Twist formulas for one-row colored A_2 webs and $\mathfrak{sl}_3$ tails of $(2, 2m)$-torus links

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Main Contents

- CJP & skein relation for A_1 and A_2
- Twist formula via the generating function of Young diagrams
- Explicit description of tails of $T(2, m)$

Intelligence of Low-dimensional Topology, May 14, 2020
at RIMS, Kyoto Univ. (Zoom)

The colored Jones polynomial.

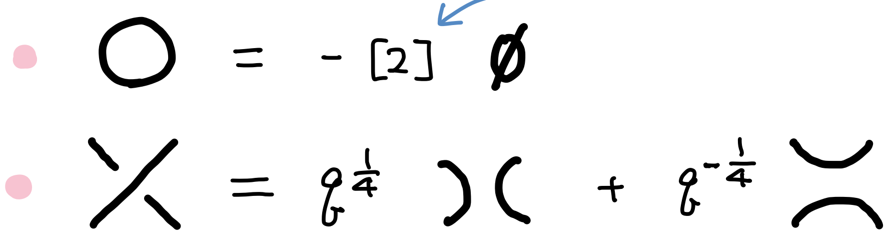
- $\mathfrak{sl}_2$: {irreducible rep's} $\longleftrightarrow$ $\{n \in \mathbb{Z}_{\geq 0}\}$

  - V_2 : the 2-dimensional irrep. $\rightsquigarrow$ the Jones polynomial $J(K) := J^{\mathfrak{sl}_2}_{V_2}(K)$
  - V_{n+1}: the $(n+1)$-dimensional irrep $\rightsquigarrow$ the $(n+1)$-dimensional colored Jones polynomial $J_n(K) := J^{\mathfrak{sl}_2}_{V_{n+1}}(K)$

Graphical calculation of $J(K)$

$$[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

the Kauffman bracket skein relation

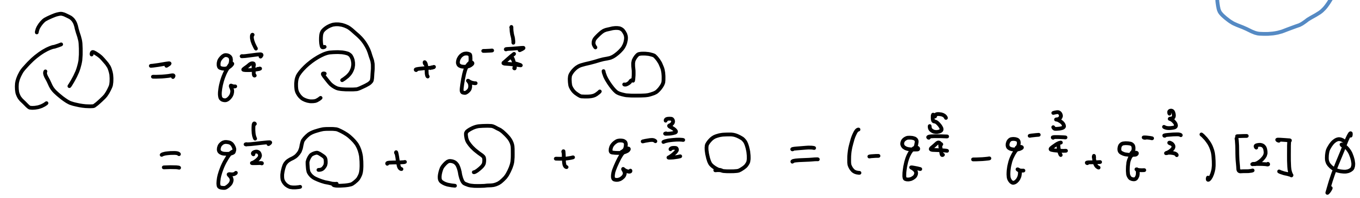


knot diagram $D(K)$ of K framed knot

$\downarrow$ KBSR

$J(K)\,\emptyset$ an invariant of framed knot in $\mathbb{Z}[q^{\pm\frac{1}{4}}]$

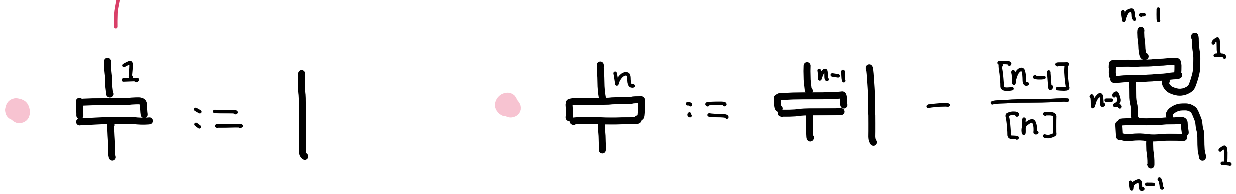
e.g.

