Floer K-theory for knots

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Intelligence of Low-dimensional Topology
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Overview in one slide

Today we consider **Seiberg–Witten** Floer *K*-theory (which is only existing Floer *K*-theory in low-dimensional topology for now)

$$\begin{tikzcd}
\text{Floer } K\text{-theory for 3-mfds with involution} \arrow[r, rightsquigarrow, "\text{through double branched cover}"] \arrow[d, rightsquigarrow, "\text{4-dimensional cobordism}"'] & \text{Floer } K\text{-theory for knots} \arrow[d, rightsquigarrow, "\text{cobordism between knots}"] \\
\text{10/8-inequality for 4-mfds with } \partial \text{ \& invol.} \arrow[r, rightsquigarrow, "\text{through double branched cover}"'] & \text{“10/8-inequality for knots”}
\end{tikzcd}$$

- 10/8-inequality: Constraint on **smooth** spin 4-mfds from SW *K*-theory (originally given by Furuta for closed 4-manifolds)
- Our “10/8-inequality for knots” detects difference between topological & smooth categories in 4D aspects of knot theory.
- Applications: stablizing numbers, relative genus bounds

Outlines

1. Applications: Stabilizing numbers and relative genus bounds
2. Floer K-theory for involutions, and for knots

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Outcomes of our framework in knot theory

Two applications of our framework to 4D aspects of knot theory:

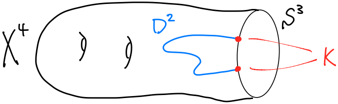
1. Topological stabilizing number vs. Smooth stabilizing number
2. Relative genus bounds

Toward stabilizing number: H-sliceness

Today we consider only *oriented* knots in S^3.

Definition

A knot K is *smoothly H-slice in a closed 4-manifold X* if K bounds a null-homologus smoothly and properly embedded disk in $X \setminus \mathrm{int} D^4$.



The *topological H-sliceness* is also defined by considering locally flat topological embeddings of disks.

Basic Question

Given a knot K and X^4, is K smoothly/topologically H-slice in X?

A quantitative question of this kind $\rightsquigarrow$ *stablizing number*

Stablizing number for a knot

- $\forall K$ is C^∞-slice in $S^2 \times S^2$ (Norman 1969), but not H-slice in general.
- But $\forall K$ is H-slice in $\#_N S^2 \times S^2$ for $N \gg 0$, if $\mathrm{Arf}(K) = 0$.
  (Cochran–Orr–Teichner (2003), Schneiderman (2010))

The *smooth/topological stabilizing numbers* are defined by

$$\mathrm{sn}(K) := \min \left\{ N \geq 0 \;\middle|\; K \text{ is smoothly H-slice in } \#_N S^2 \times S^2 \right\},$$

$$\mathrm{sn}^{\mathrm{Top}}(K) := \min \left\{ N \geq 0 \;\middle|\; K \text{ is topologically H-slice in } \#_N S^2 \times S^2 \right\}$$

for a knot K with $\mathrm{Arf}(K) = 0$.

By definition, we have $\mathrm{sn}^{\mathrm{Top}}(K) \leq \mathrm{sn}(K)$.

Question: Conway–Nagel (2020)

Is there a knot K with $\mathrm{Arf}(K) = 0$ such that

$$0 < \mathrm{sn}^{\mathrm{Top}}(K) < \mathrm{sn}(K) \quad ?$$

Our result: the affirmative answer to this question, and more:

Stablizing number: C^0 vs. C^∞

Reminder: Question by Conway–Nagel (2020)

Is there a knot K with $\mathrm{Arf}(K) = 0$ such that

$$0 < \mathrm{sn}^{\mathrm{Top}}(K) < \mathrm{sn}(K) \quad ?$$

Our result: the affirmative answer to this question, and more:

Theorem (K.–Miyazawa–Taniguchi (2021))

There exists a knot K with $\mathrm{Arf}(K) = 0$ such that

- We have $0 < \mathrm{sn}^{\mathrm{Top}}(K) < \mathrm{sn}(K)$,
- $\mathrm{sn}^{\mathrm{Top}}(\#_n K) > 0$ for all $n > 0$, and

$$\lim_{n \to \infty} (\mathrm{sn}(\#_n K) - \mathrm{sn}^{\mathrm{Top}}(\#_n K)) = \infty.$$

Concretely, $K = T(3, 11)$ (and some other torus knots) satisfies the above properties.

