Intelligence of Low-dimensional Topology at RIMS Kyoto University

Smooth homotopy 4-sphere

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This talk is an explanation of the following:

Theorem A.
Every smooth homotopy 4-sphere is diffeomorphic to the 4-sphere.

in the paper:

A. Kawauchi, Smooth homotopy 4-sphere. arxiv:1911.11904v4

A homotopy 4-sphere is a smooth 4-manifold M which is homotopy equivalent to the 4-sphere S^4.

A homotopy 4-ball is a 1-punctured manifold $M^{(0)}$ of a homotopy 4-sphere M.

The stable 4-sphere of genus n is:

$$\Sigma = \Sigma(n) = S^4 \# n(S^2 \times S^2) = S^4 \#_{i=1}^{n} S^2 \times S^2_i \,.$$

Fact: The stable 4-sphere Σ of genus n is the double branched covering space $S^4(F)_2$ of S^4 branched along a trivial surface F of genus n.

An orthogonal 2-sphere pair or simply anO2-sphere pair of Σ is a pair (S,S') of 2-spheres S and S' embedded in Σ meeting transversely at a point with the intersection numbers

$$\mathrm{Int}(S,S)=\mathrm{Int}(S',S')=0 \text{ and } \mathrm{Int}(S,S')=+1.$$

A pseudo-O2-sphere basis of Σ of genus n is the system (S_*,S'_*) of n mutually disjoint O2-sphere pairs

$$(S_i,S'_i)\ (i=1,2,\cdots, n) \text{ in } \Sigma.$$

Let $N(S_i,S'_i)$ be a regular neighborhood of the union $S_i \cup S'_i$ of the O2-sphere pair (S_i,S'_i) in Σ such that $N(S_i,S'_i)$ $(i=1,2,\cdots, n)$ are mutually disjoint.

The region of a pseudo-O2-sphere basis (S_*,S'_*) in Σ of genus n is a smooth 4-manifold $\Omega(S_*,S'_*)$ in Σ obtained from the 4-manifolds $N(S_i,S'_i)$ $(i=1,2,\cdots, n)$ by connecting them by mutually disjoint 1-handles h^1_j $(j=1,2, \cdots, n-1)$ in Σ.

Since Σ is a simply connected 4-manifold, the region $\Omega(S_*,S'_*)$ in Σ does not depend on any choices of h^1_j $(j=1,2, \cdots, n-1)$ and is uniquely determined by the pseudo-O2-sphere basis (S_*,S'_*) up to isotopies of Σ.

The <u>residual region</u>

$\Omega^c(S_*,S'_*) = cl(\Sigma - \Omega(S_*,S'_*))$

of the region $\Omega(S_*,S'_*)$ in Σ is always a homotopy 4-ball.

An <u>O2-sphere basis</u> of Σ is a pseudo-O2-sphere basis (S_*,S'_*) of Σ such that the residual region $\Omega^c(S_*,S'_*)$ is diffeomorphic to the 4-ball. The following result is a main result.

<u>Theorem B</u>. For any two pseudo-O2-sphere bases (R_*,R'_*) and (S_*,S'_*) of Σ, there is an orientation-preserving diffeomorphism $h:\Sigma \to \Sigma$ sending (R_i,R'_i) to (S_i,S'_i) for all i $(i=1,2,\ldots,n)$.

The stable 4-sphere Σ admits an O2-sphere basis.
If (R_*,R'_*) is an O2-sphere basis of Σ and (S_*,S'_*) is the image of (R_*,R'_*) by an orientation-preserving diffeomorphism $f:\Sigma \to \Sigma$, then (S_*,S'_*) is also an O2-sphere basis.
Thus, the following corollary is directly obtained from Theorem B.

Corollary C. Every pseudo-O2-sphere basis of Σ is an O2-sphere basis of Σ.

Corollary C implies Theorem A:

Theorem A. Every smooth homotopy 4-sphere is diffeomorphic to the 4-sphere.

