Concordance for higher dimensional welded objects and their Milnor invariants

(n-dimensional cut-diagram)

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MILNOR CONCORDANCE INVARIANT FOR KNOTTED SURFACES AND BEYOND

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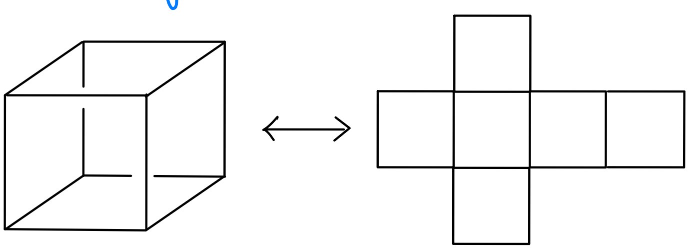
ABSTRACT. We generalize Milnor link invariants to all types of knotted surfaces in 4-space, and more generally to all codimension 2 embeddings. This is achieved by using the notion of cut–diagram, which is an higher dimensional generalization of Gauss diagrams, associated to codimension 2 embeddings in the spheres. We define a notion of group for cut-diagrams, which generalizes the fundamental group of the complement, and we extract Milnor-type invariants from the successive nilpotent quotients of this group. We show that the latter are invariant under an equivalence relation called cut-concordance, which encompasses the topological notion of concordance. We give several concrete applications of the resulting Milnor concordance invariants of knotted surfaces: we provide realization results, and compare their relative strength with previously known concordance invariants. Classification results up to link-homotopy are also obtained for Spun links. The theory of cut-diagrams is also further investigated, which provides a generalization of welded knotted objects to any dimension.

n-dim cut-diagram : (n-1)-dim diagram in n-dim with "labels"

Data of n-dim diagram in (n+1)-space

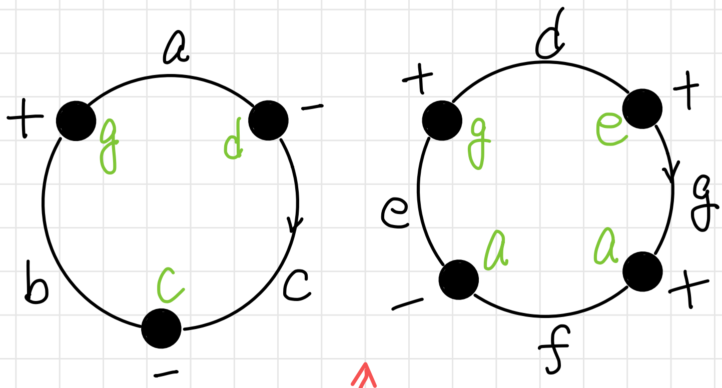
diagram of n-dim obj in (n+2)-space



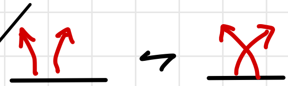
1-dim cut-diagram : 0-dim D on ori 1-dim Σ

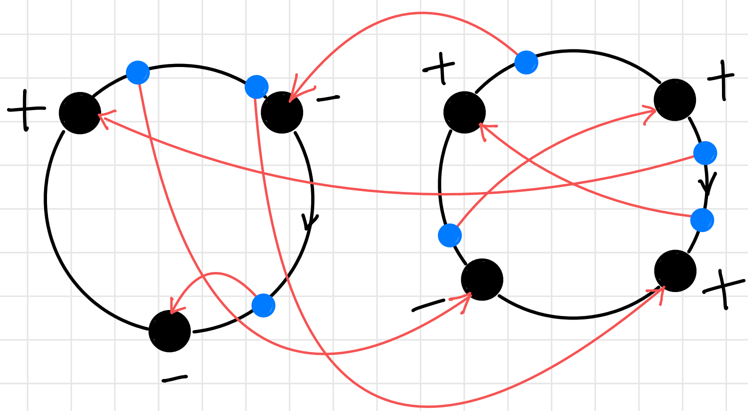
with sign & labeling map $\{\text{comp. of } D\} \to \Sigma \setminus D$

