Properties of links from the viewpoint of R. Thompson's group F

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Intelligence of Low-dimensional Topology
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Introduction

[Thompson, 1965] defined the groups F, T and V.

- They are used to construct finitely-presented groups with unsolvable word problems.
- T and V are finitely-presented infinite simple groups.

[Jones, 2017]

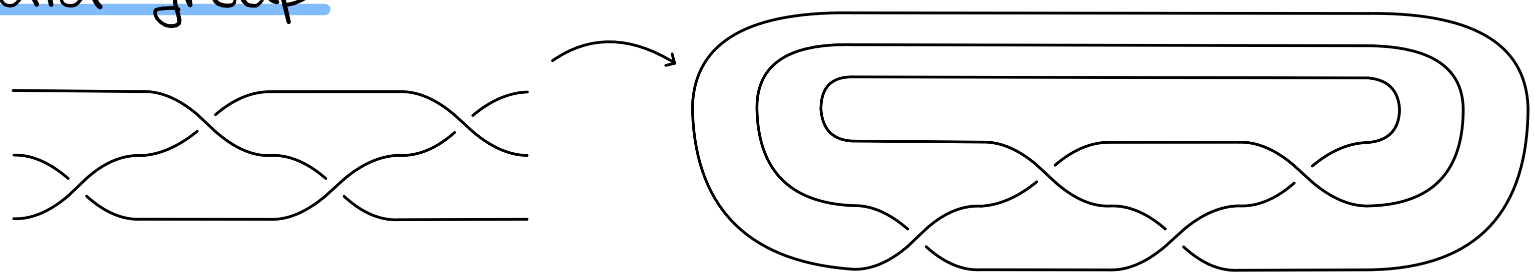
- introduced a method of constructing (oriented) links from elements of F (or $\vec{F}$).
- showed an analogue of (unoriented) Alexander's theorem

Every link can be represented as a closed braid.

and a slightly weaker result for the oriented case.

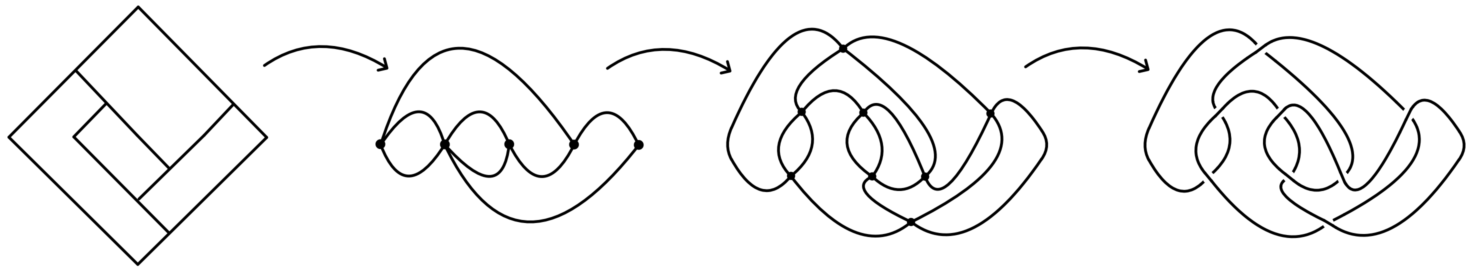
[Aiello, 2020] completely proved the therem for the oriented case.

Braid group



simple!

Thompson's group



complicated...

Main result

We constract a sequence $\{g_k\}_{k\geq 0} \subset \vec{F}$.

$L_k := \mathcal{L}(g_k)$: associated link

- L_k : alternating and fibered
- $\# L_k$, $c(L_k)$, $g(L_k)$, $b(L_k)$
- minimal genus flat Seifert surfaces

1. Definitions and Constructions
2. Examples
3. Main result

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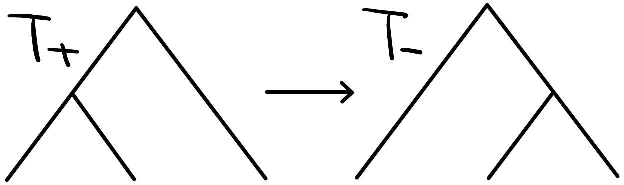
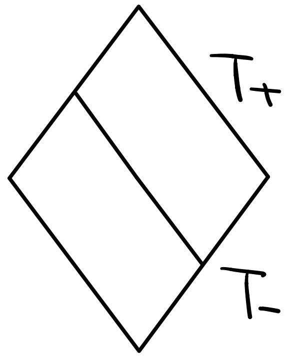
Thompson's group F

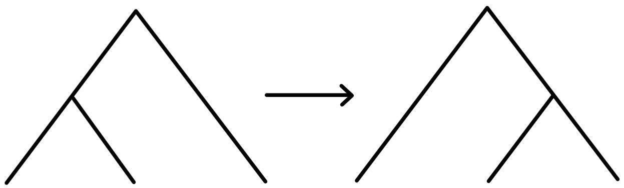
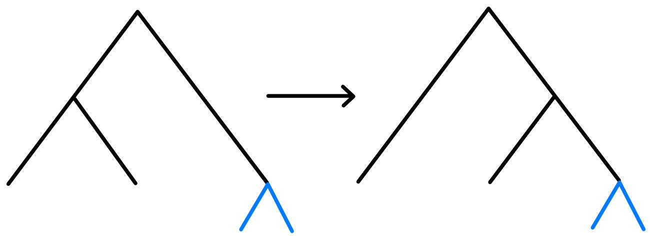
$$F := \left\{ \begin{array}{l} \text{pairs of rooted, planar, binary trees } \overbrace{(T_+, T_-)}^{\text{tree diagram}} \\ \text{with the same number of leaves} \end{array} \right\} \Big/ \sim$$

