On quantum character varieties of knots

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Outline

Introduction

Main Idea

Algebra of free ribbons on a punctured disk $\mathcal{F}_k$

$\downarrow$

Action of bottom tangle $T_b - id$ to $\mathcal{F}_k$

$\downarrow$

Ideal I_b generated by $\mathrm{Im}(T_b - id)$

$\downarrow$

Space of reprelsentations $\mathcal{A}_b = \mathcal{F}_k / I_b$

$\downarrow$

Insert skein relation $\mathcal{S}_k = \mathcal{F}/\sim$

$\downarrow$

$\widetilde{\mathcal{A}}_{b,0}$ acts on $\widetilde{\mathcal{A}}_{b,1}$

$\downarrow$

Investigate the structure of $\widetilde{\mathcal{A}}_{b,1}$ as an $\widetilde{\mathcal{A}}_{b,0}$-module

$\downarrow$

Elementary ideals of the relation matrix $\rightarrow$ $\boxed{\det(T_b - id)}$

1. Algebra of free ribbons

1.1 Free ribbon in a thickened punctured disk

K : $\mathbb{C}(t)$

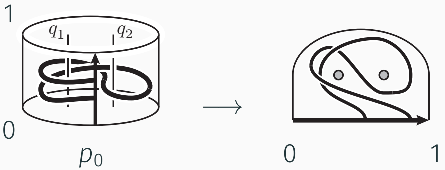
D_k : a closed disk

$q_1, \cdots, q_k$: punctures inside D_k

p_1 : a puncture on the boundary of D_k

p_0 $(\neq p_1)$: a point of ∂D_k (base point).

A ribbon in the thickened disk with punctures :



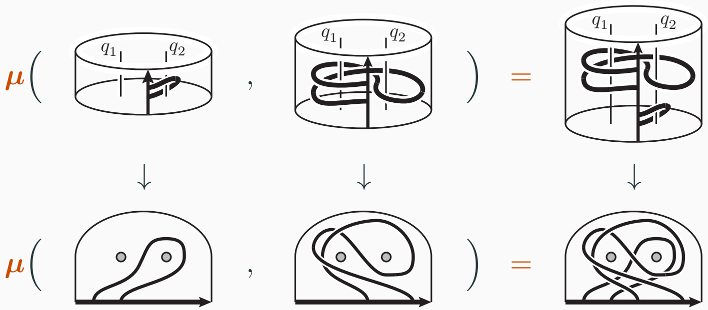
$D_2 \times [0,1]$ diagram on D_2

$\mathcal{F}_{k,1}$: K-linear combinations of the isotopy classes of ribbons in D_k.

1.2 Algebra of free ribbons

Multiplication : stacking two thickened disks with punctures

$$\boldsymbol{\mu} : \mathcal{F}_{k,n_1} \times \mathcal{F}_{k,n_2} \to \mathcal{F}_{k,n_1+n_2}$$



Product : (another mjultiplication) connect two adjacent end points of two ribbons

$$m : \mathcal{F}_{k,1} \times \mathcal{F}_{k,1} \to \mathcal{F}_{k,1}$$

2. Acton of bottom tangles

2.1 Bottom tangle action on $\mathcal{F}_{k,n}$

Definition

Let $\mathcal{T}_{k,n}$ be the subspace of $\mathcal{F}_{k,n}$, which consists of non-closed free arcs $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_n)$ such that the heights of their end points $h(\gamma_i(0))$ and $h(\gamma_i(1))$ satisfy

$$h(\gamma_1(1)) < h(\gamma_1(0)) < h(\gamma_2(1)) < \cdots < h(\gamma_n(1)) < h(\gamma_n(0)).$$

Then an element of $\mathcal{T}_{k,n}$ is called **a bottom tangle** of type (k, n).

For $T \in \mathcal{T}_{k,\ell}$ and $F \in \mathcal{F}_{\ell,n}$, the composition $T \circ F \in \mathcal{F}_{k,n}$ is defined by glueing the handles of F to the ribbons of T as follows.