Example



Gauss diagram / tail commute move





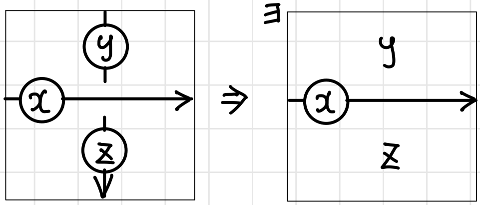
(● : under crossing   ● : over crossing)

compact, ori
possibly $\partial\Sigma \neq \emptyset$

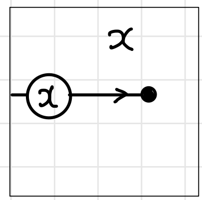
2-dim cut-diagram : 1-dim diagram D on ori 2-dim Σ with ori & labeling map $\{\text{arcs of } D\} \to \{\text{regions of } \Sigma \setminus D\}$

s.t.

(1)

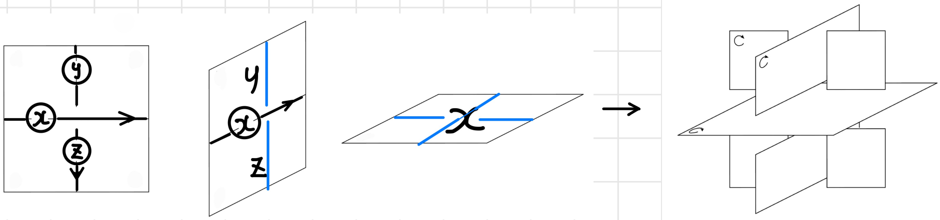


(2)

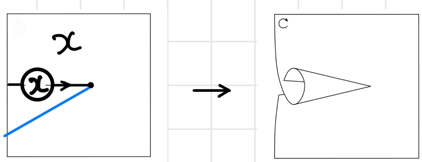


☆ it can be stuck together locally

(1)



(2)



n-dim cut-diagram : $\mathcal{C} := (\Sigma, D, \text{label})$

"diagram" D of compact ori $(n-1)$-mfd

on compact ori n-mfd Σ

Roseman '00, '04

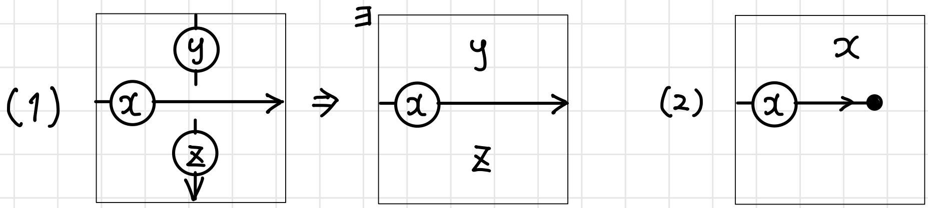
with labeling map

$\{$ "arcs" of $D \} \to \{$ regions of $\Sigma \setminus D \}$

$(n-1)$-dim

s.t. "natural" conditions similar to

(can be stuck together locally)



(We also call (Σ, D), D : cut-diagram)

Rem We can also define self-sing cut-diagram

Our works can be done within

"cut-diagram World"

cut-diagram World

1-dim

cut-diagram

concordance
& link-homotopy

2-dim

cut-diagram

concordance
& link-homotopy

3-dim

cut-diagram

concordance
& link-homotopy

⋮

results

- Milnor inv. for cut-diagram
- Milnor inv. are concordance inv. & link-homotopy inv. by combinatrial Stallings Theorem
- etc...

application

- concordance inv. & link-homotopy inv. for surface links
- etc...