Descriptions of elements of F

$F \ni (T_+, T_-) =$  or 

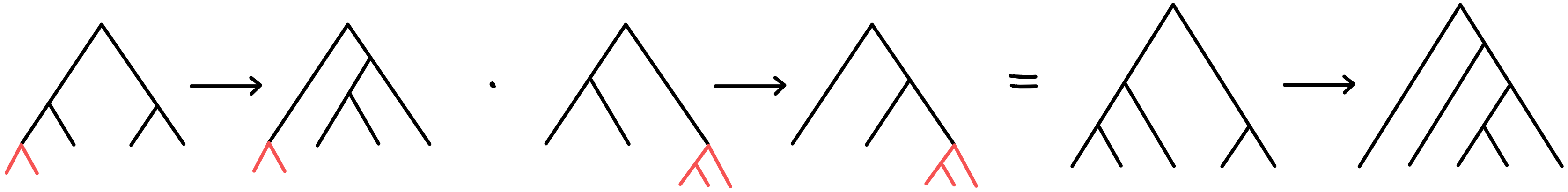
The equivalence relation

 : caret

 $=$  $\in F$

The tree diagram representing $g \in F$ without pairs of opposing carets is called the reduced tree diagram of g.

The group operation



$$(T_{1+}, T_{1-}) \cdot (T_{2+}, T_{2-}) = (T'_{1+}, T'_{2-})$$

$$(T_{1+}, T_{1-}) = (T'_{1+}, T'_{1-}), \qquad (T_{2+}, T_{2-}) = (T'_{2+}, T'_{2-})$$

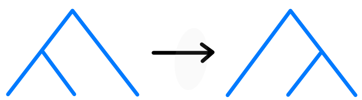
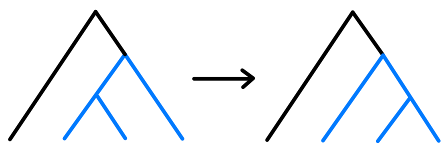
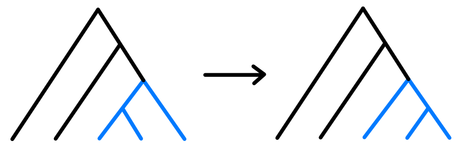
T'_{1-}, T'_{2+}: same trees

- $1 = (T, T) \in F$ ($^\forall T$: binary tree)
- $(T_+, T_-)^{-1} = (T_-, T_+)$

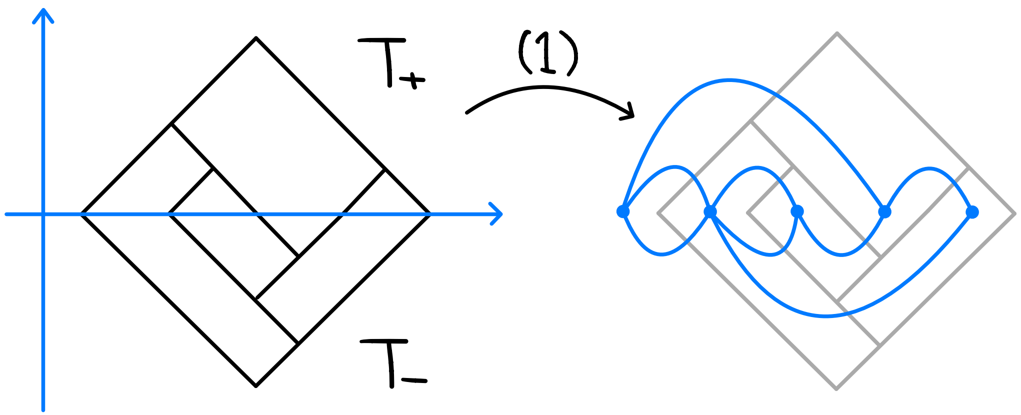
Presentations of F [McKenzie-Thompson, 1973]

$$F \cong \langle x_0, x_1, x_2, \cdots \mid x_i^{-1} x_j x_i = x_{j+1} \ (i < j) \rangle$$

$$\cong \langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle$$

$x_0 :=$  , $x_1 :=$  , $x_2 :=$  , $\cdots$

Jones' construction



$(T_+, T_-) \in F$: reduced tree diagram with $n+1$ leaves.

Place the leaves of (T_+, T_-) at $(\frac{1}{2}, 0), (\frac{3}{2}, 0), \cdots, (\frac{2n+1}{2}, 0)$.

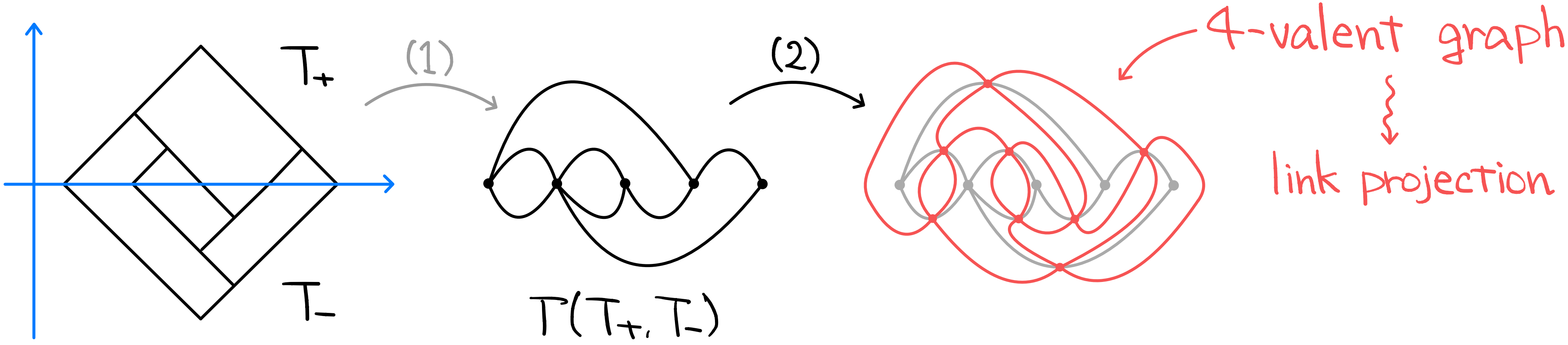
Note that $T_+ \subset \mathbb{R} \times \mathbb{R}_{\geq 0}$ and $T_- \subset \mathbb{R} \times \mathbb{R}_{\leq 0}$.