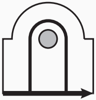
$T \in \mathcal{T}_{2,2}$ $\quad$ $F \in \mathcal{F}_{2,2}$

The composition of a bottom tangle $T \in \mathcal{T}_{k,\ell}$ and an element $F \in \mathcal{F}_{\ell,n}$ of the algebra of free ribbons in the case $k = n = \ell = 2$.

2.2 Braided Hopf algebra structure of bottom tangles

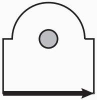
A braided Hopf algebra structure is given to bottom tangles as follows (Habiro).

$\mathbf{1}$
identity for tensor



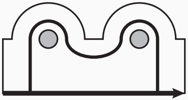
id
identity for composition

μ
multiplication



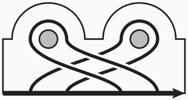
η
unit

S
antipode



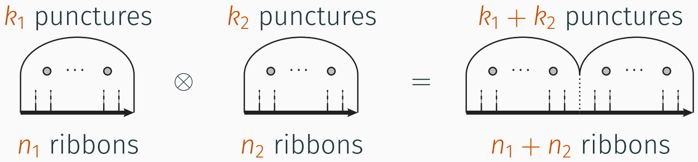
Δ
coproduct

ε
counit



ψ
braiding

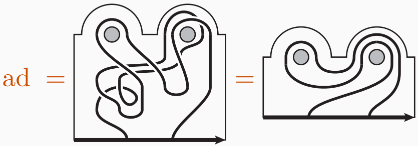
Tensor product $\otimes$ **:** $\mathcal{F}_{k_1,n_1} \otimes \mathcal{F}_{k_2,n_2} \to \mathcal{F}_{k_1+k_2,n_1+n_2}$



Multiplication $\boldsymbol{\mu}$ **:** $\mathcal{F}_{k,n_1} \otimes \mathcal{F}_{k,n_2} \to \mathcal{F}_{k,n_1+n_2}$

$$\boldsymbol{\mu} = \underbrace{(\mu \otimes \cdots \otimes \mu)}_{k} \circ \Psi_{2k-2} \circ (\Psi_{2k-4} \circ \Psi_{2k-3}) \circ \cdots \circ (\Psi_4 \circ \Psi_5 \circ \cdots \circ \Psi_{k+1}) \circ (\Psi_2 \circ \Psi_3 \circ \cdots \circ \Psi_k)$$

Adjoint ad **:** $\mathrm{ad} = \mu_2 \circ \Psi_1 \circ (S \otimes \Delta) \circ \Delta \in \mathcal{T}_{2,1}$.

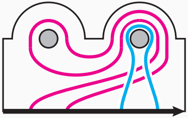


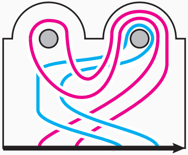
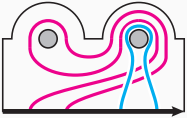
Proposition (braided commutativity)

$$\mu_2 \circ (\mathrm{ad} \otimes id) = \mu_2 \circ \Psi_1 \circ (id \otimes \mathrm{ad}) \circ \Psi \in \mathcal{T}_{2,2}.$$

where $\mu_2 = id \otimes \mu$ *and* $\Psi_1 = \Psi \otimes id$.

Proof.

$\mu_2 \circ (\mathrm{ad} \otimes id) =$  ,

$\mu_2 \circ \Psi_1 \circ (id \otimes \mathrm{ad}) \circ \Psi =$  $=$  .