Milnor inv for Classical links

(Milnor '54, '57)

$$\begin{tikzcd} L \arrow[d] \\ G(L) = \pi_1(S^3 \setminus L) \arrow[d] \arrow[dd, bend right] \\ G(L)/G(L)_2 \\ G(L)/G(L)_q \end{tikzcd}$$

qth nilpotent quotient

$$\begin{tikzcd} \begin{array}{|l|} \hline i\text{th longitude } l_i \\ i\text{th meridian } m_i \\ \hline \end{array} \arrow[d] \\ \begin{array}{|c|} \hline [l_i] \\ [m_i] \\ \hline \end{array} \arrow[d] \arrow[dd, bend right] \\ \begin{array}{|c|} \hline [l_i]_2 \\ [m_i]_2 \\ \hline \end{array} \\ \boxed{\begin{array}{c} [l_i]_q \\ [m_i]_q \end{array}} \end{tikzcd}$$

peripheral system

linking number

"Milnor inv."

qth nilpotent peripheral system

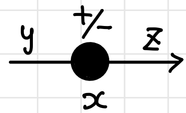
Group of $\mathcal{C}$ ($n = 1$ or 2)

$\mathcal{C} = (\Sigma, D, \mathit{label})$: cut-diagram

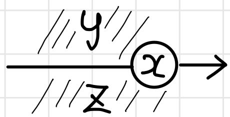
group of $\mathcal{C}$: $G(\mathcal{C}) := \langle X \mid R \rangle$

where $X := \{\text{regions of } \Sigma \setminus D\}$

$R :=$

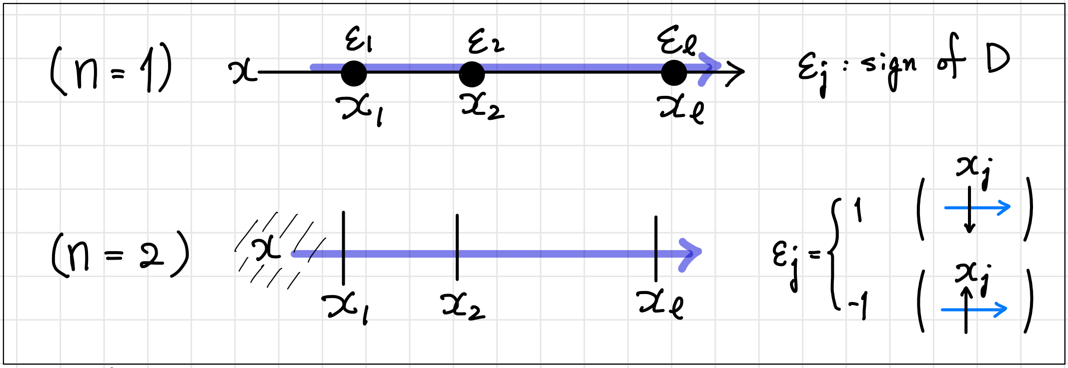
$\{ z^{-1} y^{x^{\pm 1}} \mid$ 

$\}$ ($n = 1$)

$\{ z^{-1} y^{x} \mid$ 

$\}$ ($n = 2$)

ori curve γ on Σ_i (i th comp. of Σ)



$$W_\gamma := x^{-|\gamma|} x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_\ell^{\varepsilon_\ell} \in G(\mathcal{C})$$

where $|\gamma| = \sum_J \varepsilon_j$ ($J = \{ j \mid \text{region } x_j \subset \Sigma_i \}$)

<u>Rem</u> γ, γ' : curves on Σ_i

$\gamma \underset{\text{homotopic}}{\sim} \gamma'$ (rel ∂) in Σ_i

$\Rightarrow W_\gamma = W_{\gamma'}$ in $G(\mathcal{C})$

Milnor inv for 1-dim cut-diagrams

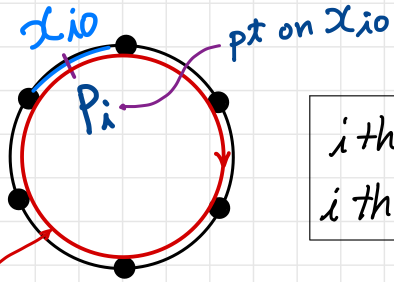
(i.e. for welded links)