(1) Construct the planar graph $\Gamma(T_+, T_-)$:

vertex : $(0,0), (1,0), \cdots, (n,0)$

edge : intersects transversally just once an edge $/$ of T_+ or $\backslash$ of T_- and does not do the other edges of (T_+, T_-).

Jones' construction

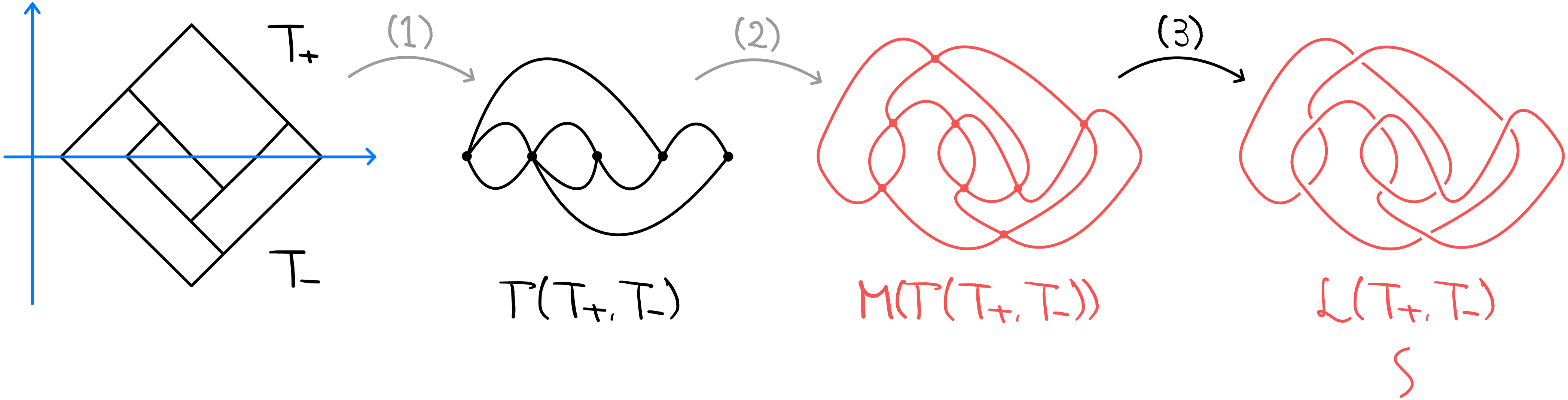


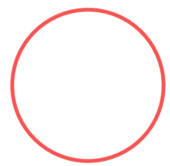
(2) Construct the medial graph $M(T(T_+, T_-))$.

Let G be a connected planar graph.
Its medial graph $M(G)$ is obtained as follows:

- vertex : on every edge of G
- edge: two vertices are joined if the corresponding edges are adjacent on a face of G.

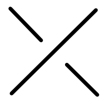
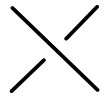
Jones' construction





(3) Construct the link diagram $\mathcal{L}(T_+, T_-)$:

replace vertices in

- $\mathbb{R}\times\mathbb{R}_+$ with 
- $\mathbb{R}\times\mathbb{R}_-$ with  .

Jones' subgroup $\vec{F}$

$\vec{F} := \{(T_+, T_-) \in F \mid \Gamma(T_+, T_-) \text{ is } \underline{\text{2-colorable}}\}$: Jones' subgroup

or

oriented Thompson's group

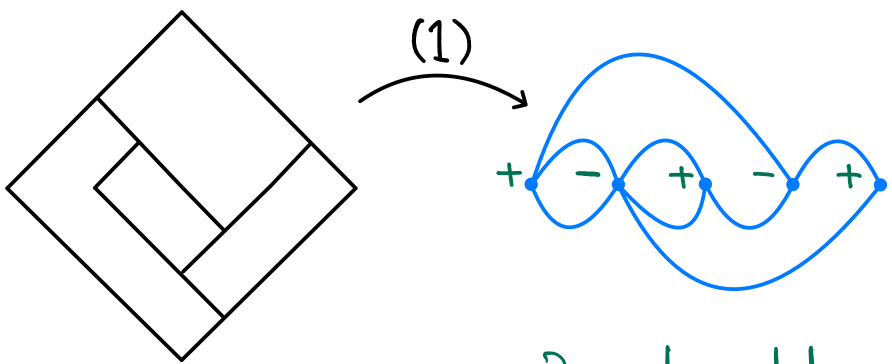
A graph G is 2-colorable

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}f : V(G) = \{\text{vertices of } G\} \longrightarrow \{+, -\}$

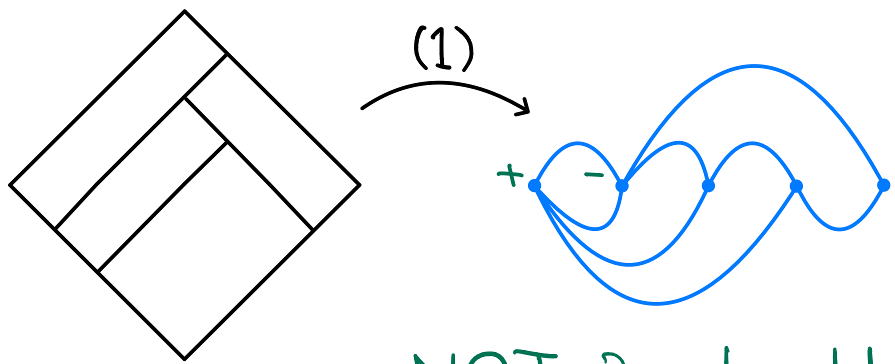
s.t. $v_1, v_2 \in V(G)$ are joined $\Longrightarrow f(v_1) \neq f(v_2)$.

