On keen bridge splittings of links

(joint work with A. Ido and T. Kobayashi)

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Intelligence of Low-dimensional Topology

Main Results

Thm 1 $\forall g \geq 0$, $\forall b \geq 1$, $\forall n \geq 1$
except for $(g,b) = (0,1)$, $(g,b,n) = (0,3,1)$,
$\exists$ strongly keen (g,b)-splitting with distance n.

Thm 2 Any $(0,3)$-splitting with distance 1
cannot be keen.

Contents

§1. (Strongly) keen Heegaard splittings

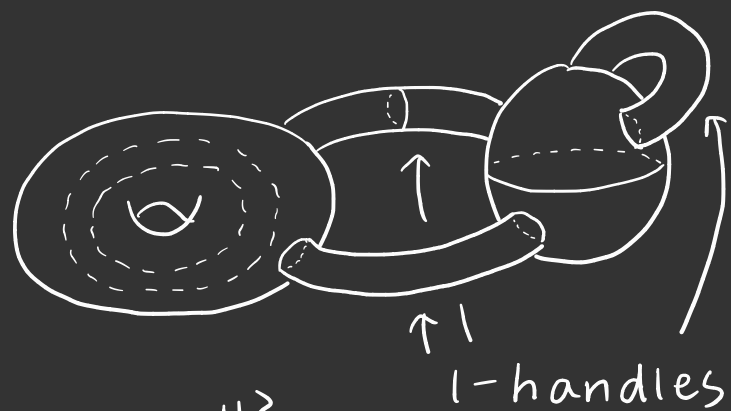
Heegaard splittings

- C : compression-body

$\overset{def}{\Longleftrightarrow}$ C is a connected 3-manifold obtaind from $S \times [0,1] \cup$ (3-ball) by attaching "1-handles" to $S \times \{1\} \cup \partial$(3-ball)
(S : closed orientable surface, possibly $S = \phi$)

$\partial_- C := S \times \{0\}$
$\partial_+ C := \partial C \setminus \partial_- C$

The genus of $\partial_+ C$ is called the genus of C



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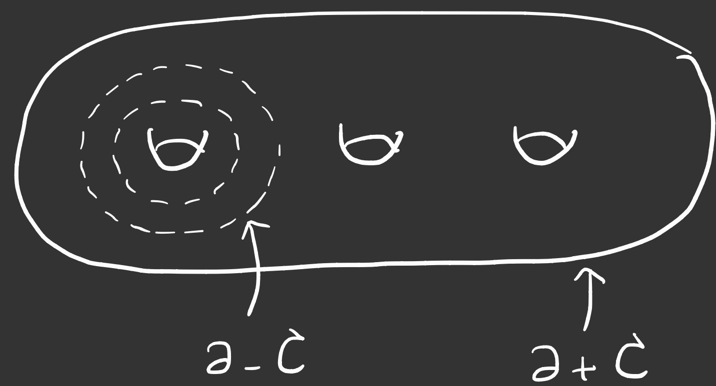


- C : handlebody

$\overset{def}{\Longleftrightarrow}$ C : compression-body & $\partial_- C = \phi$

- M : compact orientable 3-manifold

$C_1 \cup_\Sigma C_2$: (genus-g) Heegaard splitting of M

$\overset{\text{def}}{\Longleftrightarrow}$
- C_1, C_2 : genus-g compression-bodies
- $C_1 \cup C_2 = M$
- $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = \Sigma$ ← Heegaard surface

<u>Fact</u> Any compact orientable 3-manifold admits a Heegaard splitting. (Moise '52)

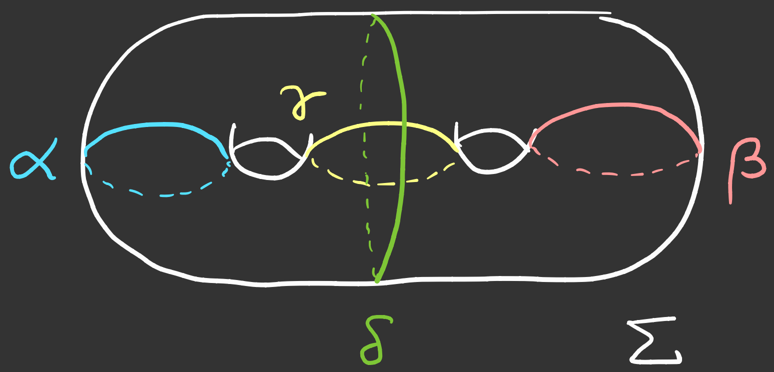
(Hempel) distance of Heegaard splitting

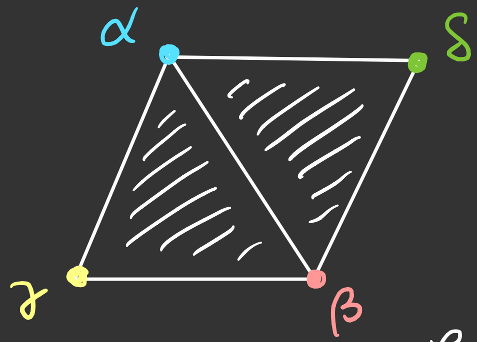
- Σ : closed orientable surface of genus ≥ 2

The curve complex $\mathcal{C}(\Sigma)$ of Σ
is the simplicial complex s.t.

$$\begin{cases} \text{0-simplex} \longleftrightarrow \text{(isotopy class of) an essential simple closed curve on } \Sigma \\ \text{n-simplex } (n \geq 1) \longleftrightarrow (n+1) \text{ s.c.c.s on } \Sigma \text{ which are mutually disjoint} \end{cases}$$

