On the kernel of the surgery map

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joint work with
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Intelligence of Low-dimensional Topology

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[NSS1] arXiv: 2001.09825 Geom. Topol (2022)
☆ [NSS2] arXiv: 2103.07086 J. Topol (2022)

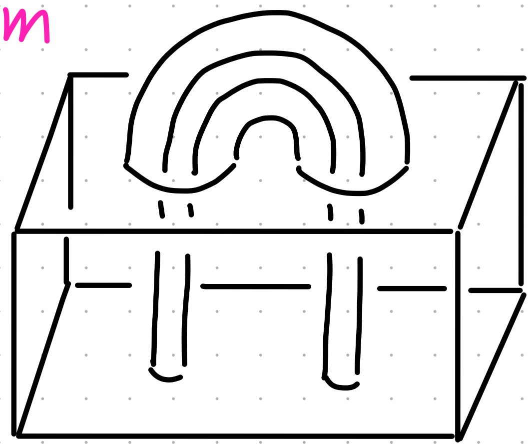
§1. Introduction

M: connected oriented compact 3-mfd

$m : \partial(\Sigma_{g,1} \times [-1,1]) \xrightarrow{\cong} \partial M \quad (m \rightsquigarrow m_+, m_-)$

$$(M, m) \sim (M', m') \underset{\text{def}}{\iff} \begin{tikzcd} M \arrow[rr, "\exists \cong"] & & M' \\ & \partial(\Sigma_{g,1} \times [-1,1]) \arrow[ul, "m"] \arrow[ur, "m'"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

cobordism



$g=1$

Def

(M, m) is a homology cylinder over $\Sigma_{g,1}$

$\iff (m_+)_* = (m_-)_* : H_*(\Sigma_{g,1}) \xrightarrow{\cong} H_*(M)$.

$\mathcal{IC}_{g,1} := \{\text{homology cylinders over } \Sigma_{g,1}\}$

$\shortparallel$

$\mathcal{IC}$ is a monoid by $M \circ M' = \begin{array}{|c|}\hline M' \\ \hline M \\ \hline\end{array}$

◎ Our motivation for studying $\mathcal{IC}$

Torelli group

・ monoid hom $c: \mathcal{I} \hookrightarrow \mathcal{IC}$

$f \longmapsto (\Sigma_{g,1}\times[-1,1],\ f\times 1 \cup \mathrm{id}\times(-1))$

・ $\mathcal{IC} = \left\{\begin{array}{l}\text{homology cylinders obtained from} \\ \Sigma_{g,1}\times[-1,1] \text{ by clasper surgery}\end{array}\right\}$

・ $\mathcal{IH} := \mathcal{IC}/\sim_H$ the homology cobordism group

$(\mathcal{IH}_{0,1} \cong \Theta^3_{\mathbb{Z}})$

§2. Preliminaries

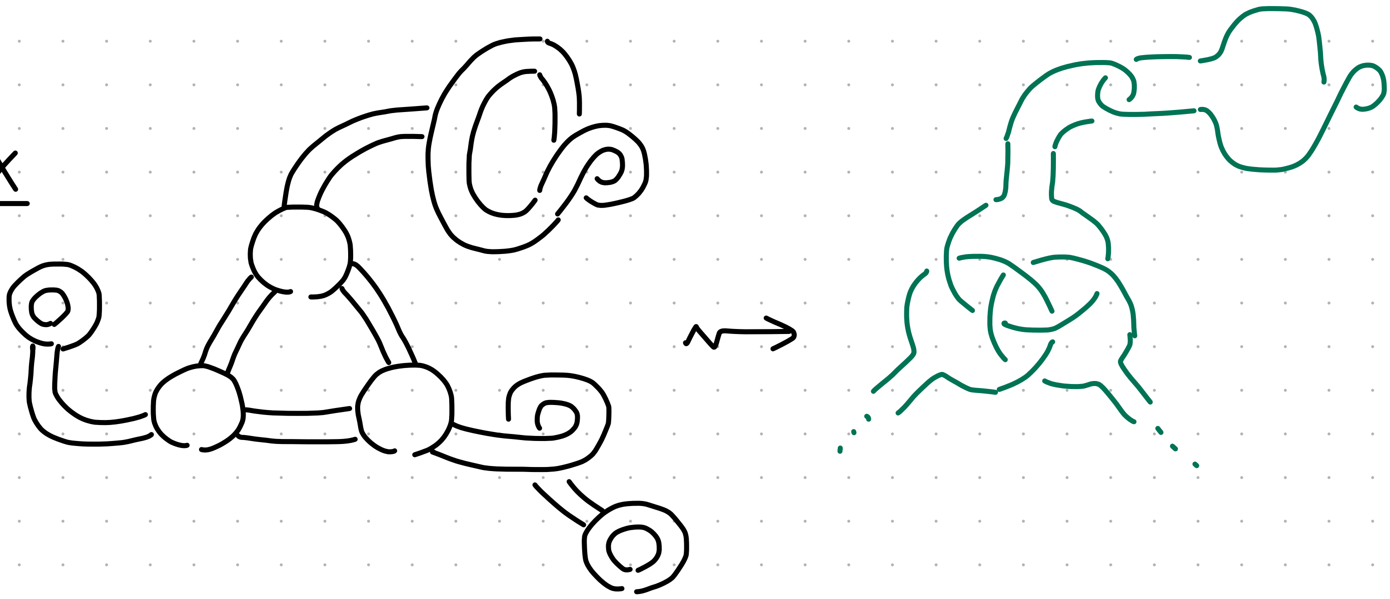
◎ Clasper surgery (Goussarov '99, Habiro '00)