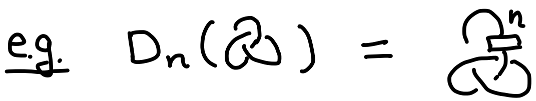


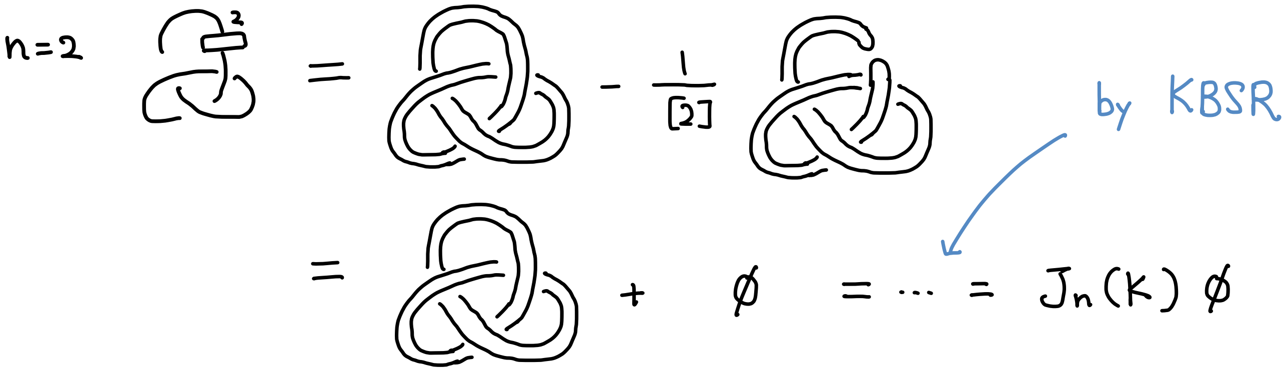
Graphical calculation of $J_n(K)$

KBSR + the Jones-Wenzl projector  coloring

(Knot diagram with Jones-Wenzl projector $D_n(K) \xrightarrow{\text{KBSR}} J_n(K)$)



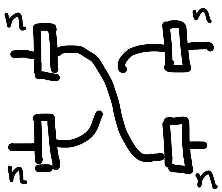




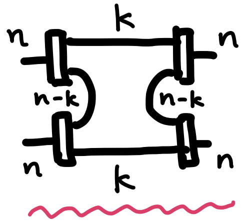
Twist formulas (useful for compute $J_n(K)$)

$(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$

- half twist
  (Yamada 1989)



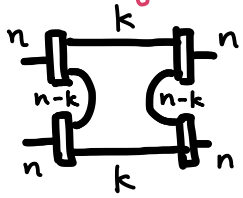
$$= \sum_{k=0}^{n} q^{\frac{1}{4}(-n^2+2k^2)} \frac{(q)_n}{(q)_k (q)_{n-k}}$$



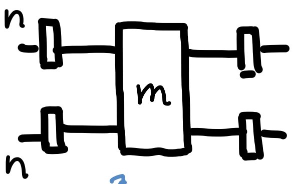
basis web (crossingless diagram)

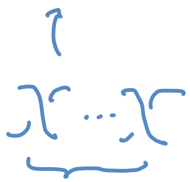
- full twist
  (Masbaum 2003)

$$= \sum_{k=0}^{n} (-1)^{n-k} q^{\frac{1}{2}(-n^2-n+2k^2+k)} \frac{(q)_n^2}{(q)_k^2 (q)_{n-k}}$$



- m half twist
  (Y. 2017)

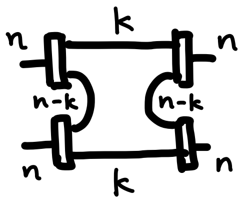




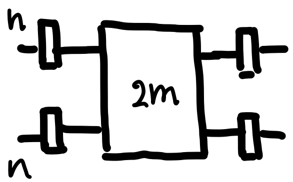
m crossings

$$= \sum_{n \geq k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} (-1)^{n-k_m} q^{\frac{n-k_m}{2}} (-1)^{\sum_{i=1}^{m} k_i} q^{\frac{1}{2}\sum_{i=1}^{m}(k_i^2+k_i)}$$

$$\times \frac{(q)_n}{(q)_{n-k_1}(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}}$$

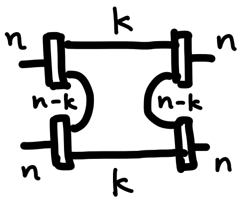


- m full twist
  (Y. 2017)