Outcomes of our framework in knot theory

Two applications of our framework to 4D aspects of knot theory:

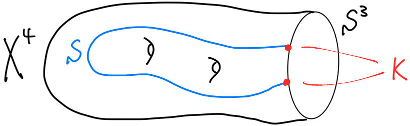
1. Topological stabilizing number vs. Smooth stabilizing number
2. Relative genus bounds

4-manifold genus of a knot

- The *4-genus* or *slice genus* of a knot K is defined as the minimal genus of surfaces bounded by K in D^4. This is a classical invariant of knots (1962, Fox).
- A natural generalization of 4-genus is defined by replacing D^4 with a given 4-manifold:

Definition: 4-manifold genus of a knot

K : knot, X : closed 4-mfd, $\alpha \in H_2(X;\mathbb{Z}) \cong H_2(X \setminus \mathrm{int} D^4, S^3;\mathbb{Z})$
$g_{X,\alpha}(K)$: min of genus of an (oriented, cpt, properly and) smoothly emb. surface S in $X \setminus \mathrm{int} D^4$ with $\partial S = K, [S, \partial S] = \alpha$
$g^{\mathrm{Top}}_{X,\alpha}(K) \rightsquigarrow$ defined by locally flat topologically embedded surfaces



Big difference between topological and smooth 4-genera

Reminder: Definition of 4-manifold genus of a knot

K : knot in S^3, X : closed 4-manifold, $\alpha \in H_2(X;\mathbb{Z})$

$g_{X,\alpha}(K)$: min of genus of an (oriented, cpt, properly and) smoothly emb. surface S in $X \setminus \mathrm{int} D^4$ with $\partial S = K, [S, \partial S] = \alpha$

- $g_{S^4,0}(K) =$(4-genus of K)
- Study of $g_{X,\alpha}(U)$ = minimal genus problem for closed surfaces (a classical problem in 4D topology)
- Many known results on bounds on $g_{X,\alpha}$ and $g^{\mathrm{Top}}_{X,\alpha}$

Remark: Big difference between topological and smooth 4-genera

$$\lim_{n\to\infty} \left(g_{S^4,0}(K_n) - g^{\mathrm{Top}}_{S^4,0}(K_n) \right) = \infty \quad \text{for} \quad K_n = T(3, 12n-1)$$

(Follows from the solution to the Milnor conjecture by Kronheimer–Mrowka (1993), and upper bounds on g^{Top} by Baader–Banfield–Lewark (2020))

Big difference between 4-manifold genera

Remark: Big difference between topological and smooth 4-genera

$$\lim_{n\to\infty}\left(g_{S^4,0}(K_n) - g^{\mathrm{Top}}_{S^4,0}(K_n)\right) = \infty \quad \text{for} \quad K_n = T(3, 12n-1)$$

Instead of $(S^4, 0)$, the same claim holds for a **negative-definite** X and every $\alpha \in H_2(X;\mathbb{Z})$ (using the τ-invariant by Ozváth–Szabó)

Our result: Find a big difference also for indefinite X

Theorem (K.–Miyazawa–Taniguchi (2021))

There exists a knot K' with the following property:
$\forall X$: oriented closed smooth 4-manifold with $H_1(X;\mathbb{Z}) = 0$,
$\forall \alpha \in H_2(X;\mathbb{Z})$ with $2|\alpha$ and $\alpha/2 = PD(w_2(X)) \mod 2$,
$\forall K$: knot,

$$\lim_{n\to\infty}\left(g_{X,\alpha}(K\#(\#_n K')) - g^{\mathrm{Top}}_{X,\alpha}(K\#(\#_n K'))\right) = \infty$$

e.g. $K' = T(3, 11)$ (and some other torus knots) is the case.

Summary of applications to knots

Two applications of our framework to 4D aspects of knot theory:

1. Topological stabilizing number vs. Smooth stabilizing number
   $\cdots$ we proved these two notions are essentially different.
2. Relative genus bounds
   $\cdots$ we showed $g^{\mathrm{Top}}_{X,\alpha}$ and $g_{X,\alpha}$ have a big difference for all X with $H_1(X;\mathbb{Z}) = 0$, without any restriction on the intersection form.

