of $\pi_1(S^3 \setminus K)$

An $\mathrm{SL}(2)$ representation ρ of $\pi_1(S^3 \setminus K)$ is determined by $\rho(y_1)$, $\cdots$, $\rho(y_n) \in \mathrm{SL}(2)$ satisfying

$$\Phi_1(\rho(y_1), \cdots, \rho(y_n)) = \rho(y_1),$$
$$\cdots,$$
$$\Phi_n(\rho(y_1), \cdots, \rho(y_n)) = \rho(y_n).$$

Let I_b be the ideal in the tensor $\mathbb{C}[\mathrm{SL}(2)]^{\otimes n}$ of the coordinate space of $\mathrm{SL}(2)$ generated by the above relations.

Theorem
The quotient $\mathbb{C}[\mathrm{SL}(2)]^{\otimes n}/I_b$ does not depend on the presentation of $\pi_1(S^3 \setminus K)$ and is called **the $\mathrm{SL}(2)$ representation space** *of $\pi_1(S^3 \setminus K)$.*

Hopf algebra interpretation

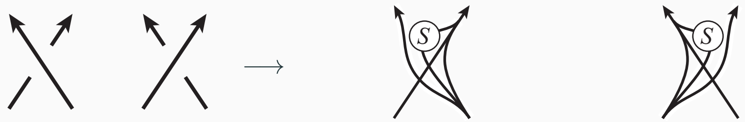
$\mathbb{C}[\mathrm{SL}(2)]$ is generated by a, b, c, d representing $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$\mathbb{C}[\mathrm{SL}(2)]$ has natural Hopf algebra structure coming from the group structure of SL(2).

$\Delta : \mathbb{C}[\mathrm{SL}(2)] \to \mathbb{C}[\mathrm{SL}(2)] \otimes \mathbb{C}[\mathrm{SL}(2)]$ with $\Delta(f)(x \otimes y) = f(xy)$,
$S : \mathbb{C}[\mathrm{SL}(2)] \to \mathbb{C}[\mathrm{SL}(2)]$ with $S(f)(x) = f(x^{-1})$,
$\varepsilon : \mathbb{C}[\mathrm{SL}(2)] \to \mathbb{C}$ with $\varepsilon(f) = f(1)$.

Let $\Phi^* : \mathbb{C}[\mathrm{SL}(2)]^{\otimes n} \to \mathbb{C}[\mathrm{SL}(2)]^{\otimes n}$ be the dual map of $\Phi = (\Phi_1, \cdots, \Phi_n)$. At a crossing, Φ^* acts as follows.

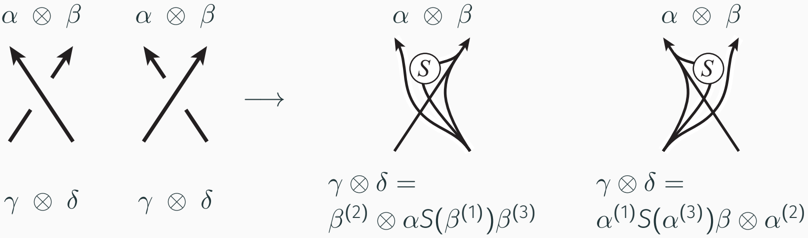
$\alpha \otimes \beta$ $\quad$ $\alpha \otimes \beta$ $\quad$ $\alpha \otimes \beta$ $\quad$ $\alpha \otimes \beta$



$\gamma \otimes \delta$ $\quad$ $\gamma \otimes \delta$ $\quad$ $\gamma \otimes \delta = \beta^{(2)} \otimes \alpha S(\beta^{(1)})\beta^{(3)}$ $\quad$ $\gamma \otimes \delta = \alpha^{(1)} S(\alpha^{(3)})\beta \otimes \alpha^{(2)}$

Hopf algebra interpretation 2

Let $\Phi^* : \mathbb{C}[\mathrm{SL}(2)]^{\otimes n} \to \mathbb{C}[\mathrm{SL}(2)]^{\otimes n}$ be the dual map of $\Phi = (\Phi_1, \cdots, \Phi_n)$. At a crossing, Φ^* acts as follows.



Theorem
Let J_b be the ideal generated by the image of $\Phi^* - id$, then J_b is equal to the previous ideal I_b and $\mathbb{C}[\mathrm{SL}(2)]^{\otimes n}/J_b$ is the $\mathrm{SL}(2)$ representation space of $\pi_1(S^3 \setminus K)$.