Proof: Let M be a smooth homotopy 4-sphere . By Wall, it is known that there is an orientation-preserving diffeomorphism

$$\kappa : M\# \Sigma \to \Sigma$$

for some n. Let $M\#\Sigma$ be the union $M^{(0)} \cup \Sigma^{(0)}$. Corollary C means that the image $\kappa(\Sigma^{(0)})$ is the region $\Omega(S_*, S'_*)$ of an O2-sphere basis (S_*, S'_*) of Σ. Hence $\Omega^c(S_*, S'_*) = \kappa(M^{(0)})$ is a 4-ball and hence $M^{(0)}$ is the 4-ball D^4. By $\Gamma_4=0$ by Cerf or $\pi_0(\mathrm{Diff}^+(S^3))=0$ by Hatcher, the diffeomorphism $M^{(0)} \to D^4$ extends to a diffeomorphism $M \to S^4$. //

How to prove Theorem B.

Idea: Use the concept of an O2-handle pair in the paper:

(1) A. Kawauchi, Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link.
(2) A. Kawauchi, Supplement to ribbonness of a stable-ribbon surface-link, I: A revised proof of uniqueness of an orthogonal 2-handle pair on a surface-link.

Both (1) and (2) are in arxiv:1804.02654v11

Let F be a surface-knot F in S^4.

An <u>O2-handle pair</u> $(D\times I, D'\times I)$ on a surface-knot F in S^4 is a pair of 2-handles $D\times I$, $D'\times I$ on F embedded in S^4 such that

(i) $D\times I$ and $D'\times I$ meet F only at the attaching annuli $(\partial D)\times I$ and $(\partial D')\times I$,
(ii) ∂D and $\partial D'$ meet transversely at just one point q for a core disk pair (D,D') of $(D\times I, D'\times I)$,
(iii) $(\partial D)\times I\cap(\partial D')\times I$ is the square $Q=\{q\}\times I\times I$.

An O2-handle basis of a surface-knot F of genus n in S^4 is a system $(D_* \times I, D'_* \times I)$ of n mutually disjoint O2-handle pairs $(D_i \times I, D'_i \times I)$ $(i=1,2,\ldots, n)$ on F in S^4.

Let $p: S^4(F)_2 \to S^4$ be the double branched covering projection branched along a surface-knot F in S^4.
Let α be the non-trivial covering involution of the double branched covering space $S^4(F)_2$.

The preimage $p^{-1}(F)$ in Σ of F which is the fixed point set of α and diffeomorphic to F is also written by the same notation as F in Σ.

The following result is a standard result.

Lifting Lemma.
For a standard O2-handle basis $(D_* \times I, D'_* \times I)$ of a trivial surface-knot F of genus n in S^4, there is an orientation-preserving diffeomorphism f: $S^4(F)_2 \to \Sigma$ sending the 2-sphere pair system

$$(S(D_*),S(D'_*)) = \{ (S(D_i),S(D'_i)) \mid i=1,2, \ldots ,n \}$$

to the standard O2-sphere basis $(S^2 \times 1_*, 1 \times S^2_*)$ of the stable 4-sphere Σ of genus n.
In particular, the 2-sphere pair system $(S(D_*),S(D'_*))$ is an O2-sphere basis of Σ.

An <u>n-rooted disk family</u> is the triplet (d, d_*, b_*) where d is a disk, d_* is a system of n mutually disjoint disks $d_i (i=1,2,\ldots,n)$ in the interior of d and b_* is a system of n mutually disjoint bands $b_i (i=1,2,\ldots,n)$ in the n-punctured disk $cl(d-d_*)$ such that b_i spans an arc in the loop ∂d_i and an arc in the loop ∂d. Let b'_* denote the centerline system of the band system b_*.

Rooted family lemma. Let $(D_* \times I, D'_* \times I)$ be a standard O2-handle basis of a trivial surface-knot F of genus n in S^4, and (d, d_*, b_*) an n-rooted disk family. Then there is an embedding $\varphi : (d, d_*, b_*) \times I \to (S^4, D_* \times I, D'_* \times I)$ called a bump embedding such that

(1) $F = \partial V$ for the handlebody $V = \varphi(cl(d-d_*) \times I)$ of genus n,

(2) there is an identification

$$\varphi(d_* \times I, d_*) = (\varphi(d_*) \times I, \varphi(d_*)) = (D_* \times I, D_*)$$

as 2-handle systems on F, and

(3) there is an identification

$$\varphi(b_* \times I, b'_* \times I) = (D'_* \times I, D'_*)$$

as 2-handle systems on F.