$M = (M, m) \in \mathcal{IC}$
$\cup$
G : graph clasper $\rightsquigarrow$ L_G : framed link in M

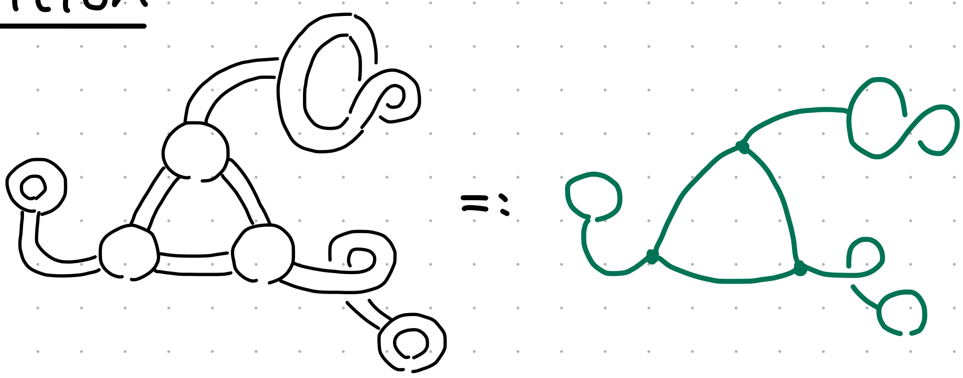
(a surface consisting of disks, bands and annuli)

$M_G := M_{L_G} \in \mathcal{IC}$

Ex



Convention



Def

M is Y_n-equivalent to M'

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists\, G_1, \cdots, G_r \subset M$

s.t. $M \coprod_{j=1}^{r} G_j = M'$ & $\deg G_j = n$ (# of disks)

$Y_n\mathcal{IC} := \{ M \in \mathcal{IC} \mid M \underset{Y_n}{\sim} \Sigma_{g,1} \times [-1,1] \}$

submonoid

$\mathcal{IC} = Y_1\mathcal{IC} \supset \cdots \supset Y_n\mathcal{IC} \supset \cdots$

: the Y-filtration

abelian group

$\bigoplus_{i=1}^{\infty} Y_n\mathcal{IC}/\sim_{Y_{n+1}}$ is regarded as an approximation of $\mathcal{IC}$

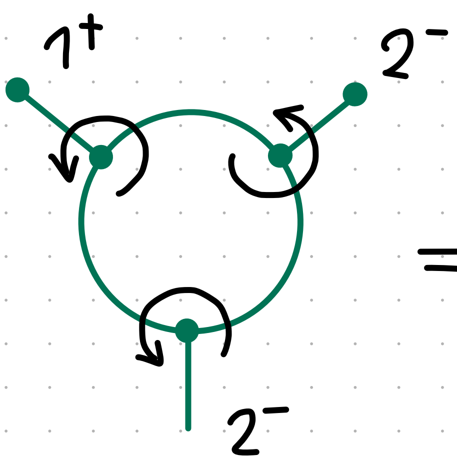
Jacobi diagrams

Def

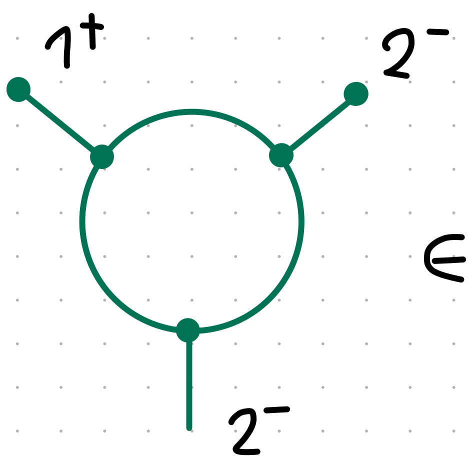
A Jacobi diagram colored with $\{1^+, \cdots, g^+, 1^-, \cdots, g^-\}$ is a uni-trivalent graph s.t. each

$$\begin{cases} \text{univalent vertex is colored} \\ \text{trivalent vertex has a cyclic order} \end{cases}$$

Ex



$=:$



$\in \mathcal{A}^c_{3,1}$

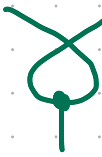
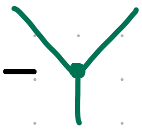
$$\mathcal{A}_n^c := \mathbb{Z}\left\{ \begin{matrix} \text{conn. Jacobi diagrams} \\ \text{of i-deg} = n \end{matrix} \right\} \Big/ \sim$$