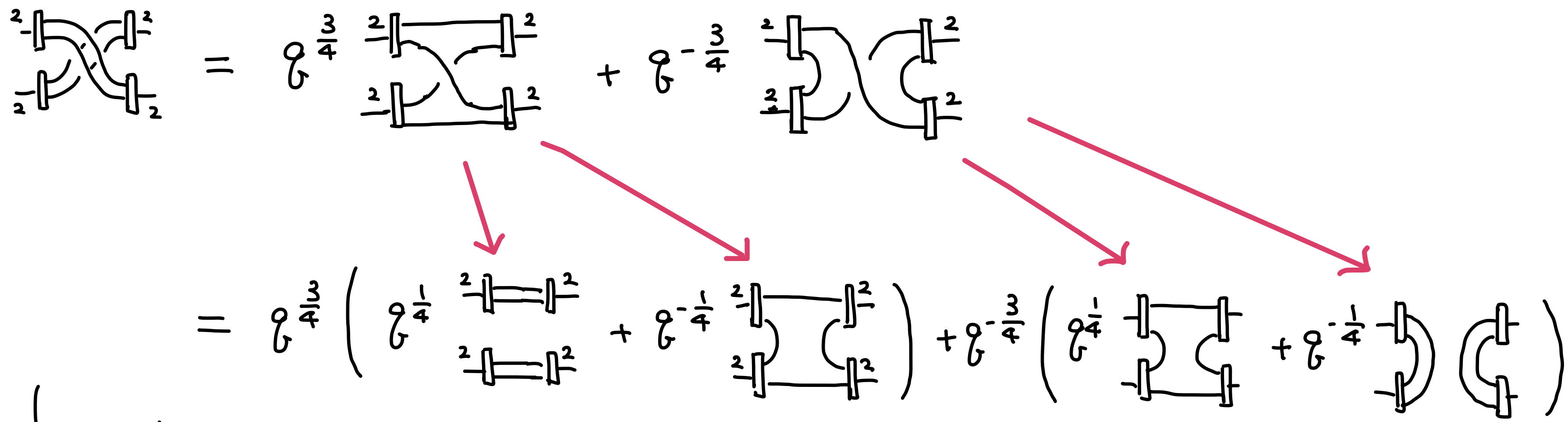
$$= q^{-\frac{m}{2}(n^2+2n)} \sum_{n \geq k_1 \geq \cdots \geq k_m \geq 0} (-1)^{n-k_m} q^{\frac{n-k_m}{2}} q^{\sum_{i=1}^{m}(k_i^2+k_i)}$$

$$\times \frac{(q)_n^2}{(q)_{n-k_1}(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}^2}$$

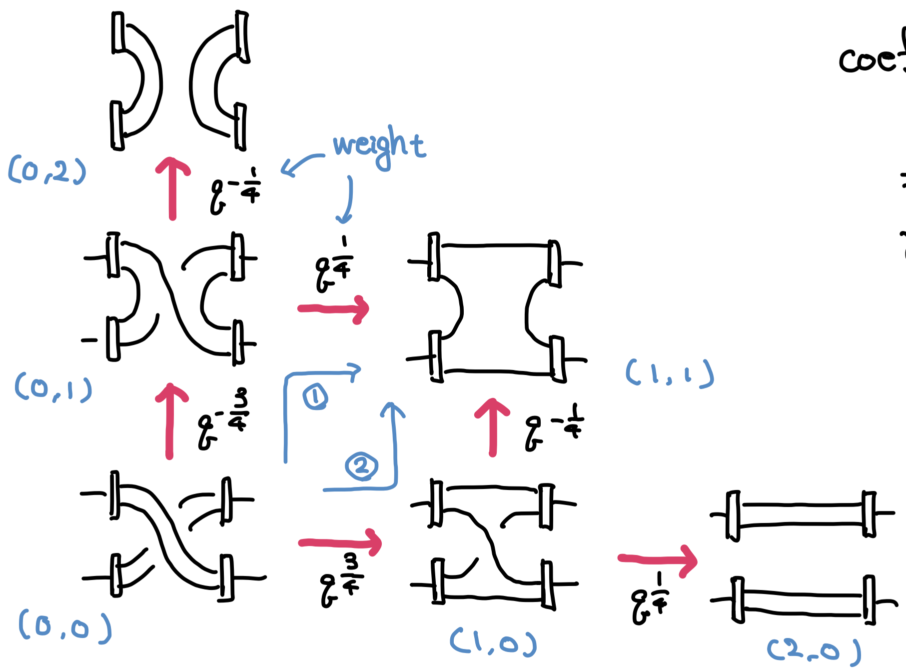


How to derive twist formulas (Y. 2017)

e.g.