(Chrisman '20, Miyazawa - Wada - Y '20)

$C = (\Sigma, D)$, $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$

May regard $l_i \in \pi_1(\Sigma_i, p_i)$

Σ_i :



l_i : curve from p_i to p_i

i th longitude	$\underline{l_i}$
i th meridian	x_{i0}

$$G(C) \longrightarrow G(C)/G(C)_q$$

$$\boxed{\begin{matrix} w_{l_i} \\ x_{i0} \end{matrix}} \longrightarrow \boxed{\begin{matrix} [w_{l_i}]_q \\ [x_{i0}]_q \end{matrix}}$$

peripheral system

"Milnor inv."

qth nilpotent quotient

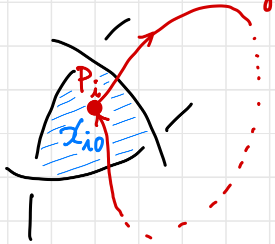
qth nilpotent peripheral system

Milnor inv for 2-dim cut-diagrams

$\mathcal{C} = (\Sigma, D)$ (for simplicity, suppose $\partial \Sigma_i = \phi$)

γ : P_i-based loop

Σ_i :



ith longitudes $\{\gamma \mid \gamma \in \pi_1(\Sigma_i, P_i)\}$
ith meridian x_{i0}

$\downarrow$

$G(\mathcal{C})$ $\qquad$ $\boxed{\begin{matrix}\{W_\gamma\} \\ x_{i0}\end{matrix}}$ peripheral system

$\downarrow$ $\qquad$ $\downarrow$

$G(\mathcal{C}) / G(\mathcal{C})_q$ $\qquad$ $\boxed{\begin{matrix}\{[W_\gamma]_q\} \\ [x_{i0}]_q\end{matrix}}$ $\leftarrow$ "Milnor inv."

qth nilpotent quotient $\qquad$ qth nilpotent peripheral system

The idea is easy to get.

Difficulty : How to get practical inv.

Chen-Milnor pres.

results

Set

$m_i := [x_{i0}]_q$ & $X = \{m_i\}_i$

$\ell_{ij} := [w_{ij}]_q$

where $\{w_{ij}\}$: generators of $\pi_1(\Sigma_i, p_i)$

Theorem (Chen-Milnor type pres.)

$$G(C)/G(C)_q \cong \langle X \mid \{[m_i, \ell_{ij}]\}_{i,j}, \langle X \rangle_q \rangle$$

Stallings Theorem for cut-diagram

$\mathcal{C}_0 = (\Sigma, D_0)$, $\mathcal{C}_1 = (\Sigma, D_1)$: cut-diagrams

$\mathcal{C}_0$ and $\mathcal{C}_1$ are (self-sing.) cut-concordant

$\overset{\text{def}}{\iff}$ $\mathcal{C}_0$ and $\mathcal{C}_1$ "bounds" (self-sing) cut-diagram on $\Sigma \times [0,1]$

Theorem $\mathcal{C}_0, \mathcal{C}_1$: cut-concordant

They bounds cut-diagram $\mathcal{C}$

$\Rightarrow$ "natural" inclusions $\mathcal{C}_i \to \mathcal{C}$ $(i=0,1)$ induce the iso.