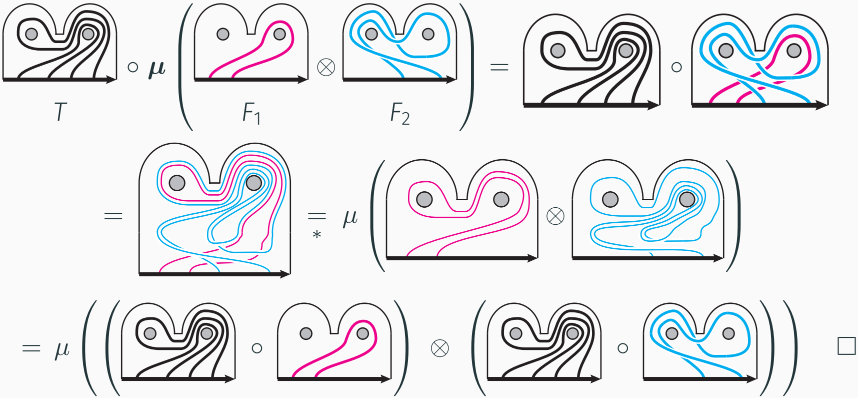
□

2.5 Flat bottom tangle

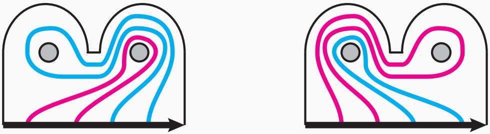
Definition. $\mathcal{T}^F_{k,n} := \{T \mid \text{flat bottom tangle}\}$
Flat bottom tangle $\Leftrightarrow$ $^\exists$projection without crossings.

Proposition. *$T \in \mathcal{T}^F_{k,n}$ commutes with the multiplication*
$\boldsymbol{\mu} : \mathcal{F}_{n,l_1} \otimes \mathcal{F}_{n,l_2} \to \mathcal{F}_{n,l_1+l_2}$.

Proof.



3. Universal representation space



T_σ $T_{\sigma^{-1}}$

Explain the braid action by bottom tangles.

$$T_\sigma = \mu_2 \circ \Psi_1 \circ (id \otimes \mathrm{ad}),$$
$$T_{\sigma^{-1}} = \mu_1 \circ \Psi_1^{-1} \circ \Psi_2^{-1} \circ \Psi_1^{-1} \circ S_2^{-1} \circ (\mathrm{ad} \otimes id).$$

T_σ and $T_{\sigma^{-1}}$ are flat bottom tangles, so they commute with $\boldsymbol{\mu}$ and are algebra automorphisms of $\mathcal{F}_k = \oplus_{n=0,1,2,\dots}\mathcal{F}_{k,n}$.

3.2 Universal representation space

L : a knot

b : a braid in B_k s.t. $\hat{b}$ is isotopic to L

T_b : the bottom tangle corresponding to b

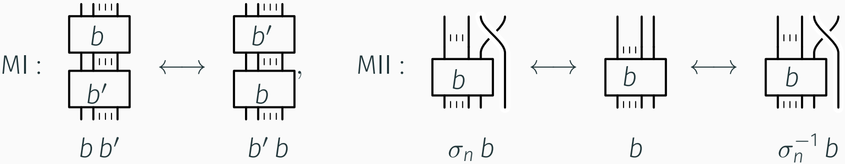
I_b : the **ideal** generated by the image of $T_b - id^{\otimes k}$

$$I_b = \mathrm{Im}\left(\boldsymbol{\mu} \circ (id^{\otimes k} \otimes (T_b - id^{\otimes n}))\right) \quad \left(= \mathrm{Im}\left(\boldsymbol{\mu} \circ ((T_b - id^{\otimes n}) \otimes id^{\otimes k})\right)\right)$$

$\mathcal{A}_b : = \mathcal{F}_k / I_b$

Theorem. If the closures of two braids b_1 and b_2 are isotopic, then $\mathcal{A}_{b_1}$ and $\mathcal{A}_{b_2}$ are isomorphic as graded rings.

Proof. Use the Morkov move.



4. Skein algebra of a punctured disk

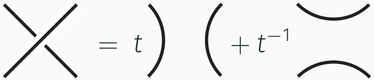
4.1 Skein algebra $\mathcal{S}_k$

The skein module $\mathcal{S}_{k,n}$ is defined by

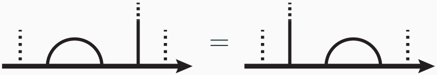
$$\mathcal{S}_{k,n} = \mathcal{F}_{k,n}/\sim$$

where $\sim$ is generated by the following two relations.