By convention, the vertex $(0,0)$ has the color $+$.

Example

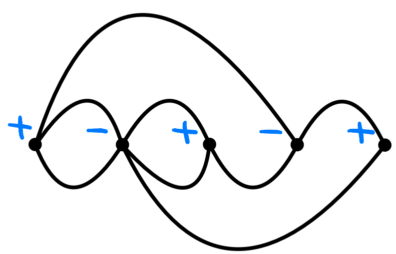


2-colorable

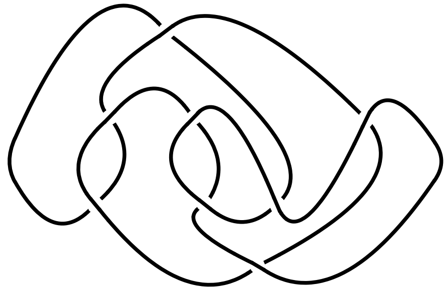


NOT 2-colorable

$(T_+, T_-) \in \vec{F} \rightsquigarrow \mathcal{L}(T_+, T_-)$ is naturally oriented

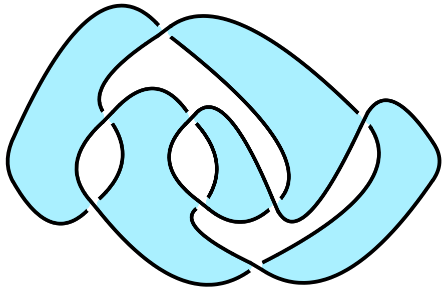


$\Gamma(T_+, T_-)$

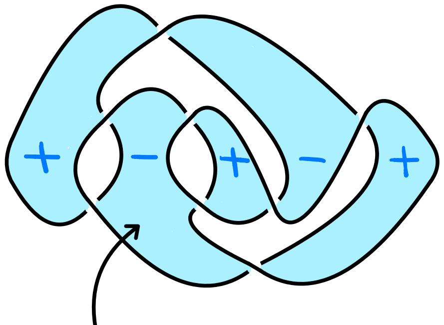


$\mathcal{L}(T_+, T_-)$

checkerboard coloring

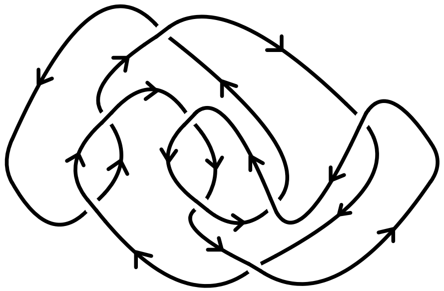


$\mathcal{L}(T_+, T_-)$



$S(T_+, T_-)$

∵ checkerboard surface obtained from blue regions



- orientable
  (∵ $\Gamma(T_+, T_-)$ is 2-colorable)
- $\partial S(T_+, T_-) = \mathcal{L}(T_+, T_-)$

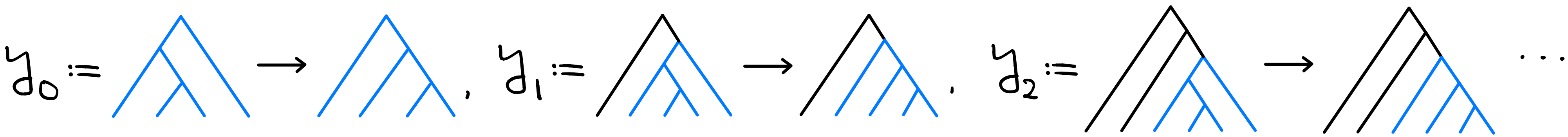
∴ $S(T_+, T_-)$ is a Seifert surface of $\mathcal{L}(T_+, T_-)$

Generators of $\vec{F}$

Thm [Golan-Sapir, 2017]

- $\vec{F}$ is generated by $X := \{x_i x_{i+1} \mid i \geq 0\}$ and
  $X' := \{x_i^{n+1} x_{i+1} x_{i+2}^{-n} \mid i \geq 0, n \geq 1\}$.
- $\vec{F}$ is generated by x_0x_1, x_1x_2 and x_2x_3.
- $\vec{F} \cong \langle y_0, y_1, y_2, \cdots \mid y_i^{-1} y_j y_i = y_{j+2}\ (i<j) \rangle \cong F_3 \quad (y_i := x_i x_{i+1})$.

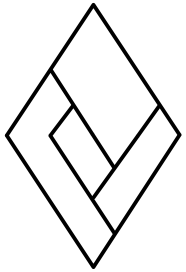
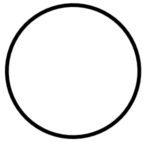
Remark $F = F_2 \ni (T_+, T_-)$: binary trees.
$F_3 \ni (T_+, T_-)$: 3-ary trees, $\cdots$, $F_k \ni (T_+, T_-)$: k-ary trees.