by skein tree



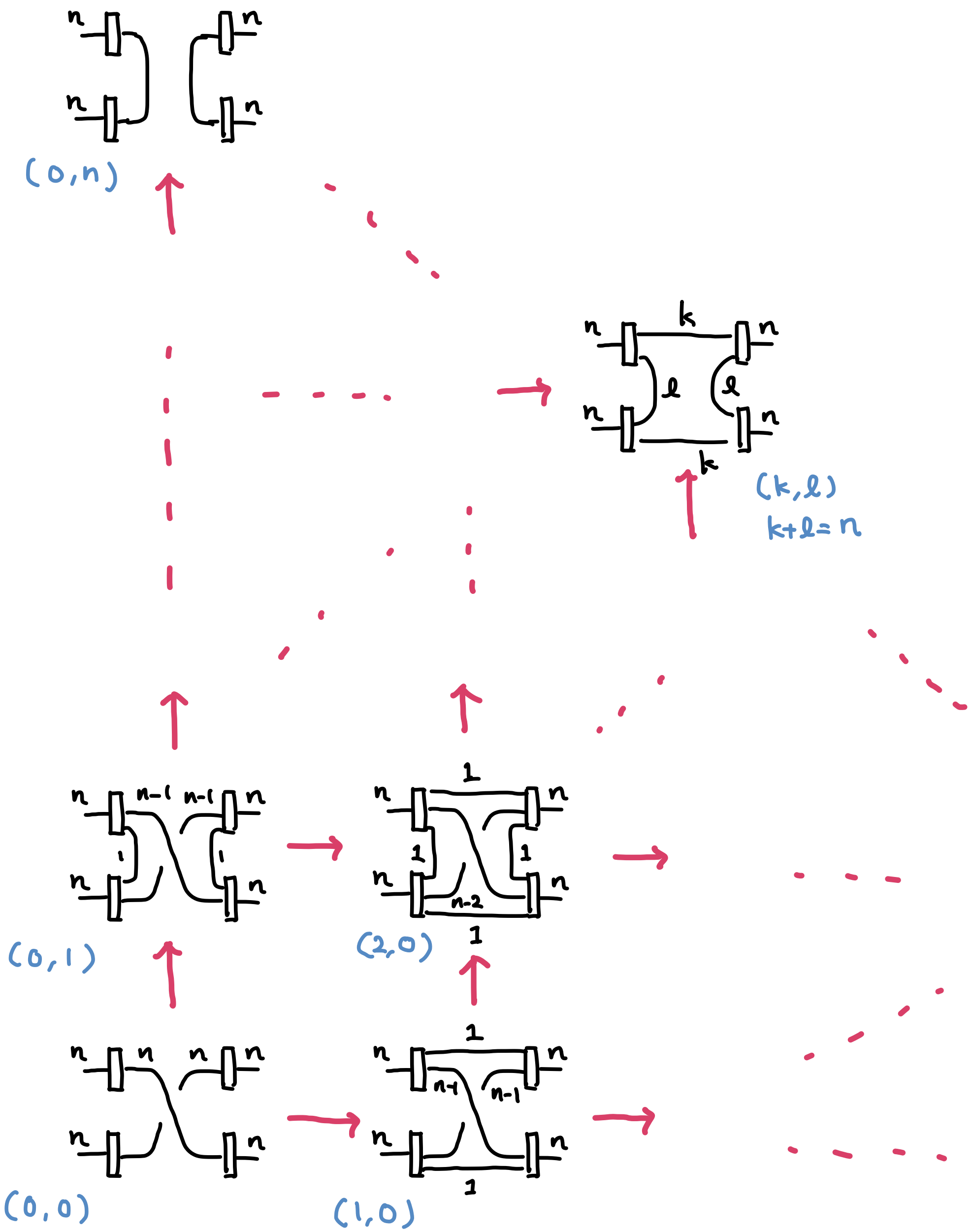
coefficient of basis web on (1, 1)

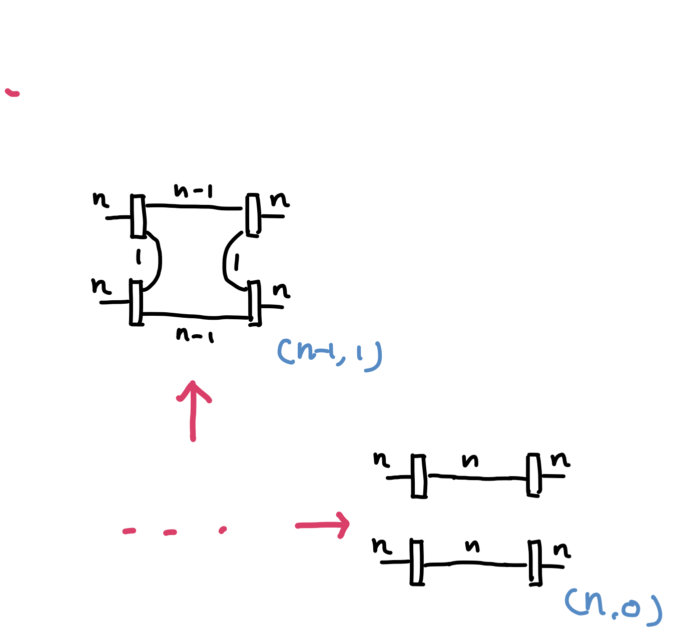
$$= \sum_{\gamma:\ \text{paths from } (0,0) \text{ to } (1,1)} \ \prod_{w:\ \text{weight on } \gamma} w$$

$$= \underbrace{q^{-\frac{3}{4}} q^{\frac{1}{4}}}_{①} + \underbrace{q^{\frac{3}{4}} q^{-\frac{1}{4}}}_{②}$$

$\times q$

(half twist formula)