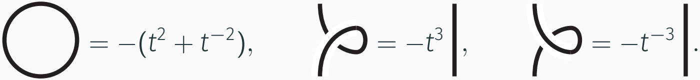
Kauffman bracket skein relation :



Boundary parallel relation :



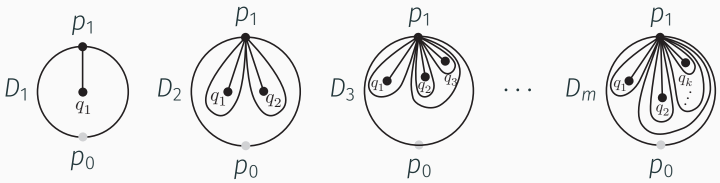
The Kauffman bracket skein relation relations implies that



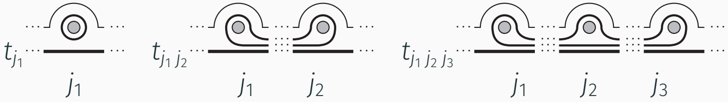
Let $\mathcal{S}_k = \oplus_{n=0,1,\dots} \mathcal{S}_{k,n}$.

4.2 Structure of $\mathcal{S}_k$

- As a K-linear space, $\mathcal{S}_{k,n}$ is spanned by $\mathcal{T}^F_{k,n}$ (flat bottom tangles).
- Standard triangular decomposition of D_k.



- Flat bottom tangle $\underset{1:1}{\longleftrightarrow}$ numbers of intersection points at edges.
- $\mathcal{S}_k$ is a $\mathcal{S}_{k,0}$-(two-sided) module.
- $\mathcal{S}_{k,0}$ is a K algebra generated by $t_{j_1 \cdots j_m}$ $(j_1 < \cdots < j_m, m \leq 3)$.
  (classical case by Bullok)



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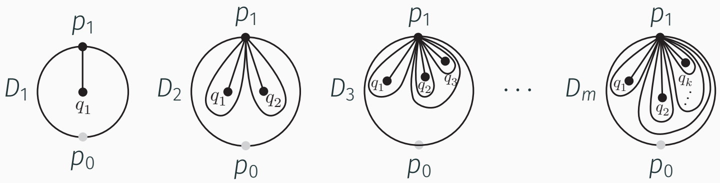

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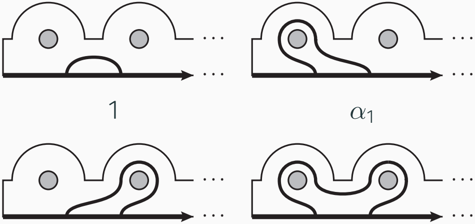
- $\mathcal{S}_{k,0}$ is a K algebra generated by $t_{j_1\cdots j_m}$ $(j_1 < \cdots < j_m, m \leq 3)$.
  (classical case by Bullok)
- $\mathcal{S}_{k,0}$ is an integral domain.
  (by Diamond lemma or Buchberger algorithm)

4.3 Structure of $\mathcal{S}_k$ (cont.)

- Let $\widetilde{\mathcal{S}}_{k,0} = \mathcal{S}_{k,0}[t^{-1}_{j_1\ldots j_m}]$ and $\widetilde{\mathcal{S}}_{k,n} = \widetilde{\mathcal{S}}_{k,0} \otimes_{\mathcal{S}_{k,0}} \mathcal{S}_{k,n}$.
- $\widetilde{\mathcal{S}}_{k,0}$ is generated by $t_{j\ldots j+m}$ $(m = 0, 1, 2)$.
- Let $1 = \eta^{\otimes k} \circ \varepsilon$, $\alpha_i = \eta^{\otimes(i-1)} \otimes id \otimes \eta^{\otimes(n-i)}$. Then $\widetilde{\mathcal{S}}_{k,1}$ is a $\widetilde{\mathcal{S}}_{k,0}$ algebra spanned by 1, α_1 if $k = 1$ and 1, α_1, α_2, $\alpha_1\,\alpha_2 = m(\alpha_1 \otimes \alpha_2)$ if $k \geq 2$.