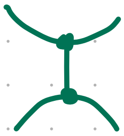
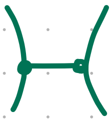
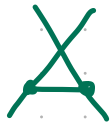
i-deg: # of trivalent vertices (internal)

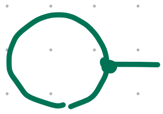
$$\mathcal{A}_n^c = \bigoplus_{\ell \geq 0} \mathcal{A}_{n,\ell}^c$$

ℓ: first Betti number

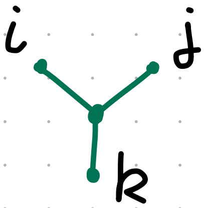
$\sim$

 $= -$    AS

 $-$  $+$  $= 0$ IHX

 $= 0$ self-loop

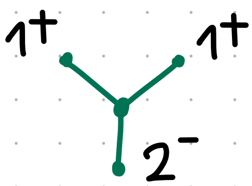
Ex

$\mathcal{A}_1^c = \langle$ 

's $\rangle$

$$\cong (\mathbb{Z}^{2g})^{\otimes 3} \Big/ x_1 \otimes x_2 \otimes x_3 \sim \mathrm{sgn}(\sigma)\, x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}$$

$$\cong \Lambda^3 \mathbb{Z}^{2g} \oplus \underline{(\mathbb{Z}^{2g})^{\otimes 2} \otimes \mathbb{Z}/2}$$

e.g., 

$(\neq 0)$

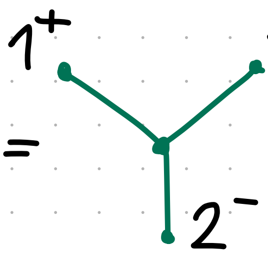
symmetry
⇝
torsion

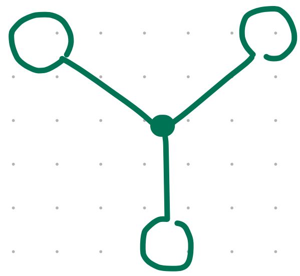
◍ Surgery map

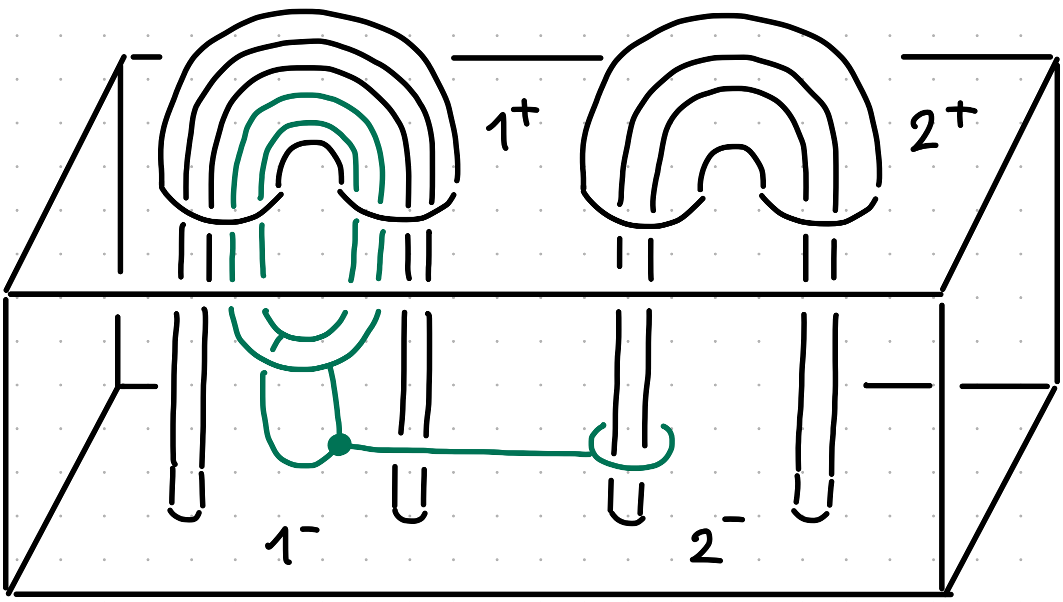
The surgery map $\mathfrak{s}_n : \mathcal{A}_n^c \xrightarrow{\text{hom}} Y_n\mathcal{IC}/\sim_{Y_{n+1}}$

is defined by ...

<u>Ex</u> $(n=1,\ g=2)$

$J =$ 

$\rightsquigarrow$  $=: G_J \subset \Sigma_{2,1} \times [-1,1]$



$\mathfrak{s}_1(J) = (\Sigma_{2,1} \times [-1,1])_{G_J}$ is well-defined up to $\sim_{Y_2}$

Fact Jacobi diag. homology cyl.

$$\begin{tikzcd} \mathcal{A}_n^c \arrow[r, "S_n"'] \arrow[d] & Y_nI\mathcal{C}/Y_{n+1} \arrow[d] \\ \mathcal{A}_n^c \otimes \mathbb{Q} \arrow[r, "\cong"'] & Y_nI\mathcal{C}/Y_{n+1} \otimes \mathbb{Q} \end{tikzcd}$$

is surjective if $n \geq 2$

Cheptea-Habiro-Massuyeau '08

Let us focus on

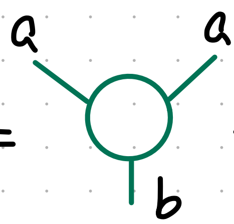
$$0 \to \operatorname{Ker} S_n \to \mathcal{A}_n^c \xrightarrow{S_n} Y_nI\mathcal{C}/Y_{n+1} \to 0$$