Remark. This construction can be generalized to any commutative Hopf algebra.

3. Braided Hopf algebra

Definition

An algebra A is called **a braided Hopf algebra** if it is equipped with following linear maps satisfying the relations given in the next picture.

multiplication

$$\mu : A \otimes A \to A,$$

comultiplication

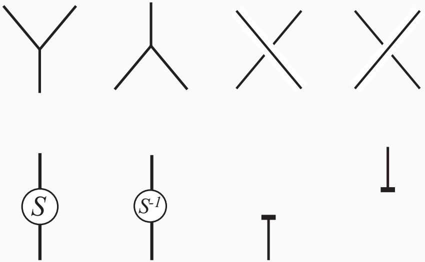
$$\Delta : A \to A \otimes A,$$

unit $1 : k \to A$,

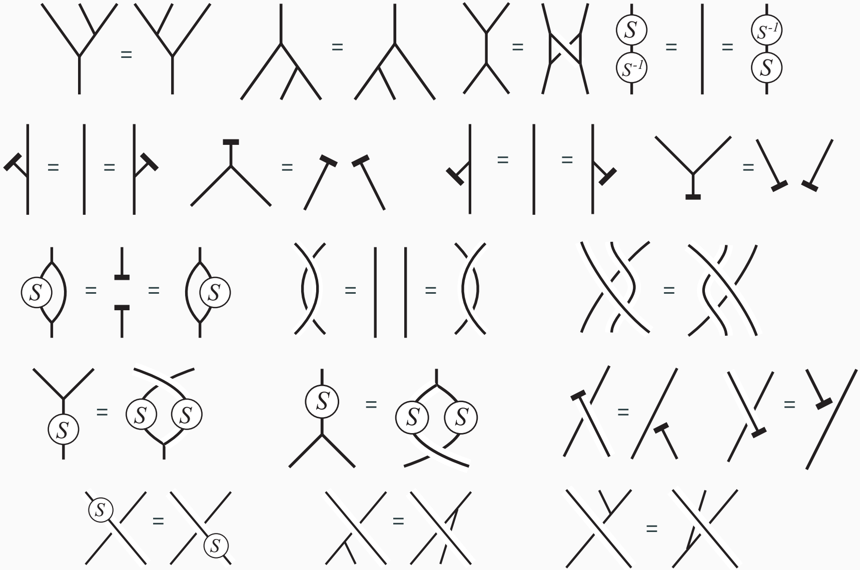
counit $\varepsilon : A \to k$,

antipode $S : A \to A$,

braiding $\Psi : A \otimes A \to A \otimes A$.



Relations of a braided Hopf algebra

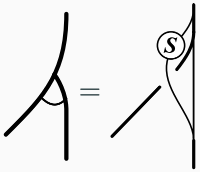


Adjoint coaction

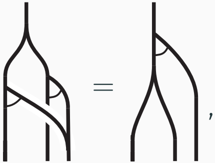
Definition
The adjoint coaction $\mathrm{ad} : A \to A \otimes A$ is defined by

$$\mathrm{ad}(x) = (id \otimes \mu)(\Psi \otimes id)(S \otimes \Delta)\Delta(x).$$

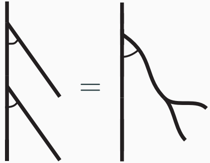
The adjoint coaction ad satisfies the following.



definition



$\mathrm{ad} \otimes \mathrm{ad}$ and Δ



two ad to $\mu \otimes \mathrm{ad}$

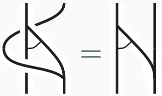
Braided commutativity is a weakened version of the commutativity.

Definition

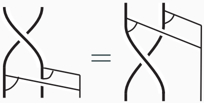
A braided Hopf algebra A is **braided commutative** if it satisfies

$$(id \otimes \mu)(\Psi \otimes id)(id \otimes \mathrm{ad})\Psi = (id \otimes \mu)(\mathrm{ad} \otimes id).$$

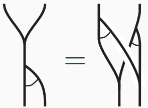
If A is braided commutative, the following relations hold.