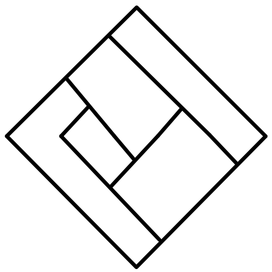
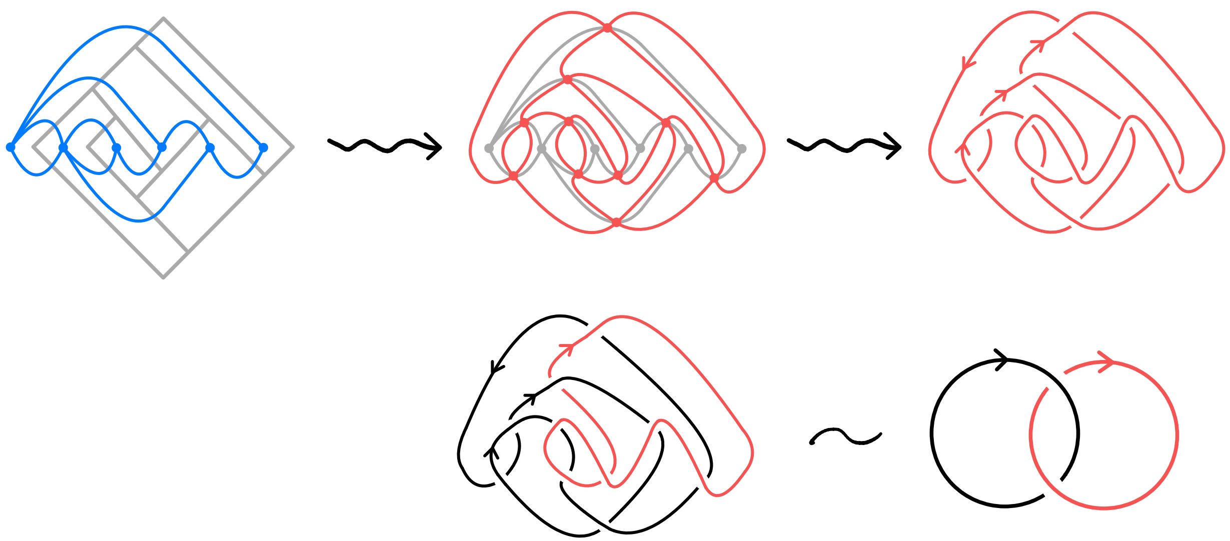
1. Definitions and Constructions

2. Examples

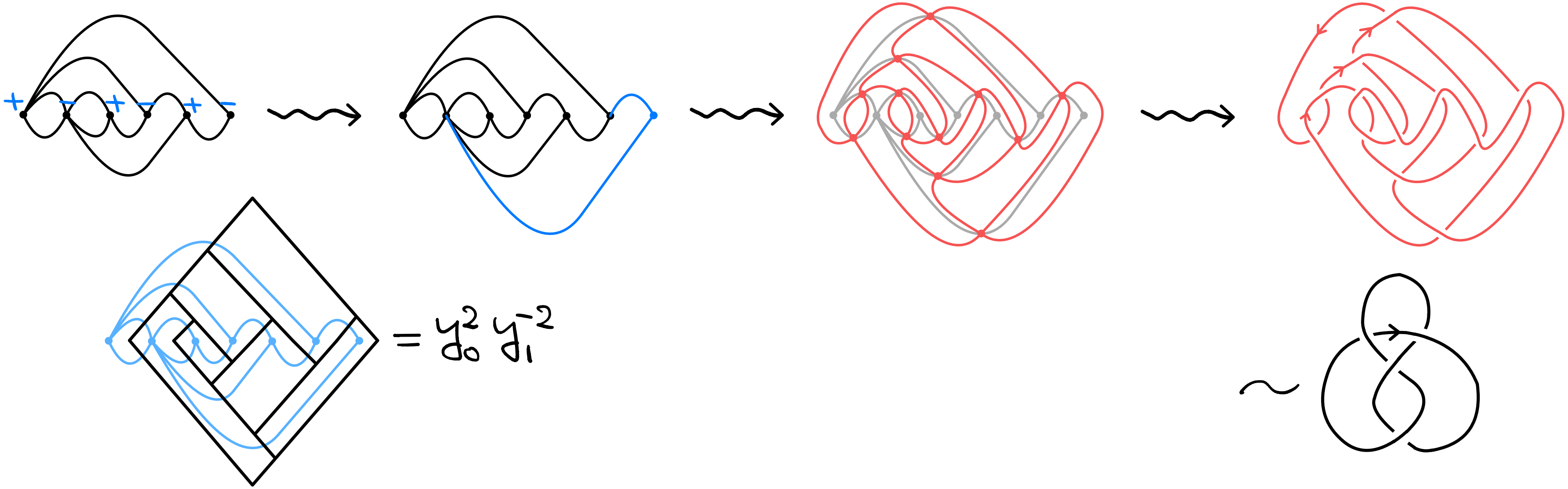
3. Main result

[1] $y_0 y_1^{-1}$ =  $\rightsquigarrow$ $\mathcal{L}(y_0 y_1^{-1})$ = 

<u>Fact</u> All links obtained from elements of $\vec{F}$ with ≤ 5 leaves are trivial.