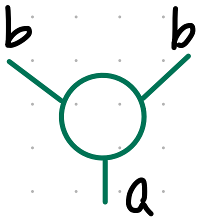
$$\operatorname{Ker} S_n \subset \operatorname{tor} \mathcal{A}_n^c$$

Related results

① $\operatorname{Ker} S_n$ & $Y_nI\mathcal{C}/Y_{n+1}$ was determined for

- $n = 1, 2$ Massuyeau-Meilhan '03 '13
- $n = 3, 4$ [NSS 1&2]

Ex $0 \neq$ 

$+$ 

$\xmapsto{S_3} 0$

pair of torsions $\in$ kernel

Question 1

$\exists ?$ torsion in $Y_n\mathcal{IC}/Y_{n+1}$ of order > 2

Question 2

$\exists ?$ torsion in $\mathcal{A}_n^c$ of order > 2

Rem

The answer for Q1 is Yes $\Rightarrow$ Q2 is Yes

Question 4

$$\mathrm{tor}\,\mathcal{A}_n^c \overset{?}{\subset} \mathcal{A}_n^{c,s}, \qquad \mathrm{Ker}\, \mathfrak{s}_n \overset{?}{\subset} \mathcal{A}_n^{c,s}$$

cf. Bar-Natan, Dasbach '97,
Moskovich-Ohtsuki '06, Ishikawa, ...

Question 5 $\mathrm{Ker}\, \mathfrak{s}_{2m+1,0} \overset{?}{=} \mathrm{Im}\, \Delta_{m,0}$

i.e., $\mathcal{A}_{m,0}^c \xrightarrow{\Delta_{m,0}} \mathcal{A}_{2m+1,0}^c \xrightarrow{\mathfrak{s}_{2m+1,0}} Y_{2m+1}\mathcal{IC}/Y_{2m+2}$

exact ?

② Ker $s_{n,0}$ was investigated by Conant-Schneiderman-Teichner '16

$$\begin{tikzcd}
\bigoplus_{\ell\geq 0} \mathcal{A}^c_{n,\ell} \arrow[r, "s_n = \bigoplus s_{n,\ell}"] & Y_n I\mathcal{C}/Y_{n+1} \arrow[r, two heads, "q"] \arrow[d, "\boxed{\bar{z}_{n+1}}"] & Y_n I\mathcal{H}/Y_{n+1} \\
& \mathcal{A}^c_{n+1} \otimes \mathbb{Q}/\mathbb{Z} &
\end{tikzcd}$$

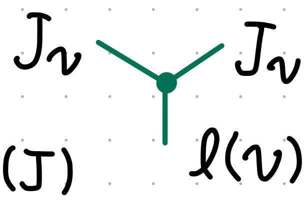
$$\begin{cases} \operatorname{Ker} s_{2m,0} = \{0\} \quad [\mathrm{CST}\,'12] \\ \operatorname{Ker} s_{2m+1,0} \subset \operatorname{Ker}(\bar{z}_{2m+2} \circ s_{2m+1,0}|_{\mathrm{tor}}) \end{cases}$$

$\boxed{\text{"=" if } m = 1, 2}$

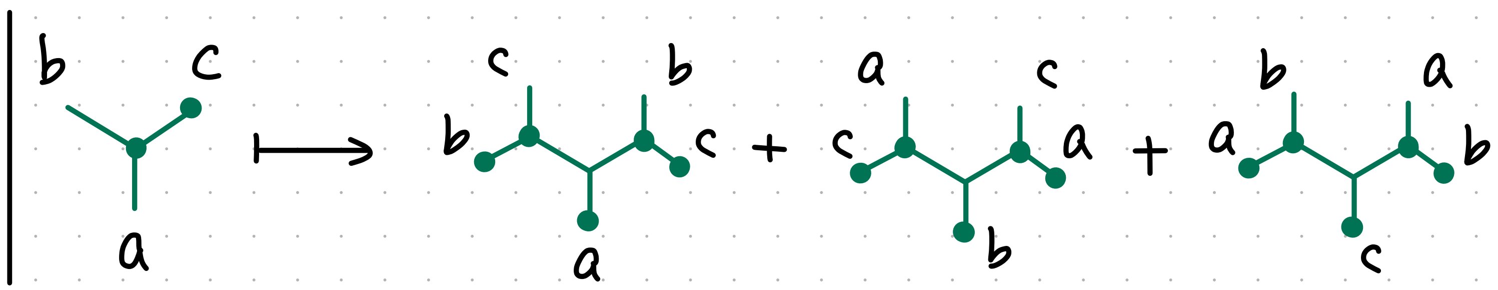
$= \operatorname{Im} \Delta_{m,0}$ $\boxed{[\mathrm{NSS}\,1]}$