1 $\quad$ α_1 $\quad$ α_2 $\quad$ $\alpha_1\,\alpha_2$

- $\boldsymbol{\mu} : \mathcal{S}_{k,1}^{\otimes n} \to \mathcal{S}_{k,n}$ is surjective.

4.4 Action of braids

Let L be a knot, $b \in B_k$ is a braid whose closure is isotopic to L, and $\widetilde{I}_b = \widetilde{\mathcal{S}}_k \otimes_{\mathcal{S}_k} I_b$.

Definition. Let $\widetilde{I}_b = \widetilde{\mathcal{S}}_k \otimes_{\mathcal{S}_k} (I_b/\sim)$, $\widetilde{I}_{b,n} = \widetilde{\mathcal{S}}_k \otimes_{\mathcal{S}_k} (I_b \cap \mathcal{F}_{k,n}/\sim)$,
$\widetilde{\mathcal{A}}_b = \widetilde{\mathcal{S}}_k/\widetilde{I}_b$, $\widetilde{\mathcal{A}}_{b,n} = \widetilde{\mathcal{S}}_{k,n}/\widetilde{I}_{b,n}$.
$\widetilde{\mathcal{A}}_b$: **the space of quantum $SL(2)$ representations** of L.

Proposition. The ideal $\widetilde{I}_{b,1}$ is generated by $T_b(\alpha_1) - \alpha_1, \cdots,$ $T_b(\alpha_{k-1}) - \alpha_{k-1}$ as a left $\mathcal{S}_{k,0}$-module.

Remark. By definition, $\widetilde{I}_{b,1}$ is generated by $T_b(x) - x$ for all $x \in \mathcal{F}_{k,1}$. But $x = \alpha_1, \cdots, \alpha_{k-1}$ are good enough.

Remark. α_i $(i > 2)$ is explained as a linear combination of $1, \alpha_1, \alpha_2, \alpha_1\alpha_2$ with coefficients in $\widetilde{\mathcal{S}}_{k,0}$.

5. Quantum character varitety

5.1 Structure of $\widetilde{\mathcal{A}}_{b,0}$ and $\widetilde{\mathcal{A}}_{b,1}$

From now on, we consider the case that $k = 2$ (# of the punctures).

- $T_b(\alpha_j) - \alpha_j$ induces $\widetilde{\mathcal{S}}_{2,0}$ algebra endomorphisms on $\widetilde{\mathcal{S}}_{2,0}$ and $\widetilde{\mathcal{S}}_{2,1}$.
- $\widetilde{\mathcal{A}}_{b,0} = \widetilde{\mathcal{S}}_{2,0}/\widetilde{I}_{b,1}$. If L is a one-component knot, $\bar{t}_2 = \bar{t}_1 \in \widetilde{\mathcal{A}}_{2,0}$.
- $\widetilde{\mathcal{A}}_{b,1} = \widetilde{\mathcal{S}}_{2,n}/\widetilde{I}_{b,1}$ where $\widetilde{I}_{b,1}$ is spanned by the images of $T_b(\alpha_1) - \alpha_1$.
- $M_b \in M_4(\widetilde{\mathcal{A}}_{2,0})$: matrix representing the right action of $T_b(\alpha_1) - \alpha_1$ with respect to the basis $1, \alpha_1, \alpha_2$ and $\alpha_1\alpha_2$.
- M_b is the matrix for the relations of $\widetilde{\mathcal{A}}_{b,1}$ as an $\widetilde{\mathcal{A}}_{2,0}$ module.

Theorem. *The elementary ideals of M_b are invariants of L.*

Corollary. $\det M_b$ *(product of the elementary divisors) is an invariant of L.*

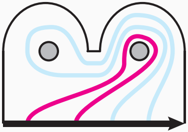
By putting $A = -1$, we can recover the classical case.

$k = 2$ (number of the punctures)

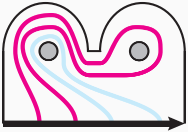
Definition. The **quantum character variety** of L is the algebraic variety defined by $\sqrt{\det M_b}$ (radical).

Reason. By putting $A = -1$, $\sqrt{\det M_b}$ coincides (?) with the classical character variety of L.