$$G(\mathcal{C}_0)/G(\mathcal{C}_0)_q \xrightarrow{\cong} G(\mathcal{C})/G(\mathcal{C})_q \xleftarrow{\cong} G(\mathcal{C}_1)/G(\mathcal{C}_1)_q$$

combinatrial Stallings Theorem

Milnor number

$$\gamma \in \bigcup_j \pi_1(\Sigma_j, p_j)$$

words in $\langle X \rangle$ $(X = \{m_i\})$

$$[W_\gamma]_q \in G(\mathcal{C}) / G(\mathcal{C})_q \quad , \quad \underline{\omega_\gamma} \in [W_\gamma]_q$$

q: sufficiently large

Magnus exp. (i.e. $E(m_j) = 1 + X_j$)

$$E(\omega_\gamma) = 1 + \sum \underline{\mu_{\mathcal{C}}(j_1 \cdots j_k ; \gamma)}\, X_{j_1} \cdots X_{j_k}$$

$$\underline{m_{\mathcal{C}}(I\, i)} := \gcd \{ \mu_{\mathcal{C}}(I ; w_{i j}) \mid j = 1, \dots, \beta_1(\Sigma_i) \}$$

$$\underline{\Delta_{\mathcal{C}}(I)} := \gcd \{ m_{\mathcal{C}}(J) \mid J \xleftarrow[\text{permutate cyclicly}]{\text{delete \&}} I \}$$

Theorem

$\Delta_{\mathcal{C}}(I)$ is inv. for $\mathcal{C}$

$\gamma \in \pi_1(\Sigma_i, p_i)$,

$$\underline{\bar{\mu}_e(I;\gamma)} := \mu_e(I;\gamma) \bmod \Delta_e(I_i)$$

$\lambda \in H_1(\Sigma_i)$,

$\gamma \in \pi_1(\Sigma_i, p_i)$ s.t. γ represents λ

$$\underline{\bar{\mu}_e(I;\lambda)} := \bar{\mu}_e(I;\gamma)$$

Theorem

$\underline{\bar{\mu}_e(I;\lambda)}$ $(\lambda \in H_1(\Sigma_i))$ are well-defined

independent of the choice of $\{p_i\}_i$

Milnor inv for $\mathcal{C}$

Milnor map

$$\begin{array}{rccc} M_{\mathcal{C}}^{Ii} : H_1(\Sigma_i) & \to & \mathbb{Z}/\Delta_{\mathcal{C}}(Ii)\mathbb{Z} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \lambda & \longmapsto & \bar{\mu}_{\mathcal{C}}(I;\lambda) \end{array}$$

Milnor loop-inv.

$$\nu_{\mathcal{C}}(Ii) := \gcd\{\Delta_{\mathcal{C}}(Ii),\ m_{\mathcal{C}}(Ii)\}$$

Milnor inv = Milnor map & Milnor loop-inv.

Theorem

(1) Milnor inv. are cut-concordance inv.

(2) Ii is nonrepeated seq

$\Rightarrow M_F^{Ii}$ & $\nu_F(Ii)$ are self-sing. cut-concordance inv.

Milnor inv for surface links

F : surface link

$\mathcal{C}$: 2-dim cut-diagram obtained from a diagram of F

$$M_F^{Ii} := M_{\mathcal{C}}^{Ii}, \quad \mu_F(Ii) := \mu_{\mathcal{C}}(Ii)$$

Cor

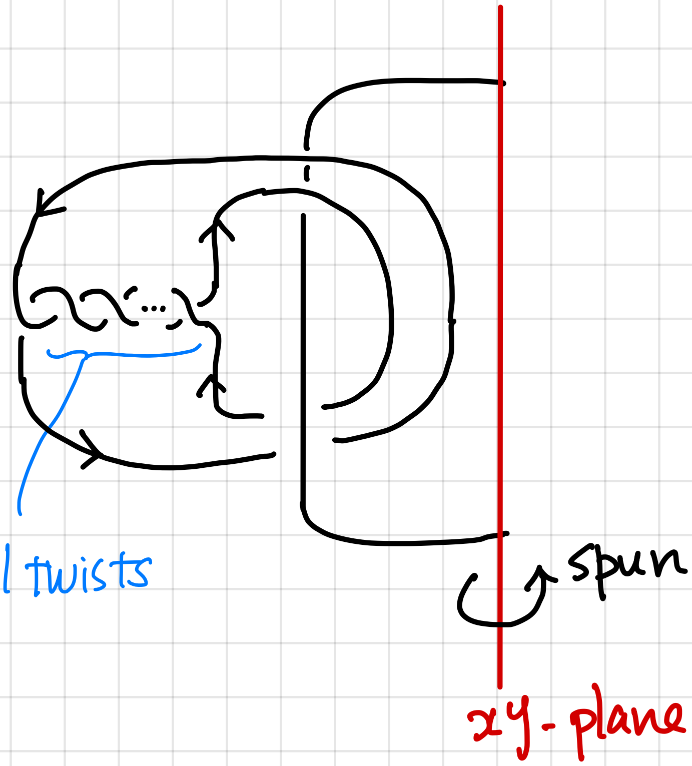
(1) M_F^{Ii} & $\mu_F(Ii)$ are concordance inv.