$\subset \operatorname{Ker}(q \circ s_{2m+1,0})$

"=" if m is odd [CST '16]

$\Delta_{m,0} : \mathcal{A}^c_{m,0} \longrightarrow \mathcal{A}^c_{2m+1,0}, \quad J \longmapsto \sum_{v \in U(J)}$ 

Ex ($m = 1$)

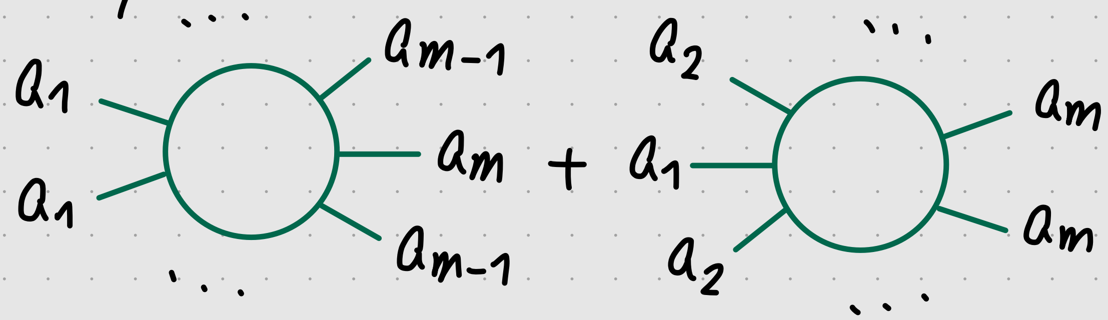


§3. Main results

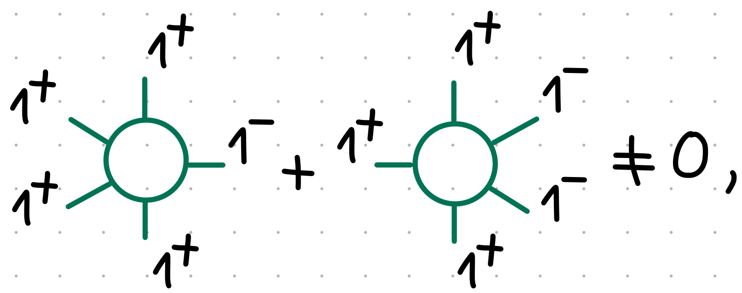
Thm 1 (NSS 2) The kernel of

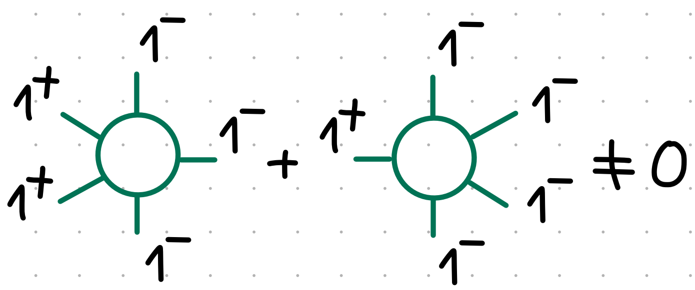
$$\mathcal{A}_{2m-1,1} \xrightarrow{\mathfrak{s}_{2m-1,1}} Y_{2m-1}\mathcal{IC}/Y_{2m} \xrightarrow{\text{"}\pi\text{"}} \left(Y_{2m-1}\mathcal{IC}/Y_{2m}\right)/\sim$$

is free $\mathbb{Z}/2$-module of rank $\frac{1}{2}\left((2g)^m - (2g)^{\lceil \frac{m}{2} \rceil}\right)$ generated by

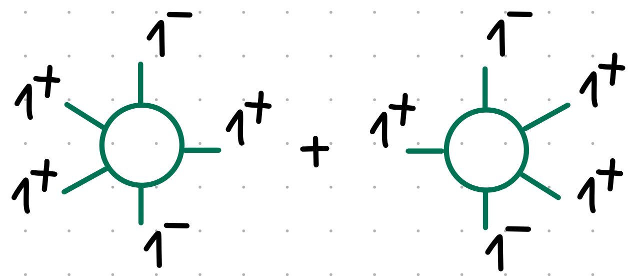


Ex ($g=1$, $m=3$)