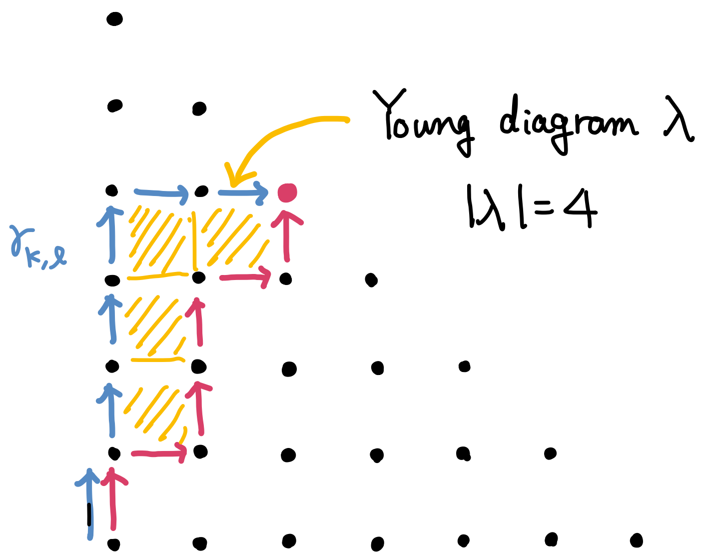


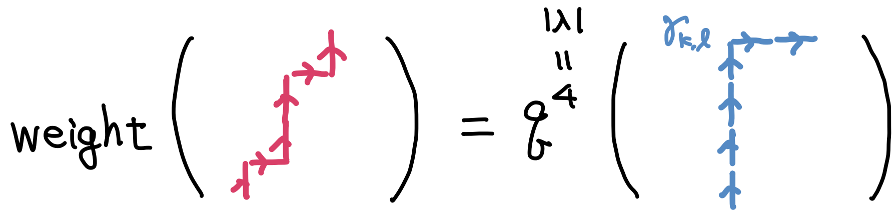
<u>Lemma</u>

$$\begin{tikzcd}
(k,\ell+1) \arrow[r, "X'"] & (k+1,\ell+1) \\
(k,\ell) \arrow[u, "Y'"] \arrow[r, "X"] & (k+1,\ell) \arrow[u, "Y"]
\end{tikzcd}$$

X, Y, X', Y' : weights

then $XY = q\, Y'X'$





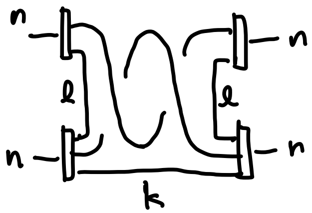
◎ coefficient of (k, ℓ) $(k+\ell = n)$

$$= \sum_{\substack{\gamma:\ \text{path from} \\ (0,0)\ \text{to}\ (k,\ell)}} \prod_{\substack{w:\ \text{weight} \\ \text{on}\ \gamma}} w$$

$$= \left(\prod_{\substack{w:\ \text{weight} \\ \text{on}\ \gamma_{k,\ell}}} w \right) \left(\sum_{\substack{\lambda:\ \text{Young diagram} \\ \#\,\text{row} \leq k \\ \#\,\text{column} \leq \ell}} q^{|\lambda|} \right)$$

$$= \left(\prod_{\substack{w:\ \text{weight} \\ \text{on}\ \gamma_{k,\ell}}} w \right) \frac{(q)_n}{(q)_k (q)_{n-k}}$$

(full twist formula)

$(k, \ell) \longleftrightarrow$ 

& the same method.

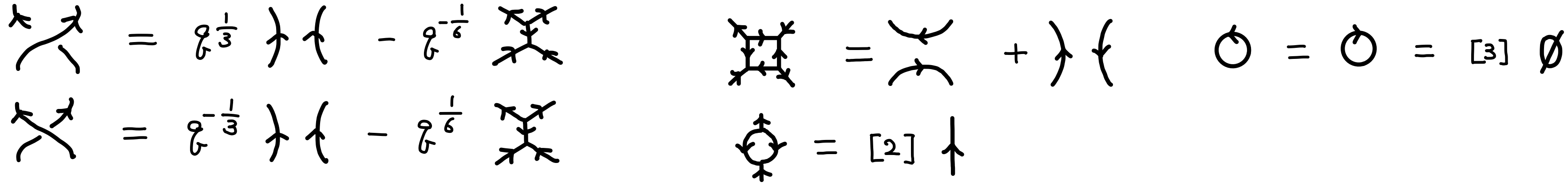
The $\mathfrak{sl}_3$ colored Jones polynomial

- $\mathfrak{sl}_3$: $\{\text{irreducible rep's}\} \longleftrightarrow \{(n_1, n_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\}$