Braid action. $T_\sigma(\alpha_1) = \alpha_2, \quad T_{\sigma^{-1}}(\alpha_1) = \alpha_1\,\alpha_2\,\alpha_1^{-1}$
$T_\sigma(\alpha_2) = \alpha_2^{-1}\,\alpha_1\,\alpha_2, \quad T_{\sigma^{-1}}(\alpha_2) = \alpha_1$



T_σ



$T_{\sigma^{-1}}$

6. Examples

The matrices N_1 and N_2 corresponding to the right actions of α_1 and α_2 are given as follows.

$$N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -t^4 & -t^2\,t_1 & 0 & 0 \\ -t^4\,t_1\,t_2 - t^6\,t_{12} & -t^2\,t_2 & -t^2\,t_1 & -t^4 \\ t^2\,t_2 & -t^2\,t_{12} & 1 & 0 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -t^4 & 0 & -t^2\,t_2 & 0 \\ 0 & -t^4 & 0 & -t^2\,t_{12} \end{pmatrix},$$

To compute $T_b(\alpha_1)$, express $T_b(\alpha_1)$ as a word of α and α_2 and then compute the matrix corresponding to this word.

$b = \sigma_1^3, \quad T_b(\alpha_1) = \alpha_1^{-1}\,\alpha_2^{-1}\,\alpha_1^{-1}\,\alpha_2\,\alpha_1\,\alpha_2,$
$X = t_1 = t_2, \quad Y = t_{12},$

$$\begin{aligned}&\det(N_1^{-1}\,N_2^{-1}\,N_1^{-1}\,N_2\,N_1\,N_2 - I_4) = \\ &\quad \frac{1}{t^{12}}, \left(A^4\,Y^2 + (t^6 + t^2)(X^2 - 2)\,Y + X^4 - 4\,t^4\,X^2 + t^8 + 2\,t^4 + 1\right) \\ &\qquad\qquad \left(t^4\,Y^2 + (t^6 + t^2)\,Y + t^8 - t^4 + 1\right)^2.\end{aligned}$$

By putting $t = -1$, we have

$$\left(Y + X^2 - 2\right)^2\,(Y + 1)^4.$$

6.3 Figure eight knot

T_b − identity $\longrightarrow \quad \alpha_1\,\alpha_2^{-1}\,\alpha_1^{-1}\,\alpha_2 - \alpha_2\,\alpha_1\alpha_2^{-1}\,\alpha_1^{-1}\,\alpha_2\,\alpha_1^{-1},$
$X = t_1 = t_2, \quad Y = t_{12},$

$$\begin{gathered}
\det(N_1\,N_2^{-1}\,N_1^{-1}\,N_2 - N_2\,N_1N_2^{-1}\,N_1^{-1}\,N_2\,N_1^{-1}) = \\
\left(t^4\,Y^2 + (t^6+t^2)\,(X^2-2)\,Y + X^4 - 4\,t^4\,X^2 + t^8 + 2\,t^4 + 1\right) \\
\Big(t^8\,Y^4 + (t^6+t^{10})\,(X^2+1)\,Y^3 + \\
(t^8\,X^4 + 2\,(t^{12}+t^8+t^4)\,X^2 + t^4 - 3\,t^8 + t^{12})\,Y^2 + \\
(2\,(t^6+t^{10})\,X^4 + (t^2+t^{14})\,X^2 + t^2 - 2\,t^6 - 2\,t^{10} + t^{14})\,Y + \\
+(t^4+2\,t^8+t^{12})\,X^4 - 2\,(t^4+t^{12})\,X^2 + 1 - t^4 + t^8 - t^{12} + t^{16}\Big)^2 .
\end{gathered}$$

By putting $t = -1$, we have

$$(Y + X^2 - 2)^2\,(Y^2 + (X^2+1)\,Y + 2\,X^2 - 1)^4.$$

7. Problems

Problems

1. Does the quantum character ring split into abelian and non-abelian factors?
2. Is the non-abelian part irreducible?
3. How to relate to the $\hat{A}$ polynomial?
4. What is the geometry of the quantum character variety?