Outlines

Recall: Overview

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- 10/8-inequality: Constraint on **smooth** spin 4-mfds from SW *K*-theory (originally given by Furuta for closed 4-manifolds)
- Our “10/8-inequality for knots” detects difference between topological & smooth categories in 4D aspects of knot theory.
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Three Backgrounds

The 10/8-inequality is a constraint on **spin** smooth 4-manifolds from Seiberg–Witten *K*-theory

- The original one is due to Furuta (2001) for closed spin 4-manifolds.
- Manolescu (2014) extended Furuta's 10/8-inequality to spin 4-manifolds with ∂ using Seiberg–Witten Floer *K*-theory.
- On the other hand, Y. Kato (2022) gave a "with involution" version of the 10/8-inequality.

Our construction of Floer *K*-theory for involutions is a hybrid of Manolescu' construction and Kato's.

$$\begin{tikzcd}
\text{Closed 4-manifolds (Furuta)} \arrow[r, Rightarrow, "\substack{\text{extend to}\\ \text{with } \partial}"] \arrow[d, Rightarrow, "\text{with involution}"'] & \text{4-manifolds with } \partial \text{ (Manolescu)} \arrow[d, Rightarrow, "\text{with involution}"] \\
\text{Closed 4-manifolds with invol.(Kato)} \arrow[r, Rightarrow, "\substack{\text{extend to}\\ \text{with } \partial}"'] & \text{4-manifolds with } \partial \text{ and invol. (K–M–T)}
\end{tikzcd}$$

Furuta's 10/8-inequality

Theorem (Furuta (2001))

W: oriented closed spin smooth indefinite 4-manifold, then

$$\frac{5}{4}|\sigma(W)| + 2 \leq b_2(W).$$

- If $\sigma(W) \leq 0$, (above inequality) $\Leftrightarrow -\sigma(W)/8 + 1 \leq b^+(W)$, ($b^+(W)$: the max-dim of positive-def. subspaces of $H_2(W)$)
- This is a strong constraint on smooth *indefinite* 4-manifolds (complementary to Donaldson's diagonalization for definite 4-manifolds)
- The proof: Apply K-theory to a finite-dim. approximation of the SW equations (called the Bauer–Furuta invariant).
  - based on the compactness of the moduli space (feature of SW)
  - No alternative proof by another gauge theory (e.g. Yang–Mills, Heegaard Floer) is known.

Manolescu's relative 10/8-inequality

Manolescu defined an invariant $\kappa(Y, \mathfrak{t}) \in \frac{1}{8}\mathbb{Z}$ of a spin rational homology 3-sphere $(Y, \mathfrak{t})$ with the following property:

Theorem (Manolescu (2014))

Let W be a smooth, compact, oriented spin cobordism from $(Y_0, \mathfrak{t}_0)$ to $(Y_1, \mathfrak{t}_1)$. Then we have

$$-\frac{\sigma(W)}{8} + \kappa(Y_0, \mathfrak{t}_0) - 1 \leq b^+(W) + \kappa(Y_1, \mathfrak{t}_1).$$

$\kappa(Y, \mathfrak{t})$ is denied by applying *K*-theory to Manolescu's Seiberg–Witten Floer stable homotopy type

Manolescu's SW Floer homotopy type and Floer *K*-theory

$$\begin{tikzcd}
 & SWF(Y,\mathfrak{t}) \arrow[d, mapsto, "H^*_{S^1}"] \arrow[dr, mapsto, "K_{Pin(2)}"] & \\
(Y,\mathfrak{t}) \arrow[ur, mapsto] \arrow[dr, mapsto] & H^*_{S^1}(SWF(Y,\mathfrak{t})) & K_{Pin(2)}(SWF(Y,\mathfrak{t})) \\
 & HM(Y,\mathfrak{t}) \arrow[u, leftrightarrow, "\wr\|\ \text{Lidman–Manolescu}"'] &
\end{tikzcd}$$