For a bump embedding φ :$(d, d_*, b_*)\times I \to (S^4, D_*\times I, D'_*\times I)$,
there is an embedding $\tilde{\varphi}$:$d\times I \to S^4(F)_2$ with $p\tilde{\varphi}=\varphi$.
The images $\tilde{\varphi}(d_*\times I)$ and $\tilde{\varphi}(b_*\times I)$ are considered as 2-handle systems $\tilde{D}_*\times I$ and $\tilde{D}'_*\times I$ on F in $S^4(F)_2$ so that
$(\tilde{D}_*\times I, \tilde{D}'_*\times I)$ is an O2-handle basis of F in $S^4(F)_2$ with
$p(\tilde{D}_*\times I,\tilde{D}'_*\times I)=(D_*\times I, D'_*\times I)$.
The embedding $\tilde{\varphi}$:$(d, d_*,b_*)\times I \to (S^4(F)_2, \tilde{D}_*\times I, \tilde{D}'_*\times I)$
has $p\tilde{\varphi}=\varphi$ and is called a lifting bump embedding of
the bump embedding φ .
The 3-ball $\tilde{\varphi}(d\times I)$ and the handlebody $\tilde{\varphi}(cl(d\text{-}d_*)\times I)$ are
denoted by B and V in $S^4(F)_2$.

The composite embedding

$$\alpha\,\tilde{\varphi}:(d, d_*, b_*)\times I \to (S^4(F)_2, \alpha\tilde{D}_*\times I, \alpha\tilde{D}'_*\times I)$$

is another lifting bump embedding of the bump embedding φ.

Denote

$$\alpha\tilde{\varphi}(d\times I)= \alpha(B) \text{ by } \bar{B} \text{ and } \alpha\tilde{\varphi}(cl(d-d_*)\times I)=\alpha(V) \text{ by } \bar{V}.$$

Then

$$V\cap\bar{V} = B \cap \bar{B} = F \text{ in } S^4(F)_2 .$$

For an O2-handle basis $(D_*\times I, D'_*\times I)$ of a trivial surface-knot F in S^4, denote the lifting O2-handle bases of F in $S^4(F)_2$ are denoted as follows:

$$(\tilde{D}_*\times I, \tilde{D}'_*\times I) \text{ by } (D_*\times I, D'_*\times I), \text{ and}$$

$$(\alpha\tilde{D}_*\times I, \alpha\tilde{D}'_*\times I) \text{ by } (\bar{D}_*\times I, \bar{D}'_*\times I).$$

Note that $S(D_i)=D_i \cup \overline{D}_i$ and $S(D'_i)=D'_i \cup \overline{D}'_i$ are 2-spheres in $S^4(F)_2$ such that $(S(D_i),S(D'_i))$ is an O2-sphere pair in $S^4(F)_2$.

<u>Unique lifting bump embedding lemma.</u>

Let $\tilde{\varphi}:(d, d_*, b_*)\times I \to (\Sigma, D_*\times I, D'_*\times I)$ be a lifting bump embedding. Let $u:\Sigma^{(0)} \to \Sigma$ be an embedding.

Assume that the image $\tilde{\varphi}(d\times I)$ is in the interior of $\Sigma^{(0)}$ to define the composite $u\tilde{\varphi}:(d, d_*, b_*)\times I \to (\Sigma, uD_*\times I, uD'_*\times I)$.

Then there is a diffeomorphism $g:\Sigma \to \Sigma$ which is isotopic to the identity such that the composite embedding

$$gu\tilde{\varphi}:(d, d_*, b_*)\times I \to (\Sigma, guD_*\times I, guD'_*\times I)$$

is identical to the lifting bump embedding

$$\tilde{\varphi}:(d, d_*, b_*)\times I \to (\Sigma, D_*\times I, D'_*\times I).$$

In this lemma, since gu defines an embedding from $B \cup \overline{B}$ with $B \cap \overline{B} = F$ into Σ and we have $gu(B,F)=(B,F)$, the complement $gu(\overline{B})-F$ does not meet the bump 3-ball B, which means that any disk interior of the disk systems $gu\overline{D}_*$ and $gu\overline{D}'_*$ does not meet the bump 3-ball $B = \widetilde{\varphi}(d \times I)$.