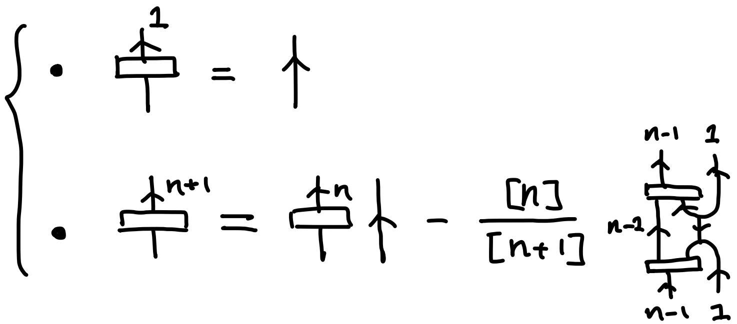
$\rightsquigarrow J^{\mathfrak{sl}_3}_{(n_1,n_2)}(K)$: the $\mathfrak{sl}_3$ colored Jones polynomial

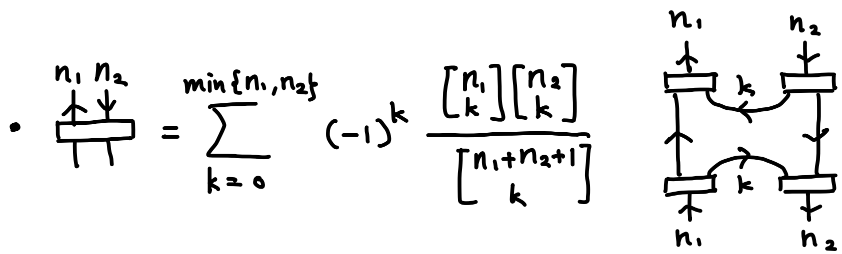
Graphical calculation

the A_2 skein relation (Kuperberg 1994, 1996)



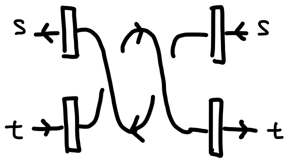
the A_2 clasps (Kuperberg 1996, Ohtsuki - Yamada '97)



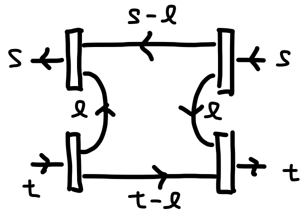


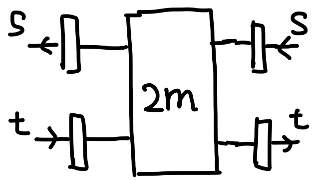
Twist formulas for A_2 skein with $(n,0)$ coloring

Theorem (Y, 2017)

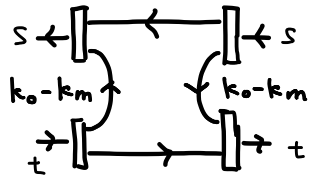


$$= q^{\frac{st}{3}} \sum_{\ell=0}^{\infty} q^{\ell^2-\ell}\, q^{-(s+t)\ell}\, \frac{(q)_s\,(q)_t}{(q)_\ell\,(q)_{s-\ell}\,(q)_{t-\ell}}$$





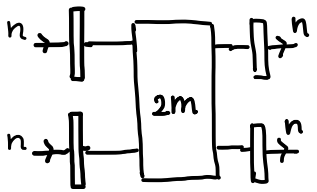
$$= q^{-\frac{2m}{3}k_0(k_0+\Delta)-2mk_0} \sum_{k_0\geq k_1\geq k_2\geq\cdots\geq k_m\geq 0} C(k_1,k_2,\ldots,k_m)$$



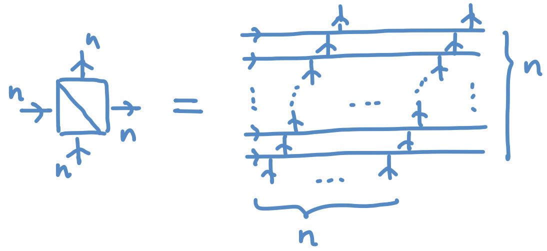
where

$$\begin{cases} C(k_1,k_2,\ldots,k_m) = q^{\sum_{i=1}^{m} k_i(k_i+\Delta)+2k_i}\, q^{k_0-k_m}\, \dfrac{(q)_{k_0+\Delta}}{(q)_{k_m+\Delta}}\, \dfrac{(q)_{k_0}}{(q)_{k_0-k_1}(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}} \\ k_0 = \min\{s,t\}, \quad \Delta = |s-t| \end{cases}$$

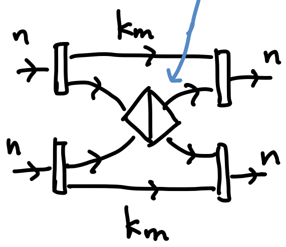
Theorem (Y. 2020)