- $(Y,\mathfrak{t})$: spin^c rational homology 3-sphere
- $SWF(Y,\mathfrak{t})$: SWF stable homotopy type (pointed "space" acted by S^1, or $Pin(2) = S^1 \sqcup jS^1 (\subset \mathbb{H})$ if $\mathfrak{t}$ is spin)
- $H^*_{S^1}(SWF(Y,\mathfrak{t}))$: (S^1-equiv) SW Floer (co)homology
- $K_{Pin(2)}(SWF(Y,\mathfrak{t}))$: ($Pin(2)$-equiv) SW Floer *K*-theory
- $HM(Y,\mathfrak{t})$: monopole Floer (co)homology
  due to Kronheimer–Mrowka

Kato's 10/8-inequality for involutions

Theorem (Kato (2021))

W : oriented closed spin smooth 4-manifold
$\iota : W \to W$: smooth involution that preserves the orientation and spin structure such that the fixed-point set W^ι is of codimension-2
Then we have

$$-\frac{\sigma(W)}{16} \leq b^+(W) - b_\iota^+(W),$$

($b_\iota^+(W)$: the max-dim of positive-definite subspaces of $H_2(W;\mathbb{R})^\iota$)

Kato defined and used an involutive symmetry on the SW equations by combing ι with the "charge conjugation" (different from usual equivariant theory, and it is crucial in applications).

Relative 10/8-inequality for involutions

- $(Y, \mathfrak{t})$: oriented spin rational homology 3-sphere.
- $\iota : Y \to Y$: smooth involution that preserves the orientation and spin structure such that the fixed-point set Y^ι is of codimension-2

We define an invariant $\kappa(Y, \mathfrak{t}, \iota) \in \frac{1}{16}\mathbb{Z}$ of the triple $(Y, \mathfrak{t}, \iota)$ using SW Floer K-theory, and derive the following property:

Main Theorem for involutions: K–Miyazawa–Taniguchi (2021)

Let $(W, \mathfrak{s}, \iota_W)$ be a compact oriented smooth spin cobordism with involution from $(Y_0, \mathfrak{t}_0, \iota_0)$ to $(Y_1, \mathfrak{t}_1, \iota_1)$ with $b_1(W) = 0$. Then:

$$-\frac{\sigma(W)}{16} + \kappa(Y_0, \mathfrak{t}_0, \iota_0) \leq b^+(W) - b^+_\iota(W) + \kappa(Y_1, \mathfrak{t}_1, \iota_1).$$

Comparison of the statements of Manolescu/Kato/KMT

Theorem (Manolescu (2014))

$W : (Y_0, \mathfrak{t}_0) \to (Y_1, \mathfrak{t}_1)$: spin cobordism. Then we have

$$-\frac{\sigma(W)}{8} + \kappa(Y_0, \mathfrak{t}_0) - 1 \leq b^+(W) + \kappa(Y_1, \mathfrak{t}_1).$$

Theorem (Kato (2021))

$\iota \curvearrowright W$: spin involution with $\mathrm{codim}\, W^\iota = 2$. Then we have

$$-\frac{\sigma(W)}{16} \leq b^+(W) - b^+_\iota(W),$$

Main Theorem for involutions (K–Miyazawa–Taniguchi (2021))

$(W, \mathfrak{s}, \iota) : (Y_0, \mathfrak{t}_0, \iota_0) \to (Y_1, \mathfrak{t}_1, \iota_1)$: spin cobordism with involution with $\mathrm{codim}\, W^\iota = 2$. Then we have

$$-\frac{\sigma(W)}{16} + \kappa(Y_0, \mathfrak{t}_0, \iota_0) \leq b^+(W) - b^+_\iota(W) + \kappa(Y_1, \mathfrak{t}_1, \iota_1).$$

Construction of Floer K-theory for involutions

$Y \rightsquigarrow CSD : \mathcal{C}(Y) \to \mathbb{R}$: functional on a Hilbert space (↔ SW eq)
$\iota : Y \to Y$ & “charge conj”$\rightsquigarrow$ invol. $I : \mathcal{C}(Y) \to \mathcal{C}(Y)$ (3D ver. of Kato’s)