(2) Ii is nonrepeated seq.

$\Rightarrow$ M_F^{Ii} & $\mu_F(Ii)$ are link-homotopy inv.

Example

$L_m :=$



$$\boxed{L_m \overset{c}{\sim} L_{m'} \iff m = m'}$$

Remark • Sato-Levin inv. for L_m vanish

N. Sato, *Cobordisms of semi-boundary links*, Topology Appl. **18** (1984), 225–234. MR0769293

• Cochran's derivations of L_m are trivial

T. Cochran, *Geometric invariants of link cobordism*, Comment. Math. Helv. **60** (1985), 291– 311. MR0800009

• Saito's inv of L_m's are all same.

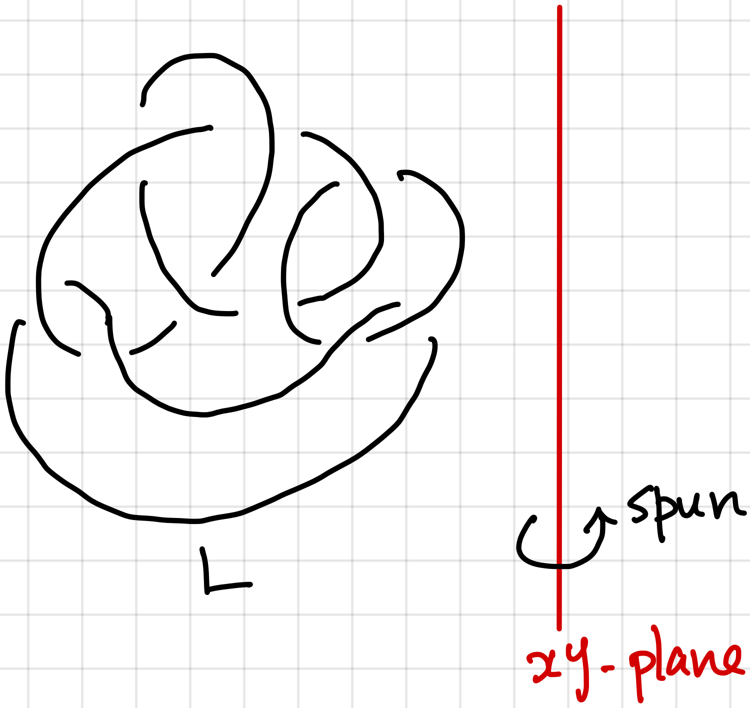
M. Saito, *A note on cobordism of surface links in S4*, Proc. Amer. Math. Soc. **111** (1991), 883–887. MR1087008

• $L_m \overset{lh}{\sim} L_{m'}$

Example

$\underline{\mathrm{spun}(L)} :=$

emb. tori



L, L': 3-comp. links

$\mathrm{Spun}(L) \overset{l.h.}{\sim} \mathrm{Spun}(L')$

$\iff L \overset{l.h}{\sim} L'$ or $\underline{-\overline{L'}}$

refrected inverse of L'