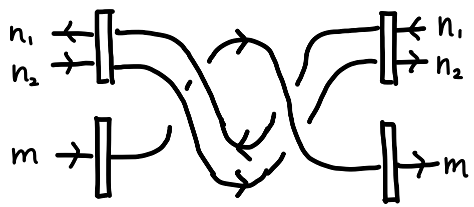


$$= q^{\frac{n}{2}} q^{-\frac{1}{3}(n^2+3n)m}$$



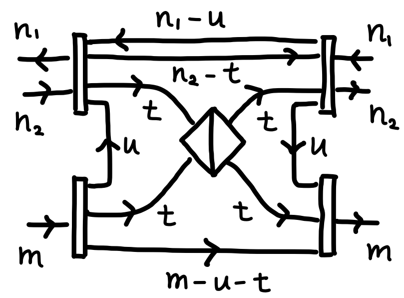
$$\times \sum_{n \geq k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} q^{k_m^2 + \frac{1}{2}k_m} q^{\sum_{i=1}^{m-1} k_i^2 + k_i} \frac{(q)_n^2}{(q)_{n-k_1}(q)_{k_1-k_2}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}}$$





$$= q^{\frac{1}{3}(n_1+2n_2)m} \sum_{u,t \geq 0} q^{-(n_1+n_2+m)u} q^{-(n_2+m)t} q^{u^2-u} q^{t^2-\frac{1}{2}t} q^{ut}$$

$$\times \frac{(q)_m (q)_{n_1} (q)_{n_2}}{(q)_{m-u-t}(q)_u (q)_t (q)_{n_1-u}(q)_{n_2-t}}$$



Apprication : tails of knot

e.g. alternating knot

K : an adequate knot

$J_n(K)$: the $(n+1)$-dim. colored Jones polynomial

$$\tilde{J}_n(K) := (\pm q^{\ast}) \cdot J_n(K) = \sum_{k \geq 0} a_k q^k \quad \text{s.t} \quad a_0 \geq 0$$

adequate　　　　alternating & more general stability

Theorem [Armond 2013, Garoufalidis - Lê 2015]

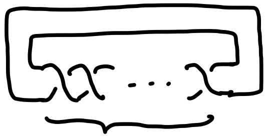
$$\tilde{J}_n(K) \equiv \tilde{J}_N(K) \mod (q^{n+1}) \quad \text{for } \forall N \geq n$$

$\rightsquigarrow \exists$ a q-series $J(K)$ s.t $J(K) \equiv \tilde{J}_n(K) \mod (q^{n+1})$

the tail of K

the tail of $(2,m)$-torus knot

$T(2,m)$: a $(2,m)$-torus knot

$T(2,m) =$ 

m crossings

Theorem using formula obtained by quantum spin network

$$J(T(2,2m+1)) = f(-q^{2m}, -q)/(1-q) \qquad \text{[Armond - Dasbach 2011]}$$

$$J(T(2,2m)) = \Psi(q^{2m-1}, q)/(1-q) \qquad \text{[Hajij 2015]}$$

where

$$f(a,b) := \sum_{i=0}^{\infty} a^{\frac{i(i+1)}{2}} b^{\frac{i(i-1)}{2}} + \sum_{i=1}^{\infty} a^{\frac{i(i-1)}{2}} b^{\frac{i(i+1)}{2}} \quad \text{: the theta series}$$

$$\Psi(a,b) := \sum_{i=0}^{\infty} a^{\frac{i(i+1)}{2}} b^{\frac{i(i-1)}{2}} - \sum_{i=1}^{\infty} a^{\frac{i(i-1)}{2}} b^{\frac{i(i+1)}{2}} \quad \text{: the false theta series}$$

Theorem

[Armond - Dasbach 2011]

$$J(T(2,2m+1)) = \frac{(q)_\infty}{1-q} \sum_{k_1 \geq k_2 \geq \cdots \geq k_{m-1} \geq 0} \frac{q^{\sum_{i=1}^{m-1} k_i^2 + k_i}}{(q)_{k_1-k_2}(q)_{k_2-k_3} \cdots (q)_{k_{m-2}-k_{m-1}}(q)_{k_{m-1}}}$$

description of R-matrix.

[Hajij 2015]

$$J(T(2,2m)) = \frac{(q)_\infty}{1-q} \sum_{k_1 \geq k_2 \geq \cdots \geq k_{m-1} \geq 0} \frac{q^{\sum_{i=1}^{m-1} k_i^2 + k_i}}{(q)_{k_1-k_2}(q)_{k_2-k_3} \cdots (q)_{k_{m-2}-k_{m-1}}(q)^2_{k_{m-1}}}$$

"bubble skein expansion formula"

Remark

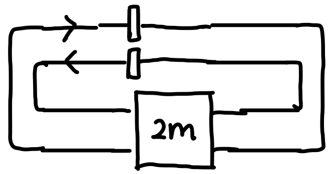
- We can obtain the above description of $J(T(2,m))$ from twist formulas !!
- These explicit formulas derive the Andrews - Gordon type identities for the (false) theta series.