$$\begin{tikzcd}
(Y,\iota) \arrow[d, mapsto] & \text{: spin rational homology 3-sphere with involution} \\
I, Pin(2) \circlearrowright (CSD : \mathcal{C}(Y) \to \mathbb{R}) \arrow[d, mapsto, "I\text{-invariant part (}cf.\text{ Kato)}"] & \text{: functional on a Hilbert space (} \leftrightarrow \text{ SW eq)} \\
\mathbb{Z}_4 \circlearrowright \left(CSD^I : \mathcal{C}(Y)^I \to \mathbb{R}\right) \arrow[d, mapsto, "\text{finite dim. approx. (}cf.\text{ Furuta, Manolescu)}"] & \text{: } I\text{-invariant part (} \leftrightarrow I\text{-invariant part of SW eq)} \\
\mathbb{Z}_4 \circlearrowright (\mathbb{R} \circlearrowright (\text{finite dim. space})) \arrow[d, mapsto, "\text{Conley index (}cf.\text{ Manolescu)}"] & \text{: finite dim. dynamical system} \\
\mathbb{Z}_4 \circlearrowright SWF(Y,\iota) \arrow[d, mapsto, "K_{\mathbb{Z}/4}"] & \text{: “}I\text{-invariant” SWF stable homotopy type} \\
K_{\mathbb{Z}/4}(SWF(Y,\iota)) & \text{: Floer } K\text{-theory for } (Y,\iota)
\end{tikzcd}$$

K-theoretic knot concordance invariant

K : a knot $\rightsquigarrow$ $\Sigma(K)$: the double branched cover of S^3 along K
$\Sigma(K)$ is a $\mathbb{Z}_2$-homology 3-sphere with covering involution ι_K.

$$\kappa(K) := \kappa(\Sigma(K), \mathfrak{t}, \iota_K) \in \frac{1}{16}\mathbb{Z},$$

where $\mathfrak{t}$ is the unique spin structure.

Basic Properties of $\kappa(K)$

- $\kappa(K)$ is a knot concordance invariant.
- $\kappa(-K) = \kappa(K)$ $\quad$ ($-K$: orientation reversal)
- $\kappa(K) + \kappa(K^*) \geq 0$ $\quad$ (K^* : mirror)
- $2\kappa(K) \equiv -\frac{\sigma(K)}{8}$ in $(\frac{1}{8}\mathbb{Z})/2\mathbb{Z} \cong \mathbb{Z}/16\mathbb{Z}$

Via double branched cover, Main Theorem for involutions implies the following key property of $\kappa(K)$:

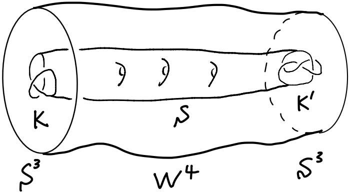
10/8-inequality for knots

Main Theorem for knots: K–Miyazawa–Taniguchi (2021)

- K, K' : knots in S^3
- W : compact oriented smooth cobordism from S^3 to S^3 with $H_1(W;\mathbb{Z}) = 0$
- S : an oriented compact smoothly embedded cobordism from K to K' in W, with $2|[S]$ and $[S]/2 = PD(w_2(W)) \mod 2$

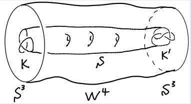
Then we have

$$-\frac{\sigma(W)}{8} + \frac{9}{32}[S]^2 + \frac{9}{16}\sigma(K) + \kappa(K) \leq b^+(W) + g(S) + \frac{9}{16}\sigma(K') + \kappa(K').$$



Computations of $\kappa(K)$

Reminder: Main Theorem for knots: KMT (2021)



$$-\frac{\sigma(W)}{8} + \frac{9}{32}[S]^2 + \frac{9}{16}\sigma(K) + \kappa(K) \leq b^+(W) + g(S) + \frac{9}{16}\sigma(K') + \kappa(K').$$

$\kappa(K)$ is computable for 2-bridge knots and many torus knots:

- $\kappa(K(p,q)) = -\sigma(K(p,q))/16$ for coprime p, q with p odd.
- $\kappa(T(p,q)) = -\bar{\mu}(\Sigma(2,p,q))/2$ for coprime odd p, q
  Here $\bar{\mu}$ is the Neumann–Siebenmann invariant (combinatorial)

For connected sums and crossing changes,

- $\kappa(K \# K') = \kappa(K) + \kappa(K')$ if K' is one of above knots.
- If K_1 is obtained from K_2 by positive crossing changes,
  $\kappa(K_2) - \kappa(K_1) \leq \frac{9}{16}(\sigma(K_1) - \sigma(K_2))$.

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