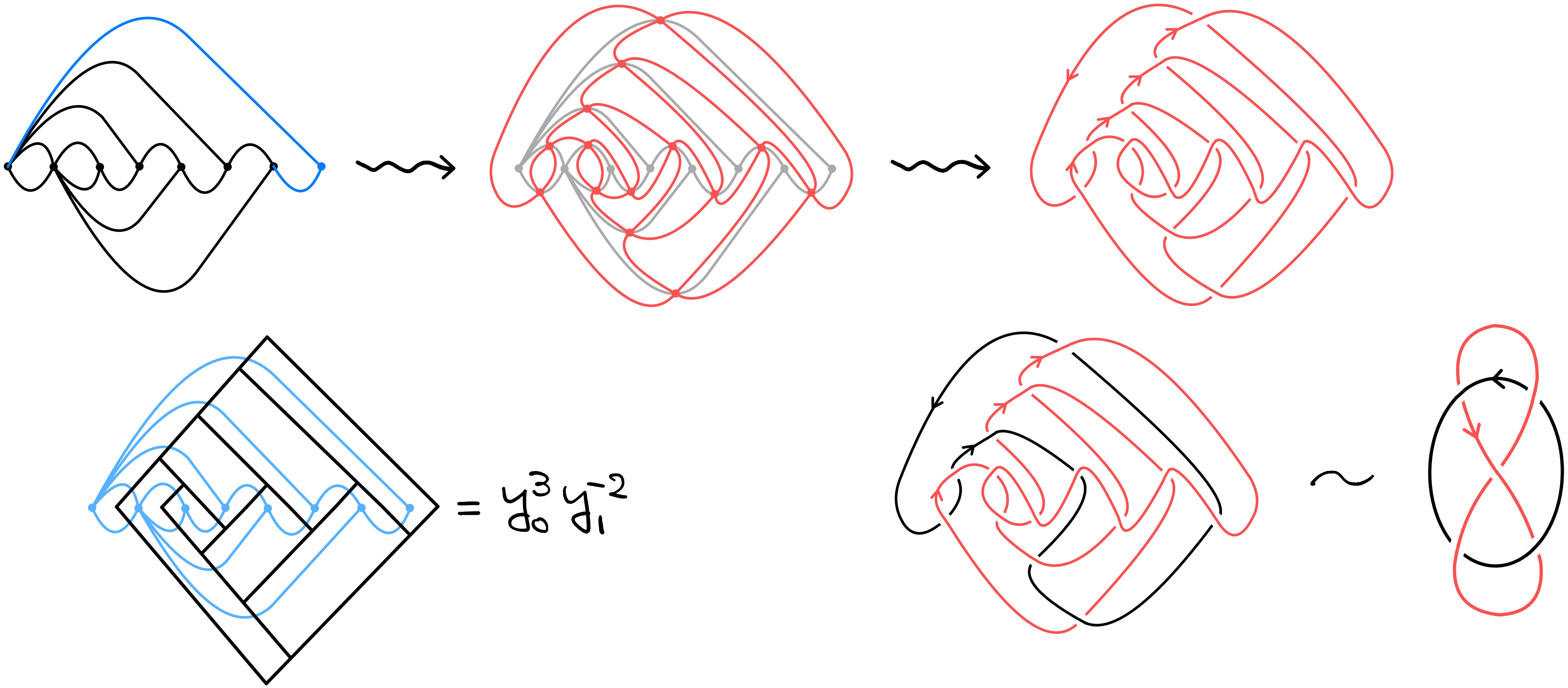
[2] $y_0^2 y_1^{-1}$ =  $\rightsquigarrow$ 

[3]



$= y_0^2 y_1^{-2}$

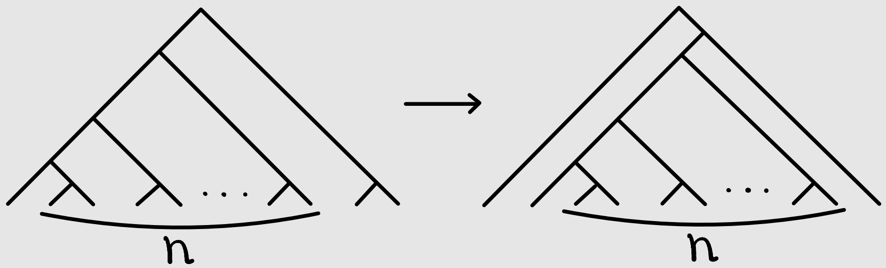
[4]



$= y_0^3 y_1^{-2}$

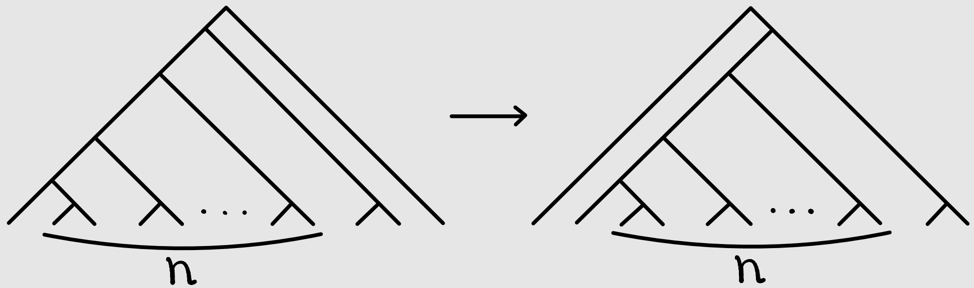
Lem [Kodama-T. 2022]

$y_0^n y_1^{-n} =$

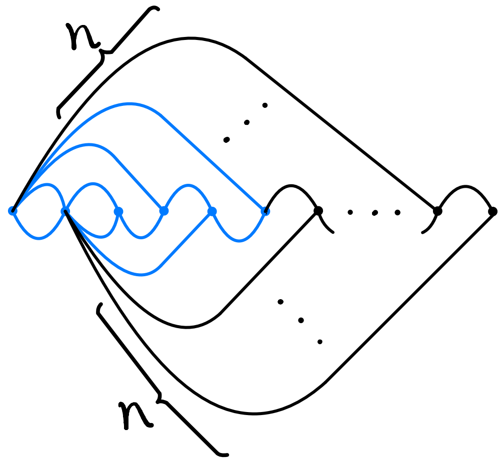


$(n \geq 1)$

$y_0^{n+1} y_1^{-n} =$

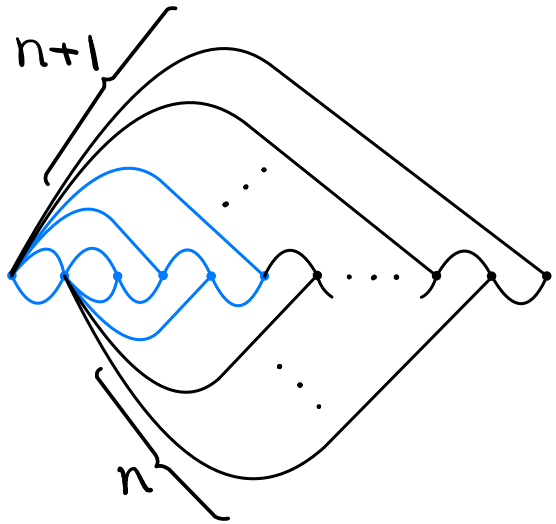


$y_0^n y_1^{-n} \rightsquigarrow$



,

$y_0^{n+1} y_1^{-n} \rightsquigarrow$



k	g_k	L	$\#L$	c	alternating	fibered	g	b
0	$y_0^2 y_1^{-1}$	$L2a1$	2	2	YES	YES	0	2
1	$y_0^2 y_1^{-2}$	4_1	1	4	YES	YES	1	3
2	$y_0^3 y_1^{-2}$	$L5a1$	2	5	YES	YES	1	3
3	$y_0^3 y_1^{-3}$	$L6a4$	3	6	YES	YES	1	3
4	$y_0^4 y_1^{-3}$	$L7a1$	2	7	YES	YES	2	3
5	$y_0^4 y_1^{-4}$	8_{18}	1	8	YES	YES	3	3

