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Chern-Simons perturbation theory and Reidemeister-Turaev torsion

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Chern-Simons perturbation theory and Reidemeister-Turaev torsion

Let ρ be a representation of the fundamental group of a closed oriented 3-manifold M such that the corresponding local system is acyclic. We give an invariant $d(\rho)$ of ρ as a 1-dimensional cohomology class of M with twisted coefficient. This invariant is deeply related to the Chern-Simons perturbation theory. In this talk, when ρ is an abelian representation, we show that $d(\rho)$ can be computed from Reidemeister-Turaev torsion of M.

A geometric description of Reidemeister-Turaev torsion of 3-manifolds

Plan of this talk

§. Main result

§. Reidemeister-Turaev torsion

§. Outline of proof (via Morse homotopy)

§. Background (Chern-Simons perturbation theory)

A geometric description of Reidemeister - Turaev torsion of 3-manifolds

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Setting

M: a closed oriented 3-mfd.

$H_1 = H_1(M; \mathbb{Z})$: rank k, torsion free (assumption)
$= \{ x_1^{n_1} \cdots x_k^{n_k} \mid n_i \in \mathbb{Z} \}$ (basis : $x_1, \dots, x_k$)

$\mathbb{R}H_1 = \{ \sum_{n_1,\dots,n_k} a_{n_1 \cdots n_k} x_1^{n_1} \cdots x_k^{n_k} \}$ the groupring of H_1.
$(= \mathbb{R}[x_1, \dots, x_k])$

$QH_1 = \{ g/f \mid g, f \in \mathbb{R}H_1 \}$, quotient field

$\rho_{ab} : H_1 \to \mathrm{Aut}(QH_1) = (QH_1)^\times$, $x \mapsto (f \mapsto xf)$, representation

$\rightsquigarrow H_*(M, \underline{\rho_{ab}}) = 0$, i.e. acyclic ($\Leftarrow$ assumption)

local system corr. to ρ_{ab} : $(\pi_1 \to) H_1 \to (QH_1)^\times$

e: Euler str. on M (a homotopy class of a non-vanishing vector field on M)

$\to \mathrm{Tor}(M, e) \in (QH_1)^\times$, Reidemeister-Turaev torsion

Setting

input

$\boxed{M}$: a closed oriented 3-mfd.

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$\boxed{e}$: Euler str. on M (a homotopy class of a non-vanishing vector field on M)

Output $\to$ $\boxed{\mathrm{Tor}(M, e)} \in (QH_1)^\times$, Reidemeister – Turaev torsion

An inv. of (M, e)

M : 3-mfd.
$H_1 = H_1(M : \mathbb{Z})$
$\mathbb{R}H_1 \cong \mathbb{R}[x_1, \cdots, x_k]$
$QH_1 \cong \mathbb{R}(x_1, \cdots, x_k)$
$\rho_{ab} : H_1 \to (QH_1)^*$
e : Euler str (non-vanishing v.f.)

$M \times M \supset \Delta = \{(x,x) \mid x \in M\}$, diagonal.

$[\Delta] \in H_3(\Delta : \mathbb{Z})$, fundamental homology class

$[\Delta] \in H_3(\Delta : \rho_{ab} \otimes \rho_{ab}^*)$

$[\Delta] \in H_3(M \times M : \rho_{ab} \boxtimes \rho_{ab}^*) = 0$ "acyclic" & Künneth.

Then $\exists\, \Sigma$: 4-chain of $C_4(M \times M ; \rho_{ab} \boxtimes \rho_{ab}^*)$

s.t. $\begin{cases} \partial\Sigma = \Delta \\ \partial\Sigma \text{ is "controlled by" } e \end{cases}$

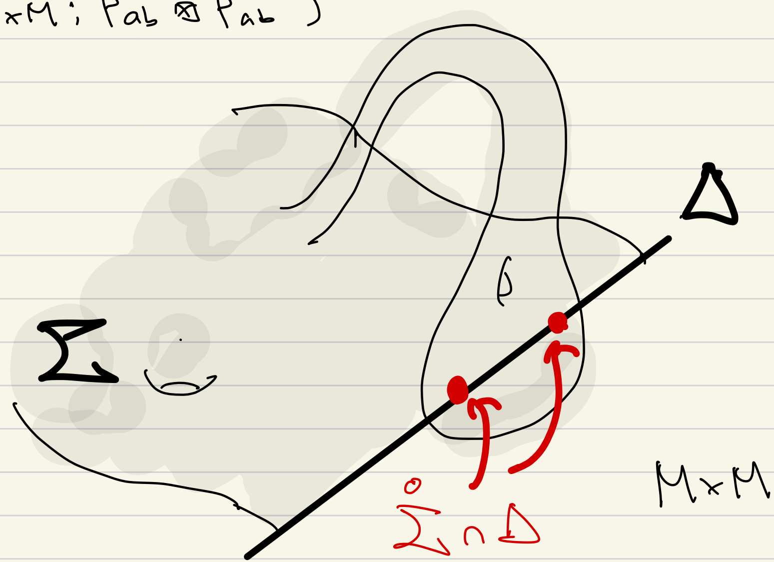
$d(\rho_{ab}) := [\mathring{\Sigma} \cap \Delta] \in H_1(\Delta : \rho_{ab} \otimes \rho_{ab}^*)$