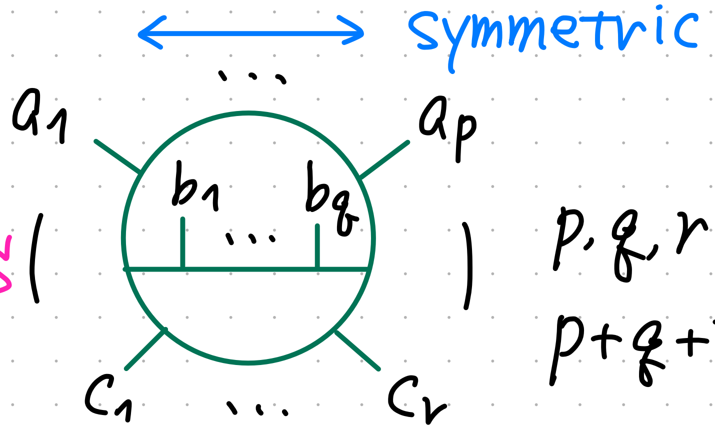


(



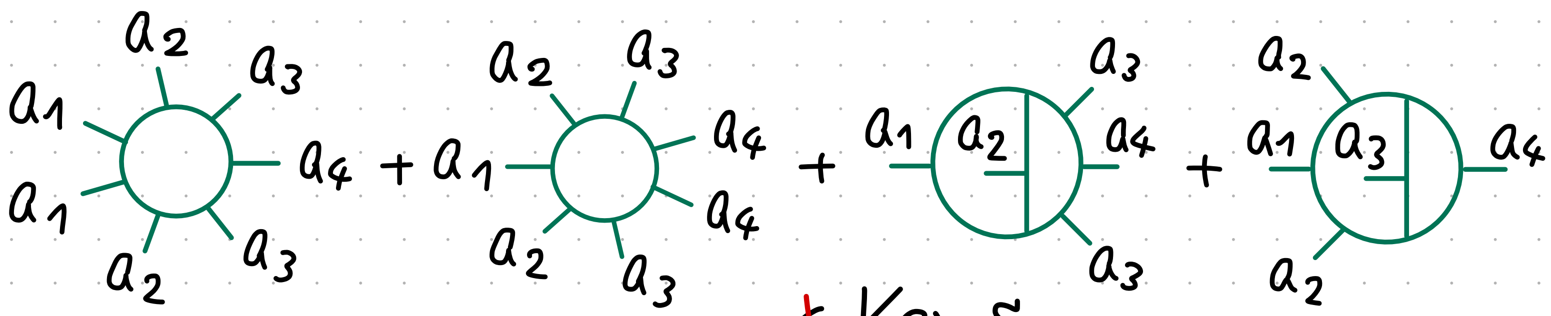
$= 0$ etc.)

$\sim$ means

$\Big[$ modulo $S\Big($  $\Big)$ $p, q, r \geq 1$, $p+q+r+2 = 2m-1$ $\Big]$

<u>Rem</u> $\operatorname{Ker} S_{2m-1,1} \subset \operatorname{Ker} \pi \circ S_{2m-1,1}$

$$\begin{cases} = & \text{if } m = 2, 3 \\ \subsetneq & \text{if } m = 4 \end{cases}$$



$\notin \operatorname{Ker} s_7$

$\rightsquigarrow$ $\boxed{\bigoplus_{\ell \geq 0} \operatorname{Ker} s_{7,\ell} \subsetneq \operatorname{Ker} s_7}$ A new/interesting phenomenon!

Key of the proof for "$\notin \operatorname{Ker} s_7$"

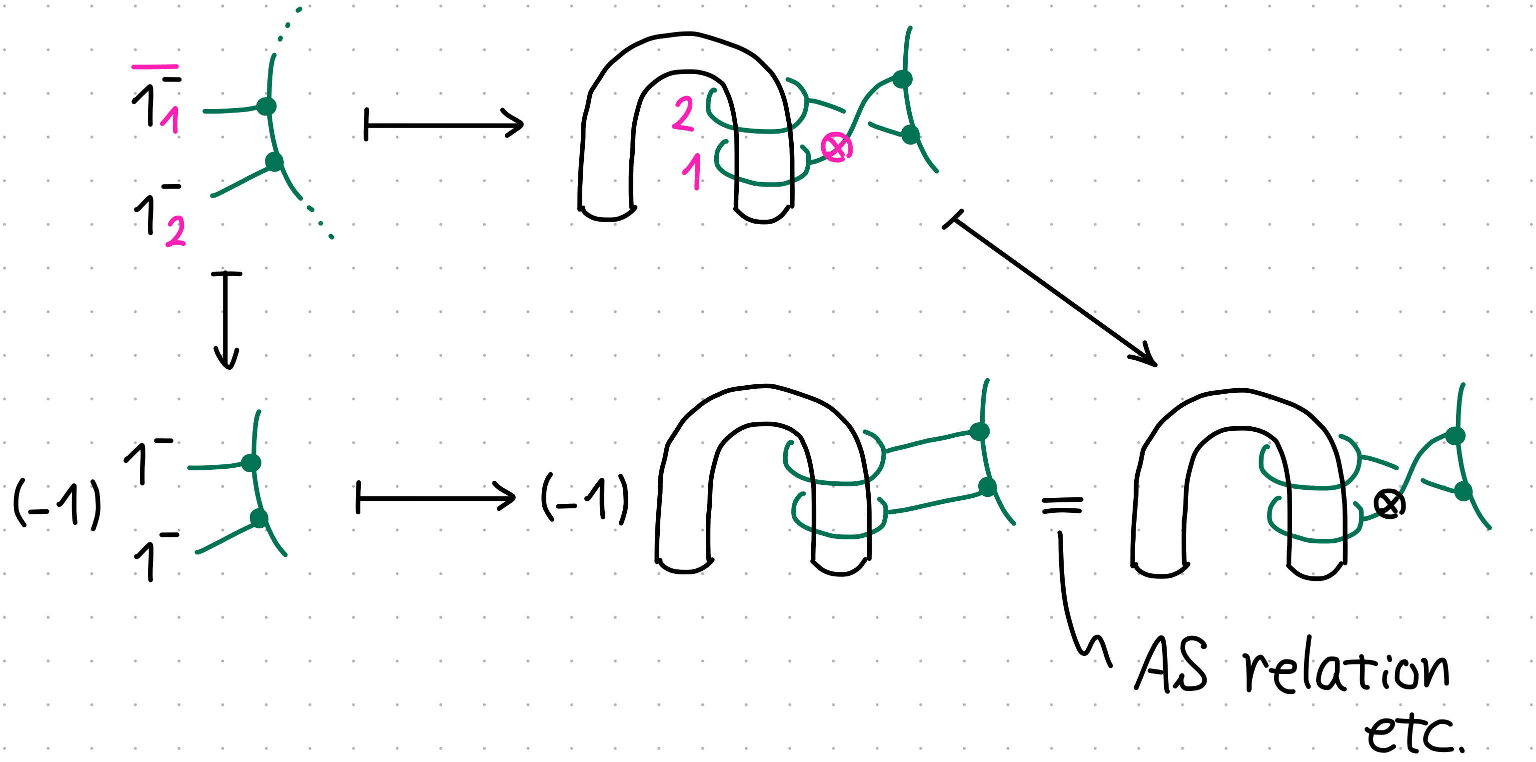
- $\mathbb{Z}_8$ [NSS 1]
- $\mathcal{A}^c_{8,2} \cong \mathbb{Z}^{32}$ Mathematica
- description of $\mathcal{A}^c_{n,2}$ [NSS 2]