$\vdots$

b : braid index
c : crossing number
g : genus

1. Definitions and Constructions

2. Examples

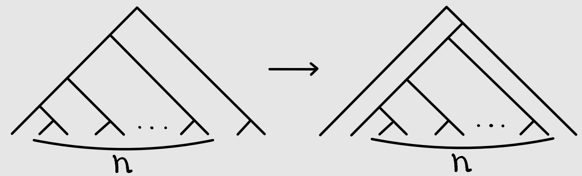
3. Main result

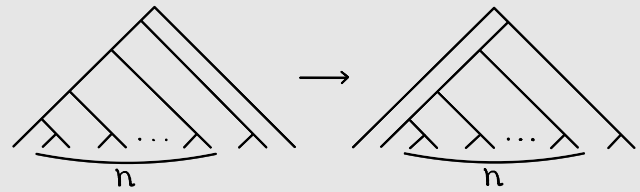
Thm [Kodama-T. 2022]

$\{g_k\}_{k\geq 0}$: sequence in $\vec{F}$ given by $g_k := \begin{cases} y_0^{m+2} y_1^{-m-1} & (k=2m) \\ y_0^{m+2} y_1^{-m-2} & (k=2m+1) \end{cases}$.

$L_k := \mathcal{L}(g_k)$.

- $\# L_k = \begin{cases} 1 & (k=6m+1, 6m+5) \\ 2 & (k=2m) \\ 3 & (k=6m+3) \end{cases}$

$y_0^n y_1^{-n} =$ 

$y_0^{n+1} y_1^{-n} =$ 

- $c(L_0) = 2$, $c(L_k) = k+3$ $(k \geq 1)$
- $^\forall k \geq 0$, L_k : alternating and fibered
- $b(L_0) = 2$, $b(L_k) = 3$ $(k \geq 1)$
- $g(L_k) = \begin{cases} m & (k=2m) \\ 3m+1 & (k=6m+1, 6m+3) \\ 3m+3 & (k=6m+5) \end{cases}$
- We can easily obtain the minimal genus flat Seifert surface of L_k from $\mathcal{S}(g_k)$.

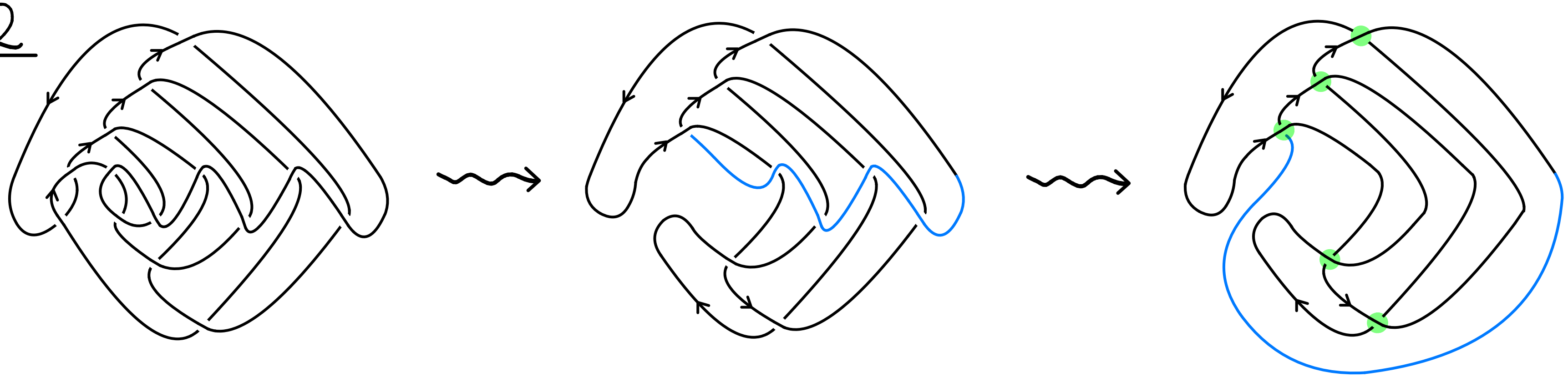
Proof (outline)

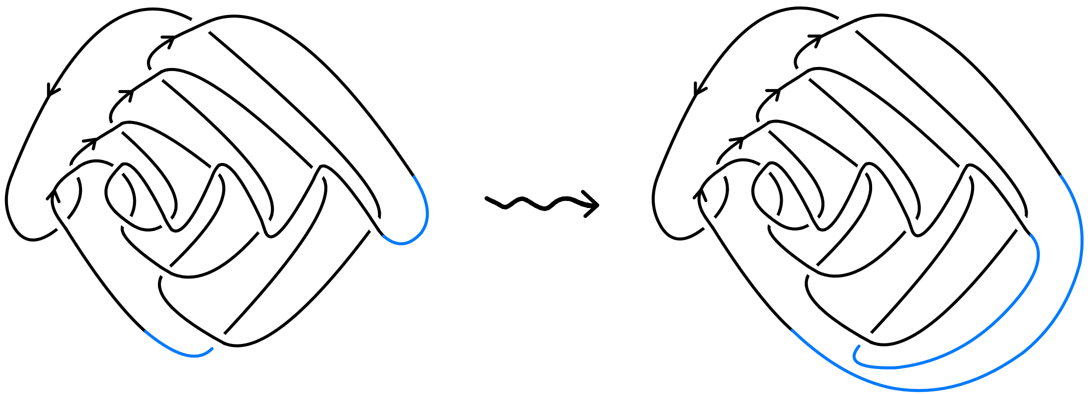
$\#L_k$

Computations using the result of [Aiello, 2019].