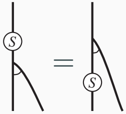
Definition



$\mathrm{ad} \otimes \mathrm{ad}$ and Ψ



ad and μ



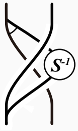
ad and S

4. Representation space from a braided Hopf algebra

Braid group representation through a braided Hopf algebra

Let A be a braided Hopf algebra which may not be braided commutative. Here we construct a representation of the braid group B_n on $A^{\otimes n}$ associated with the Wirtinger presentation. Let R and R^{-1} be elements of $\mathrm{End}(A^{\otimes 2})$ given by the following.

$R =$ 

$R^{-1} =$ 

For $\sigma_i^{\pm 1} \in B_n$, let $\rho(\sigma_i) = id^{\otimes(i1)} \otimes R \otimes id^{\otimes(n-i-1)}$ and $\rho(\sigma_i^{-1}) = id^{\otimes(i-1)} \otimes R^{-1} \otimes id^{\otimes(n-i-1)}$.

Theorem
The above ρ defined for generators of B_n extends to a representation of B_n in $\mathrm{End}(A^{\otimes n})$.

S. Woronowicz, Solutions of the braid equation related to a Hopf algebra. Lett. Math. Phys. **23** (1991), 143–145. (for usual Hopf algebra)

A representation space

From now on, we assume that the braided Hopf algebra A is **braided commutative**. For $b \in B_n$, let $\rho(b) \in \mathrm{End}(A^{\otimes n})$ be the representation of b defined as above. Let I_b be the left ideal of $A^{\otimes n}$ generated by the image of the map $\rho(b) - id^{\otimes n}$.

Proposition
The left ideal I_b is a two-sided ideal.

This proposition comes from the following lemma.

Lemma
For $x, y \in A$, we have

$$\rho(b)\mu(x \otimes y) = \mu(\rho(b)x \otimes \rho(b)y).$$

This is proved by using the braided commutativity.

Generators of I_b

Lemma
For $\mathbf{x}, \mathbf{y} \in A$*, we have*

$$\rho(b)\mu(\mathbf{x} \otimes \mathbf{y}) = \mu(\rho(b)\mathbf{x} \otimes \rho(b)\mathbf{y}).$$

Theorem
Let X *be a set of generators of* A *and* $\mathbf{x}_i = 1^{\otimes(i-1)} \otimes x \otimes 1^{\otimes(n-i)}$ *for* $x \in X$*. Then the ideal* I_b *in* $A^{\otimes n}$ *is generated by*

$$\{\rho(b)\mathbf{x}_i - \mathbf{x}_i \mid \mathbf{x} \in X,\ i = 1, \cdots, n-1\}.$$

PROOF. Since

$$d(b)\,\mu_n(\mathbf{x} \otimes \mathbf{y}) - \mu_n(\mathbf{x} \otimes \mathbf{y})$$
$$= \mu_n\Big(d(b)\,\mathbf{x} \otimes \big(d(b)\,\mathbf{y} - \mathbf{y}\big)\Big) + \mu_n\Big(\big(d(b)\,\mathbf{x} - \mathbf{x}\big) \otimes \mathbf{y}\Big).$$

and $d(b)\,\mathbf{x} - \mathbf{x}$, $d(b)\,\mathbf{y} - \mathbf{y} \in I_b$.

Main result

For a n braid b, let $A_b = A^{\otimes n}/I_b$.

Main Theorem

If the closures of two braids $b_1 \in B_{n_1}$ and $b_2 \in B_{n_2}$ are isotopic, then A_{b_1} and A_{b_2} are isomorphic algebras. Moreover, A_{b_1} and A_{b_2} are isomorphic A-comodules with adjoint coaction. In other words, A_b is an invariant of the knot (or link) $\widehat{b}$, which is the closure of b.

Definition

The quotient algebra $A_b = A^{\otimes n}/I_b$ is called **the A representation space** of the closure $\widehat{b}$.

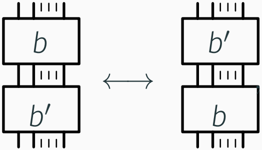
Markov equivalence

Theorem

The closures of two braids $b_1 \in B_{n_1}$ and $b_2 \in B_{n_2}$ are isotopic in S^3 if and only if there is a sequence of the following two types of moves connecting b_1 to b_2.