Proof of Thm 1

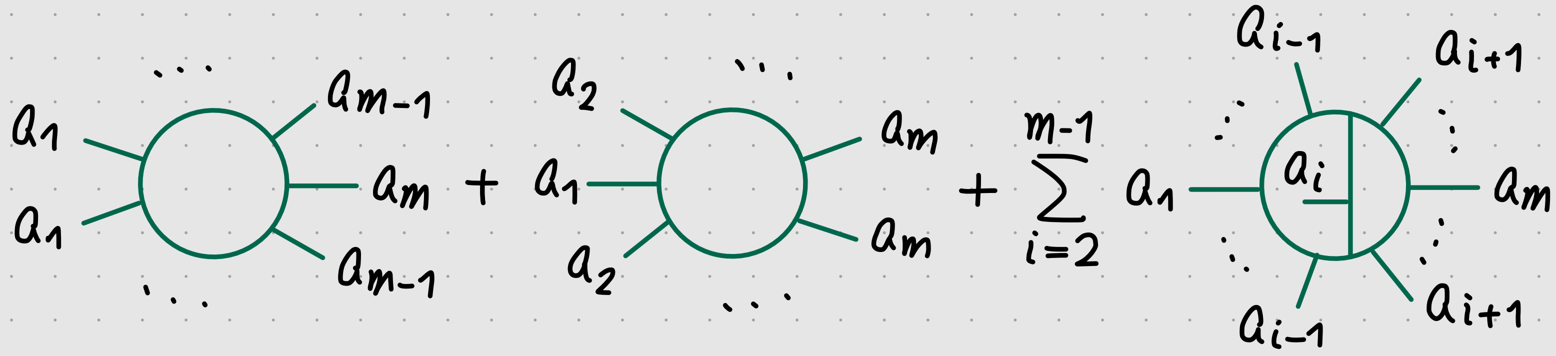
Part 1: refined surgery map $(n \geq 2)$

$$\begin{tikzcd} \mathbb{Z}\tilde{\mathcal{J}}_n^c \arrow[r, "\tilde{\mathfrak{s}}_n"] \arrow[d] & Y_nIC/Y_{n+2} \arrow[d] \\ \mathcal{A}_n^c \arrow[r, "\mathfrak{s}_n"] & Y_nIC/Y_{n+1} \end{tikzcd}$$

abelian group



Thm 2 (NSS2) The following sum lies in Ker $\mathfrak{s}_n$:



Part 2: $\bar{z}_{n+1}$ shows that the rest of the elements are not in Ker $\pi \circ \mathfrak{s}_{2m-1,1}$ ∎

Thm 1 (NSS 2) The kernel of

$$\mathcal{A}_{2m-1,1} \xrightarrow{\mathfrak{s}_{2m-1,1}} Y_{2m-1}\mathcal{IC}/Y_{2m} \overset{``\pi"}{\twoheadrightarrow} \left(Y_{2m-1}\mathcal{IC}/Y_{2m}\right)/\sim$$

— Future perspective —

We would like to

- determine the kernel without π.
- investigate the 2-loop part and more.
- determine the structures of $Y_n\mathcal{IC}/Y_{n+1}$ & $Y_n\mathcal{IH}/Y_{n+1}$.
- develope clasper calculus in $Y_n\mathcal{IC}/Y_{n+2}$.