Note: Unless $\Sigma^{(0)}$ and Σ have the same genus n, this property cannot be guaranteed.

Homological equivalence lemma.
For any pseudo-O2-sphere bases (R_*,R'_*) and (S_*,S'_*) of Σ of genus n, there is an α-invariant orientation-preserving diffeomorphism $\tilde{f}:\Sigma \to \Sigma$ which induces an isomorphism $\tilde{f}_*:H_2(\Sigma;Z) \to H_2(\Sigma;Z)$ such that

$$[\tilde{f}R_i]=[S_i] \text{ and } [\tilde{f}R_i']=[S'_i]$$

for all i.

Proof of Theorem B (outline).

Theorem B. For any two pseudo-O2-sphere bases (R_*,R'_*) and (S_*,S'_*) of Σ, there is an orientation-preserving diffeomorphism $h:\Sigma\to\Sigma$ sending (R_i,R'_i) to (S_i,S'_i) for all i $(i=1,2,\ldots,n)$.

It suffices to show this theorem when (R_*,R'_*) is an O2-sphere basis of Σ with $(R_*,R'_*)=(S(D_*),S(D'_*))$ for a standard O2-handle basis $(D_*\times I, D'_*\times I)$ of a trivial surface-knot F of genus n in S^4. Let $\Omega(S_*,S'_*)$ be the region of the pseudo-O2-sphere basis (S_*,S'_*) of Σ.

The 4-manifold obtained from $\Omega(S_*,S'_*)$ by adding a 4-ball D^4 in place of the residual region $\Omega^c(S_*,S'_*)$ is diffeomorphic to Σ. This means that there is an orientation-preserving embedding u: $\Sigma^{(0)} \to \Sigma$ such that

$$(uS(D_*),uS(D'_*))=(S_*,S'_*).$$

By Homological equivalence lemma, after applying an α-invariant orientation-preserving diffeomorphism $\tilde{f}:\Sigma \to \Sigma$, assume that the homology classes

$$[uS(D_i)]=[S_i] \text{ and } [uS(D'_i)]=[S'_i]$$

are identical to the homology classes

$$[R_i]=[S(D_i)] \text{ and } [R'_i]=[S(D'_i)]$$

for all i, respectively.

Let (B,V) be a bump pair of the O2-handle basis $(D_* \times I, D'_* \times I)$ of F in S^4. Recall that the two lifts of (B,V) to Σ under the double branched covering projection $p:S^4(F)_2 \to S^4$ are denoted by (B,V) and $(\bar{B},\bar{V})$.
For the proof of Theorem B, we provide with three lemmas.

<u>Lemma 1</u>. There is a diffeomorphism $g:\Sigma \to \Sigma$ which is isotopic to the identity such that the composite embedding $gu:\Sigma^{(0)} \to \Sigma$ preserves the bump pair (B,V) in Σ identically and has the property that every disk interior in the disk systems $gu\bar{D}_*$ and $gu\bar{D}'_*$ meets every disk in the disk systems $\bar{D}_*$ and $\bar{D}'_*$ only with the intersection number 0.

By Lemma 1, we can assume that the orientation-preserving embedding $u: \Sigma^{(0)} \to \Sigma$ sends the bump pair (B,V) to itself identically and has the property that every disk interior in the disk systems $u\bar{D}_*$ and $u\bar{D}'_*$ meets every disk in the disk systems $\bar{D}_*$ and $\bar{D}'_*$ only with intersection number 0.

Lemma 2. There is a diffeomorphism $g: \Sigma \to \Sigma$ which is isotopic to the identity such that the composite embedding $gu: \Sigma^{(0)} \to \Sigma$ sends the disk systems D_* and D'_* identically and the disk interiors of the disk systems $gu\bar{D}_*$, $gu\bar{D}'_*$ to be disjoint from the disk systems $\bar{D}_*$ and $\bar{D}'_*$ in Σ.