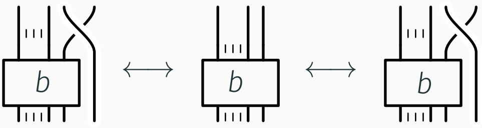
Definition

These moves are called **the Markov moves** and such b_1 and b_2 are called **Markov equivalent**.



$b\,b'$ $\quad$ $b'\,b$

MI



$\sigma_n\, b$ $\quad$ b $\quad$ $\sigma_n^{-1}\, b$

MII

The main theorem is proved by showing the invariancde under MI and MII.

Equivalent pair

Definition

For $b \in B_n$ we present $I_{\rho(b)}$ by $\rho(b) \sim \rho(1)$. Similarly, for two diagrams d_1, d_2 representing elements of $\mathrm{Hom}(A^{otimesm}, A^{\otimes n})$, $d_1 \sim d_2$ present a two-sided ideal I_{d_1,d_2} in $A^{\times n}$ generated by

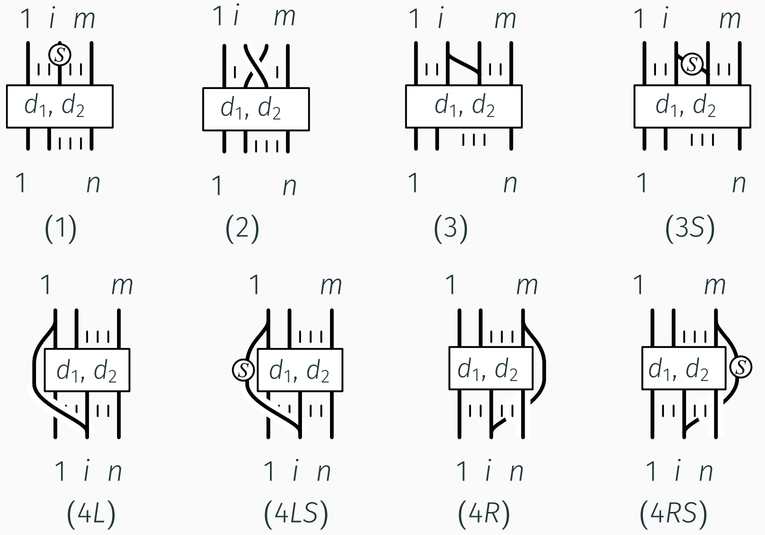
$$d_1(x_1 \otimes \cdots \otimes x_m) - d_2(x_1 \otimes \cdots \otimes x_m)$$

for $x_1, \cdots, x_m \in A$. Such d_1 and d_2 are called **the equivalent pair** of diagrams corresponding to the two-sided ideal I_{d_1,d_2} and the quotient algebra $A^{\otimes n}/I_{d_1,d_2}$.

Lemma

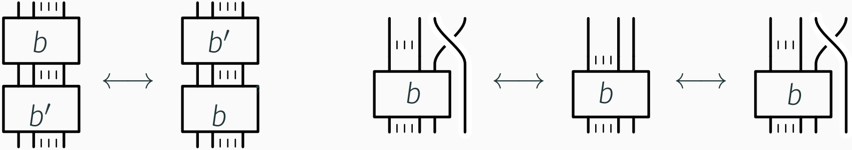
Let $d_1 \sim d_2$ be an equivalent pair and let $d_1 \sim d_2$ be the equivalent pair where d_1 and d_2 are obtained from d_1 and d_2 respectively by one of the following operations (1), (2), (3), (3S), (4L), (4LS), (4R), (4RS) illustrated in the following. Then the corresponding ideals I_{d_1,d_2} and I_{d_1,d_2} are equal.

Operations corresponding to the Tieze transformations



Invariance under Markov moves

Markov moves



$b\,b'$ $\longleftrightarrow$ $b'\,b$ (MI)

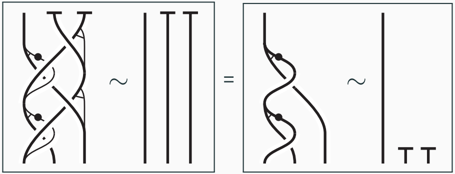
$\sigma_n\,b$ $\longleftrightarrow$ b $\longleftrightarrow$ $\sigma_n^{-1}\,b$ (MII)

Proof of the maintheorem

- To prove invariance under MI is rather easy.
- Invariance under MII is proved by using the above lemma.