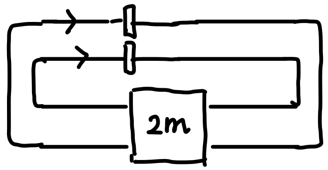
The $\mathfrak{sl}_3$ tails of $T(2,2m)$ with one-row colorings

From twist formulas for A_2 skein,

we can compute

$$\begin{cases} J^{\mathfrak{sl}_3}_{(n,0)}\left(T_{\substack{\leftarrow\\ \rightarrow}}(2,2m)\right) := \\ J^{\mathfrak{sl}_3}_{(n,0)}\left(T_{\substack{\rightarrow\\ \rightarrow}}(2,2m)\right) := \end{cases}$$





,

and

normalization

$$\tilde{J}^{\mathfrak{sl}_3}_{(n,0)}\left(T_{\substack{\leftarrow\\ \rightarrow}}(2,2m)\right) \equiv \tilde{J}^{\mathfrak{sl}_3}_{(N,0)}\left(T_{\substack{\leftarrow\\ \rightarrow}}(2,2m)\right) \mod (q^{n+1}) \quad \text{for } {}^{\forall}N \geq n$$

$$\tilde{J}^{\mathfrak{sl}_3}_{(n,0)}\left(T_{\substack{\rightarrow\\ \rightarrow}}(2,2m)\right) \equiv \tilde{J}^{\mathfrak{sl}_3}_{(N,0)}\left(T_{\substack{\rightarrow\\ \rightarrow}}(2,2m)\right) \mod (q^{n+1}) \quad \text{for } {}^{\forall}N \geq n$$

$\rightsquigarrow$ $\exists$ $\mathfrak{sl}_3$ tail $\mathcal{J}^{\mathfrak{sl}_3}(T(2,2m))$

(i.e. $\mathcal{J}^{\mathfrak{sl}_3}(T(2,2m)) \equiv \tilde{J}^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m)) \mod (q^{n+1})$)

Theorem (from twist formulas)

[Y. 2018]

$$J^{\mathfrak{sl}_3}(T_{\mp}(2,2m)) = \frac{(q)_\infty}{(1-q)(1-q^2)} \sum_{k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} \frac{q^{-2k_m} q^{\sum_{i=1}^{m} k_i^2 + 2k_i}}{(q)_{k_1-k_2}(q)_{k_2-k_3}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}^2}$$

[Y. 2020]

$$J^{\mathfrak{sl}_3}(T_{\mp}(2,2m)) = \frac{(q)_\infty}{(1-q)(1-q^2)} \sum_{k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} \frac{q^{-k_m} q^{\sum_{i=1}^{m} k_i^2 + k_i}}{(q)_{k_1-k_2}(q)_{k_2-k_3}\cdots(q)_{k_{m-1}-k_m}(q)_{k_m}^2}$$

Theorem (from "q-spin network")

[Y. 2018]

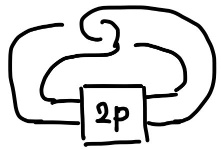
$$J^{\mathfrak{sl}_3}(T_{\mp}(2,2m)) = \sum_{i=0}^{\infty} q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3(1+q^{i+1})}{(1-q)^2(1-q^2)}$$

[Y. 2020]

$$J^{\mathfrak{sl}_3}(T_{\mp}(2,2m)) = \frac{\Psi(q^{2m-1}, q)}{(1-q^2)(1-q)^2}, \quad J^{\mathfrak{sl}_3}(T_{\mp}(2,2m+1)) = \frac{f(-q^{2m}, -q)}{(1-q^2)(1-q)^2}$$

Remark $J^{\mathfrak{sl}_3}(T_{\mp}(2,m)) \equiv J(T(2,m))$ modulo rational function twist formulas for $\mathfrak{sl}_2$ & $\mathfrak{sl}_3$ $\leftarrow\!\!\rightsquigarrow$ generating function of paths in a lattice.

Q. How about $\mathfrak{sl}_n$ with one-row coloring

Masbaum obtained an explicit description of the cyclotomic expansion of $K_p =$ 

Q. the cyclotomic expansion of $J^{\mathfrak{sl}_3}_{(n,0)}(K_{(2,m)}) = ?$

Q. slope conjecture, volume conjecture ... for $J^{\mathfrak{sl}_3}_{(n,0)}$?