For the O2-sphere basis $(S(D_*),S(D'_*))$ of Σ, let

$$q_*=\{q_i=S(D_i)\cap S(D'_i)\mid i=1,2,\ldots,n\}$$

be the transverse intersection point system between $S(D_*)$ and $S(D'_*)$.

The diffeomorphism g of Σ in Lemma 2 is deformed so that the disks guD_i and D_i are separated, and then the disks guD'_i and D'_i are separated while leaving the transverse intersection point q_i. By this deformation, we obtain a pseudo-O2-sphere basis $(guS(D_*), guS(D'_*))$ of Σ which meets the O2-sphere basis $(S(D_*),S(D'_*))$ at just the transverse intersection point system q_*.

Next, the diffeomorphism g of Σ is deformed so that a disk neighborhood system of q_* in guS(D_*) and a disk neighborhood system of q_* in S(D_*) are matched, and then a disk neighborhood system of q_* in guS(D'_*) and a disk neighborhood system of q_* in S(D'_*) are matched.
Thus, there is a diffeomorphism g:Σ → Σ which is isotopic to the identity such that the meeting part of the pseudo-O2-sphere basis (guS(D_*), guS(D'_*)) and the O2-sphere basis (S(D_*),S(D'_*)) is just a disk neighborhood pair system (d_*,d'_*) around the transverse intersection point system q_*.

Now, assume that for an embedding $u:\Sigma^{(0)} \to \Sigma$,
the meeting part of the pseudo-O2-sphere basis
$(uS(D_*),uS(D'_*))$ and the O2-sphere basis $(S(D_*),S(D'_*))$
is just a disk neighborhood pair system (d_*,d'_*) of q_*.

<u>Lemma 3</u>. There is an orientation-preserving diffeomorphism
$h: \Sigma \to \Sigma$ such that the composite embedding $hu:\Sigma^{(0)} \to \Sigma$
preserves the O2-sphere basis $(S(D_*),S(D'_*))$ identically.

Since $(uS(D_*),uS(D'_*))=(S_*,S'_*)$ and $(R_*,R'_*)=(S(D_*),S(D'_*))$,
the proof of Theorem B is completed by Lemma 3.//

The proof of Lemma 3.

A 4D solid torus is a 4-manifold $Y = S^1 \times D^3$ in S^4.
A fiber circle of ∂Y is a fiber of the S^1-bundle $\partial Y = S^1 \times S^2$.

Let $Y^c = cl(S^4 - Y)$.

Let Y_* be a system of mutually disjoint 4D solid tori Y_i, $(i=1,2,\cdots, n)$ in S^4, and $Y^c{}_*$ the system of the 4-manifolds $Y^c{}_i$ $(i=1,2,\cdots, n)$. Let $\cap Y^c{}_* = \cap_{i=1}^{n} Y^c{}_i$.

The following lemma is used for the proof of Lemma 3.

Framed light-bulb diffeomorphism lemma.
Let $D_* \times I$ be a system of mutually disjoint framed disks $D_i \times I$ ($i=1,2,\ldots, n$) in $\cap Y^c_*$ such that ∂D_i is a fiber circle of ∂Y_i and

$$(D_* \times I) \cap \partial Y^c_i = (\partial D_i) \times I$$

for all i. If $E_* \times I$ is any system of mutually disjoint framed disks $E_i \times I$ ($i=1,2,\cdots, n$) in $\cap Y^c_*$ such that

$$(E_* \times I) \cap \partial Y^c_i = (\partial E_i) \times I = (\partial D_i) \times I$$

for all i, then there is an orientation-preserving diffeomorphism $h: S^4 \to S^4$ sending Y_* identically such that $h(D_* \times I, D_*) = (E_* \times I, E_*)$.

Final Note:

In the paper:

A. Kawauchi, Smooth homotopy 4-sphere.arxiv:1911.11904v4,

it is also shown by using Framed light-bulb isotopy lemma due to D. Gabai that:

Every orientation-preserving diffeomorphism of the stable 4-sphere Σ of genus n is smoothly concordant (or piecewise-linearly isotopic) to an α-equivariant diffeomorphism of Σ.