$\cup$

$H_1(\Delta : \rho_{ab} \otimes \rho_{ab}^*) \quad H_1$

$\circlearrowleft$

$H_1 \otimes_{\mathbb{Z}} QH_1$



$\boxed{\text{fact } [\mathring{\Sigma} \cap \Delta] \text{ is an invariant of } (M, e)}$

Main theorem

$$(M, e) \rightsquigarrow \mathrm{Tor}(M,e) \in QH_1$$
3-mfd, euler str

$$\Updownarrow \text{ Theorem.}$$

$$(M, e) \rightsquigarrow [\mathring{\Sigma} \cap \Delta] \in H_1 \otimes QH_1$$

M : 3-mfd.
$H_1 = H_1(M;\mathbb{Z})$
$\mathbb{R}H_1 \cong \mathbb{R}[x_1, \cdots, x_k]$
$QH_1 \cong \mathbb{R}(x_1, \cdots, x_k)$
$\mathrm{Par} : H_1 \to (QH_1)^\times$
e : Euler str (non-vanishing v.f.)

Theorem

$$\boxed{\tilde{\Delta} \circ d\log \mathrm{Tor}(M,e) = [\mathring{\Sigma} \cap \Delta]}$$

Here,

$$QH_1 \to \text{``}\Omega^1_0(\mathbb{R}^k)\text{''} \longrightarrow H_1 \otimes QH_1$$

$$f \overset{d\log}{\longmapsto} d(\log f) \qquad dx_i \overset{\tilde{\Delta}}{\longmapsto} x_i \otimes x_i$$

Remark

- $(\tilde{\Delta} \circ d\log)$ is indep of the basis $x_1, \cdots, x_k$ of H_1
- $b_1 = 1$ … Lescop. (for special Euler str.)
- $\mathrm{Tor}(M,e) = \exp \int \tilde{\Delta}^{-1}([\mathring{\Sigma} \cap \Delta])$ in $QH_1 / \mathbb{R}^\times$
- If H_1 has torsion, then also holds.

Example

1) $M = S^1 \times S^2$, $H_1 = \mathbb{Z} = \langle x \rangle$

$$\mathrm{Tor}(S^1 \times S^2, e) = \frac{1}{(x-1)^2} \in QH_1$$

$$\Big\downarrow \ d\log$$

$$\frac{-2\,dx}{x-1}$$

$$\Big\downarrow \ \tilde{\Delta} \ (\text{i.e. } dx \mapsto x \otimes x)$$

$$[\mathring{\Sigma} \cap \Delta] = x \otimes \frac{-2x}{x-1} \in H_1 \otimes QH_1$$

2) $M = L(p,1)$, $H_1 = \mathbb{Z}/p = \langle x \mid x^p \rangle$

$$\mathrm{Tor}(L(p,1), e) = \frac{x^k}{(1-x)^2} \in QH_1$$

$$\Big\downarrow \ d\log$$

$$\frac{k x^{k-1}}{x^k}\,dx - \frac{-2}{1-x}\,dx$$

$$\Big\downarrow \ \tilde{\Delta} \ (\text{i.e. } dx \mapsto x \otimes x)$$

$$[\mathring{\Sigma} \cap \Delta] = x \otimes \left(k + \frac{2x}{1-x}\right) \in H_1 \otimes QH_1$$

A geometric description of Reidemeister - Turaev torsion of 3-manifolds

Plan of this talk

§. Main result

$\boxed{\text{§. Reidemeister - Turaev torsion}}$

§. Outline of proof (via Morse homotopy)

§. Background (Chern - Simons perturbation theory)

Reidemeister – Turaev torsion

M : a closed orid 3-mfd $\qquad$ e : an Euler str.

$\rho_{ab} : H_1 \longrightarrow QH_1$

$$
\begin{tikzcd}
0 \arrow[r] & C_3(M;\rho_{ab}) \arrow[r, "\partial_3"] & C_2(M;\rho_{ab}) \arrow[r, "\partial_2"] \arrow[l, bend left, "\exists g_3"] & C_1(M;\rho_{ab}) \arrow[r, "\partial_1"] \arrow[l, bend left, "\exists g_2"] & C_0(M,\rho_{ab}) \arrow[r] \arrow[l, bend left, "\exists g_1"] & 0
\end{tikzcd}
$$

acyclic

s.t. $\partial_i \circ g_i + (-1)^i g_{i-1} \circ \partial_{i-1} = \mathrm{id}.$

M : 3-mfd.
$H_1 = H_1(M;\mathbb{Z})$
$\mathbb{R}H_1 \cong \mathbb{R}[x_1, \cdots, x_k]$
$QH_1 \cong \mathbb{R}(x_1, \cdots, x_k)$
$\rho_{ab} : H_1 \to (QH_1)^\times$
e : Euler str (non-vanishing v.f.)