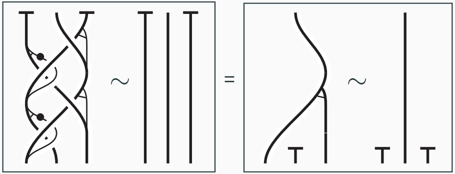
Example: Figure-eight knot

σ_2
σ_1^{-1}
σ^2
σ_1^{-1}



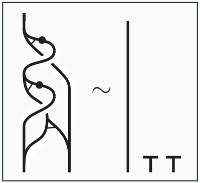
,

$d(b)(x \otimes 1 \otimes 1) \sim x \otimes 1 \otimes 1$



,

$d(b)(1 \otimes x \otimes 1) \sim 1 \otimes x \otimes 1$



5. Braided $\mathrm{SL}(2)$

Definition
A braided $\mathrm{SL}(2)$ is a one-parameter deformation of $\mathbb{C}[\mathrm{SL}(2)]$ defined by the following. It is denoted by $\mathrm{BSL}(2)$.

$$b\,a = t\,a\,b, \quad c\,a = t^{-1}\,a\,c, \quad d\,a = a\,d, \quad d\,b = b\,d + (1-t^{-1})\,a\,b,$$
$$c\,d = d\,c + (1-t^{-1})\,c\,a, \;\; b\,c = c\,b + (1-t^{-1})\,a\,(d-a), \;\; a\,d - t\,c\,b = 1,$$
$$\Delta(a) = a\otimes a + b\otimes c, \quad \Delta(b) = a\otimes b + b\otimes d, \quad \Delta(c) = c\otimes a + d\otimes c,$$
$$\Delta(d) = c\otimes b + d\otimes d, \quad S(a) = (1-t)\,a + t\,d \quad S(b) = -t\,b, \quad S(c) = -t\,c,$$
$$S(d) = a, \quad \varepsilon(a) = 1, \quad \varepsilon(b) = 0, \quad \varepsilon(c) = 0, \quad \varepsilon(d) = 1,$$
$$\Psi(x\otimes 1) = 1\otimes x, \; \Psi(1\otimes x) = x\otimes 1, \; \Psi(a\otimes a) = a\otimes a + (1-t)\,b\otimes c, \; \Psi(a\otimes b) = b\otimes a,$$
$$\Psi(a\otimes c) = c\otimes a + (1-t)(d-a)\otimes c, \; \Psi(a\otimes d) = d\otimes a + (1-t^{-1})\,b\otimes c,$$
$$\Psi(b\otimes a) = a\otimes b + (1-t)\,b\otimes(d-a), \; \Psi(b\otimes b) = t\,b\otimes b, \; \Psi(d\otimes b) = b\otimes d,$$
$$\Psi(b\otimes c) = t^{-1}\,c\otimes b + (1+t)(1-t^{-1})^2\,b\otimes c - (1-t^{-1})(d-a)\otimes(d-a),$$
$$\Psi(b\otimes d) = d\otimes b + (1-t^{-1})\,b\otimes(d-a), \; \Psi(c\otimes a) = a\otimes c, \; \Psi(c\otimes b) = t^{-1}\,b\otimes c,$$
$$\Psi(c\otimes c) = t\,c\otimes c, \; \Psi(c\otimes d) = d\otimes c, \; \Psi(d\otimes a) = a\otimes d + (1-t^{-1})\,b\otimes c,$$
$$\Psi(d\otimes c) = c\otimes d + (1-t^{-1})(d-a)\otimes c, \; \Psi(d\otimes d) = d\otimes d - t^{-1}\,(1-t^{-1})\,b\otimes c.$$

$\mathrm{BSL}(2)$ is braided commutative

Theorem
The braided Hopf algebra $\mathrm{BSL}(2)$ *is braided commutative.*

Since $\mathrm{BSL}(2)$ is an example of a braided commutative braided Hopf algebra, we have $\mathrm{BSL}(2)$ representations of K, which is $\mathrm{BSL}(2)^{\otimes n}/I_{d(b)}$. We also call it **the space of quantized $\mathrm{SL}(2, \mathbb{C})$ representations** of K. Since $\mathrm{BSL}(2)$ is Noetherian, the ideal $I_{d(b)}$ is finitely generated.

Problems

Problems

Construct following three objects and give a natural relation among them.

1. Quantum character variety
2. Quantum A polynomial
3. Quantum volume potential function