Recently, we have

constructed a "non-commutative Reidemeister-Turaev torsion" on $\mathcal{IC}$,

described a relation with

the Enomoto-Satoh trace and

the 1-loop part of the LMO functor.

$$\begin{tikzcd}
 & \mathcal{IC} \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\circlearrowleft"] & K_1(\widehat{\mathbb{Q}\pi}) \cong \mathbb{Q}^\times \oplus \prod_{n=1}^{\infty} H_{\mathbb{Q}}^{\otimes n}/\mathbb{Z}_n \arrow[d] \arrow[dr] & \\
\mathbb{Z}H \supset 1+IH \arrow[r, hook] & & K_1(\widehat{\mathbb{Q}H}) & K_1(\mathbb{Q}\pi/I_\pi^d)
\end{tikzcd}$$

、τ mod I^d is a finite-type inv.

Massuyeau-Meilhan '13

、$\tilde{\alpha}$ mod I_π^d is a finite-type inv.

[NSS, in preparation]

cf. Kricker '02,

Massuyeau-Sakasai '20,

Ohtsuki '07, ...

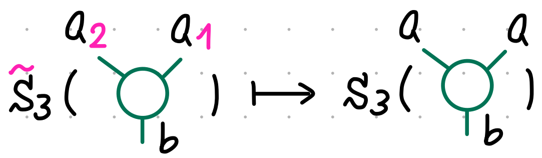
Application 1

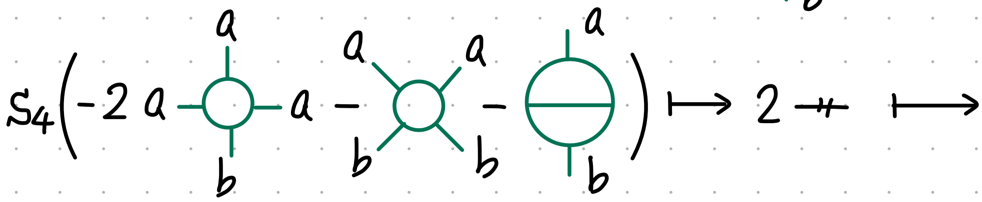
$$0 \longrightarrow Y_{n+1}\mathcal{IC}/Y_{n+2} \longrightarrow Y_n\mathcal{IC}/Y_{n+2} \longrightarrow Y_n\mathcal{IC}/Y_{n+1} \longrightarrow 0 \quad \text{(exact)}$$

Ex $(n=3)$

$$\underset{\text{tor}=0}{} \qquad\qquad\qquad\qquad \underset{\text{tor}\neq 0}{}$$

$$0 \longrightarrow Y_4\mathcal{IC}/Y_5 \longrightarrow Y_3\mathcal{IC}/Y_5 \longrightarrow Y_3\mathcal{IC}/Y_4 \longrightarrow 0$$





Cor (NSS2) The abelian group $Y_3\mathcal{IC}/Y_5$ is free

Application 2

$\mathcal{IH} = \mathcal{IC}/\sim_H$ the homology cobordism group

Cor (NSS2) The abelian group $Y_3\mathcal{IH}/Y_5$ is free

Summary:

$$\begin{array}{ccccccccc}
 & & \text{free} & & \text{free} & & \exists\,\text{torsion} & & \\
0 & \longrightarrow & Y_4\mathcal{IC}/Y_5 & \longrightarrow & Y_3\mathcal{IC}/Y_5 & \longrightarrow & Y_3\mathcal{IC}/Y_4 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_4\mathcal{IH}/Y_5 & \longrightarrow & Y_3\mathcal{IH}/Y_5 & \longrightarrow & Y_3\mathcal{IH}/Y_4 & \longrightarrow & 0 \\
 & & \text{free} & & \text{free} & & \exists\,\text{torsion} & &
\end{array}$$

Goussarov-Habiro Conjecture

Conj

$M \underset{Y_{n+1}}{\sim} M' \iff f(M) = f(M')$ for $\forall f$ "finite type inv. of $\deg \leq n$"

Cor (NSS 2) Conj is true for $n = 4$

cf. $n = 2$ Massuyeau-Meilhan '13

$n = 3$ [NSS 1]

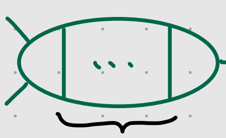
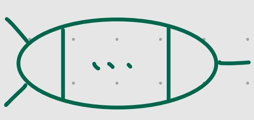
$\mathcal{IH} = \mathcal{IC}/\sim_H$ the homology cobordism group of homology cylinders

$(M, m) \underset{H}{\sim} (N, n) \overset{\text{def}}{\iff} \exists W^4$: cpt ori smooth s.t.

$\partial W = M \underset{m \circ n^{-1}}{\cup} (-N)$ & $H_*(M) \xrightarrow{\cong} H_*(W) \xleftarrow{\cong} H_*(N)$

$m : \partial(\Sigma_{g,1} \times [-1, 1]) \xrightarrow{\cong} \partial M$

Prop (NSS 2)

$s\Big($ a, a  $b \Big) =$ $s\Big($ b, b  $a \Big) \neq 0$

$\ell - 1$ clasper calculus $\mathbb{Z}_{2\ell+2}$ & weight system