$$
\begin{tikzcd}
C_2(M;\rho_{ab}) \arrow[r, "g_3"] \arrow[dr, "\partial_2"] & C_3(M;\rho_{ab}) \\
\oplus & \oplus \\
C_0(M;\rho_{ab}) \arrow[r, "g_1"'] & C_1(M;\rho_{ab})
\end{tikzcd}
$$

$$C_{even} \xrightarrow[\partial + g]{\cong} C_{odd}$$

basis $\qquad$ **basis**

e : Euler str

Definition $\qquad \mathrm{Tor}(M,e) = \det(\partial + g) \in QH_1^\times$

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Outline of proof

Theorem

(M, e) ⇝ $\mathrm{Tor}(M, e) \in QH_1$

3-mfd, euler str

$\widetilde{\Delta} \circ d\log$

⇝ $[\mathring{\Sigma} \cap \Delta] \in H_1 \otimes QH_1$

cf. S^1-valued Morse version (Hutchings–Lee, Pajitnov, Kitayama)

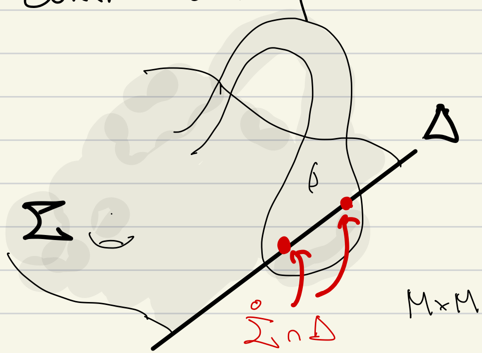
$\left[\begin{array}{c}\text{Morse function} \\ f : M \to \mathbb{R}\end{array}\right]$ ⇝ ρ_{ab}-twisted Morse–Smale cpx $C_*(f, \rho_{ab})$ ⇝ $\boxed{\mathrm{Tor}(C_*(f, \rho_{ab}), e)}$

$\shortparallel$

$\mathrm{Tor}(M, e)$

Morse homotopy
(Fukaya, Watanabe, Betz–Cohen)

concrete description of Σ
$(\partial \Sigma = \Delta)$



$\widetilde{\Delta} \circ d\log$

⇝ $\boxed{\mathring{\Sigma} \cap \Delta = \sum_{\gamma} \gamma \otimes g \cdot \gamma_*}$

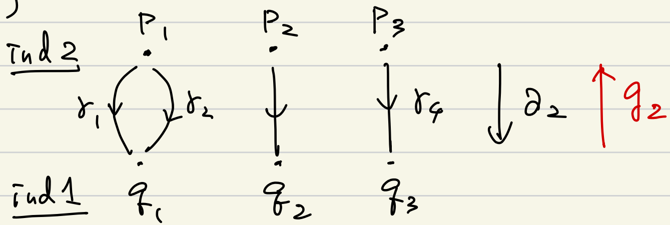
γ : trajectory

$\gamma \in C_1(M; \mathbb{Z})$ (1-chain)

$g \cdot \gamma_* \in QH$

Why "dlog" appears?

(Example)



$$\partial_2 = \begin{pmatrix} \gamma_1+\gamma_2 & & \\ & \gamma_3 & \\ & & \gamma_4 \end{pmatrix}$$

$$g_2 = \begin{pmatrix} \frac{1}{\gamma_1+\gamma_2} & & \\ & \frac{1}{\gamma_3} & \\ & & \frac{1}{\gamma_4} \end{pmatrix}$$

$$\mathrm{Tor} \longleftrightarrow \det \partial_2 = (\gamma_1+\gamma_2)\gamma_3\gamma_4$$

$$\Big\downarrow \mathrm{dlog}$$

$$\frac{d\gamma_1}{\gamma_1+\gamma_2} + \frac{d\gamma_2}{\gamma_1+\gamma_2} + \frac{d\gamma_3}{\gamma_3} + \frac{d\gamma_4}{\gamma_4}$$

$$\Big\downarrow \tilde{\Delta} \quad (dx_i \mapsto [x_i]\otimes x_i)$$

$$[\mathring{\Sigma} \cap \Delta] = \sum_{\gamma} [\gamma]\otimes g\gamma \longleftrightarrow [\gamma_1]\otimes \frac{\gamma_1}{\gamma_1+\gamma_2} + [\gamma_2]\otimes\frac{\gamma_2}{\gamma_1+\gamma_2} + [\gamma_3]\otimes 1 + [\gamma_4]\otimes 1$$

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Background of $[\mathring{\Sigma} \cap \Delta]$

Chern-Simons perturbation theory

- inv. of (3-mfd M , "acyclic" rep. of π_1 ρ)
- "propagator" g (+ Euler str.) $\xrightarrow[\text{integral}]{\text{configuration space}}$ inv. of (M, ρ)

a geometric description ↙ ↘ determinant

Σ $(\partial \Sigma = \Delta)$

↓

$(d(\rho) =)[\mathring{\Sigma} \cap \Delta]$ ⟷ today (ρ : abel) ⟷ $\mathrm{Tor}(\rho)$ Reidemeister torsion.