L_k : alternating, $c(L_k)$

$k=2$





$\sim L_3$

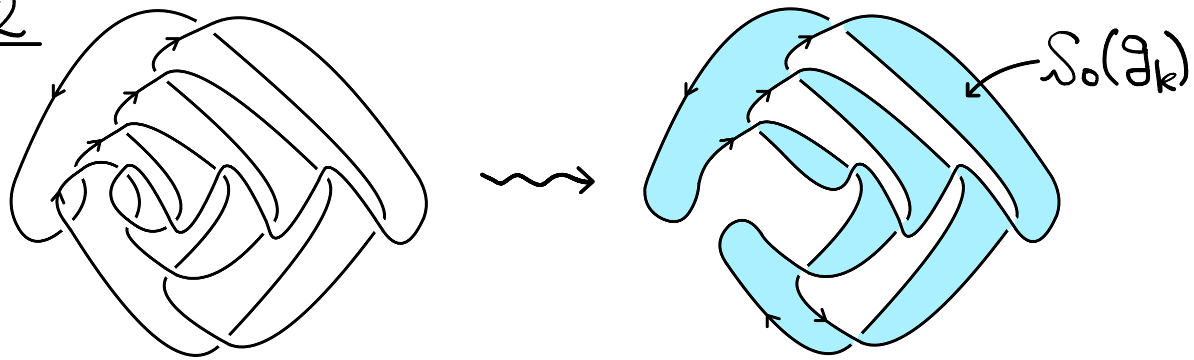
$g(L_k)$, L_k : fibered

$L \subset S^3$: link, $\Delta_L(t)$: Alexander polynomial of L.

Lem [Kodama-T. 2022]

$\forall k \geq 0$, $\deg \Delta_{L_k}(t) = k+1$ and $\Delta_{L_k}(t)$: monic.

$k=2$



$\rightsquigarrow$ Compute the Seifert matrix.

Fact [Crowell, 1959], [Murasugi, 1958], [Murasugi, 1963]

L : non-split alternating link

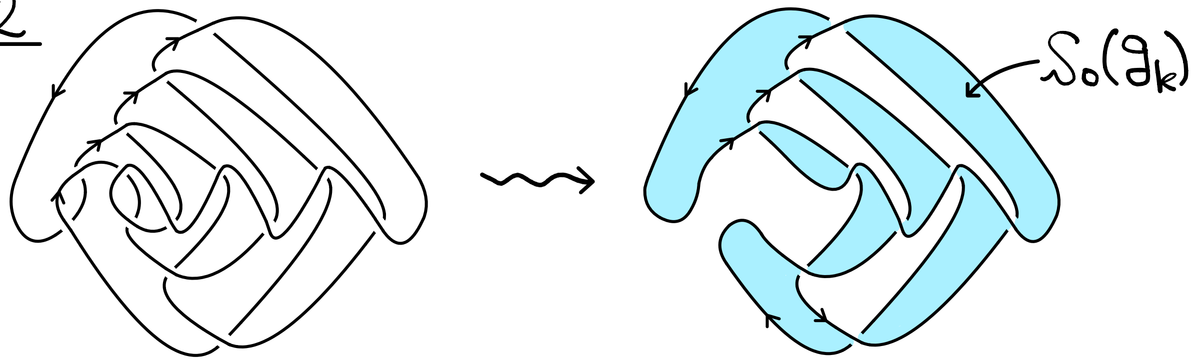
- $\deg \Delta_L(t) = 2g(L) + \#L - 1$,
- L : fibered $\Longleftrightarrow$ $\Delta_L(t)$: monic.

$b(L_k)$

$L \subset S^3$: link, $\Delta_L(t)$: Alexander polynomial of L.

Lem [Kodama-T. 2022]
$^\forall k \geq 0$, $\deg \Delta_{L_k}(t) = k+1$ and $\Delta_{L_k}(t)$: monic.

$k=2$



$\rightsquigarrow$ Compute the Seifert matrix.

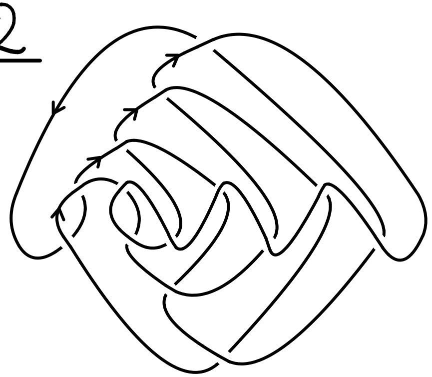
Fact [Murasugi, 1991]
L : non-split alternating fibered link
$b(L) = c(L) - \deg \Delta_L(t) + 1$.

Seifert surface

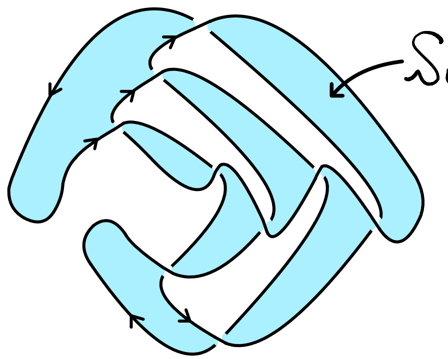
$L \subset S^3$: link, $\Delta_L(t)$: Alexander polynomial of L.

Lem [Kodama-T. 2022]
$\forall k \geq 0$, $\deg \Delta_{L_k}(t) = k+1$ and $\Delta_{L_k}(t)$: monic.

$k=2$



$\rightsquigarrow$



$S_0(g_k)$: minimal genus

$\rightsquigarrow$ Compute the Seifert matrix.

$H_1(S_0(g_k)) \cong \mathbb{Z}^{\oplus k+1}$, $k+1 = 2g(S_0(g_k)) + \#L - 1$

$\therefore g(S_0(g_k)) = g(L)$