$*$ $d : \mathcal{C}(\Sigma)^{(0)} \times \mathcal{C}(\Sigma)^{(0)} \to \mathbb{Z}_{\geq 0}$: the simplicial distance on $\mathcal{C}(\Sigma)$





$d(\alpha, \beta) = 1$

$d(\gamma, \delta) = 2$

$\mathcal{C}(\Sigma)$

$C_1 \cup_\Sigma C_2$: Heegaard splitting (of M)

$\mathcal{C}(\Sigma)$: curve complex of Σ

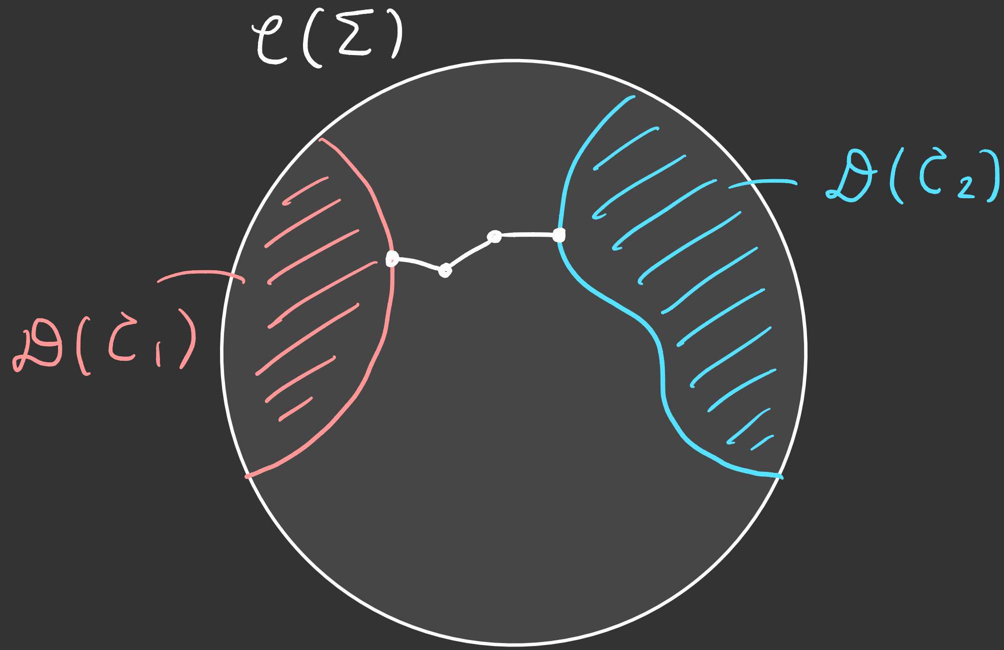
- $\mathcal{D}(C_i)$ $(\subset \mathcal{C}(\Sigma))$: disk complex of C_i

  $\overset{\text{def}}{\iff}$ $[\alpha \in \mathcal{D}(C_i)^{(0)} \iff \alpha$ bounds a disk in $C_i]$

- $d(C_1 \cup_\Sigma C_2)$

  $:= d_{\mathcal{C}(\Sigma)}(\mathcal{D}(C_1), \mathcal{D}(C_2))$

  : the distance of $C_1 \cup_\Sigma C_2$



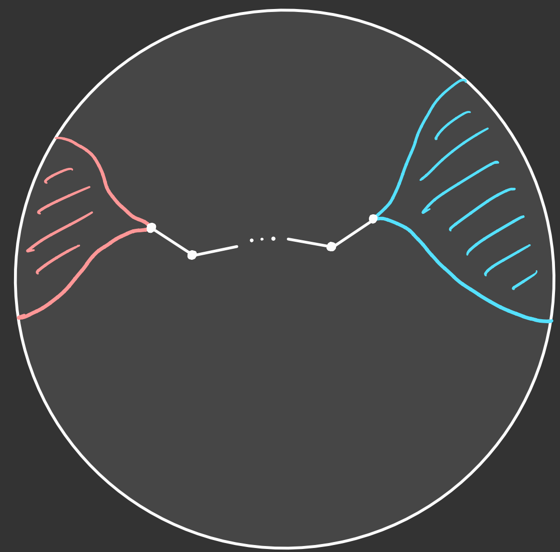
Fact $\forall n$, $\exists\, C_1 \cup_\Sigma C_2$
s.t. $d(C_1 \cup_\Sigma C_2) \geq n$.
(Hempel '01, ...)

Fact $d(C_1 \cup_\Sigma C_2)$ "reflects"
a lot of properties of M.

(Strongly) keen Heegaard splittings

- $C_1 \cup_\Sigma C_2$: keen
  $\overset{\text{def}}{\iff}$ $d(C_1 \cup_\Sigma C_2)$ is realized by a unique pair of elements of $\mathcal{D}(C_1)^{(0)}$ and $\mathcal{D}(C_2)^{(0)}$.

- $C_1 \cup_\Sigma C_2$: strongly keen
  $\overset{\text{def}}{\iff}$ $C_1 \cup_\Sigma C_2$ is keen
  & $d(C_1 \cup_\Sigma C_2)$ is realized by a unique geodesic in $\mathcal{C}(\Sigma)$.



<u>Fact</u> The "keenness" of Heegaard splitting is related with certain finiteness of "Goeritz group".
(Iguchi - Koda '20)

§2. (Strongly) keen bridge splittings

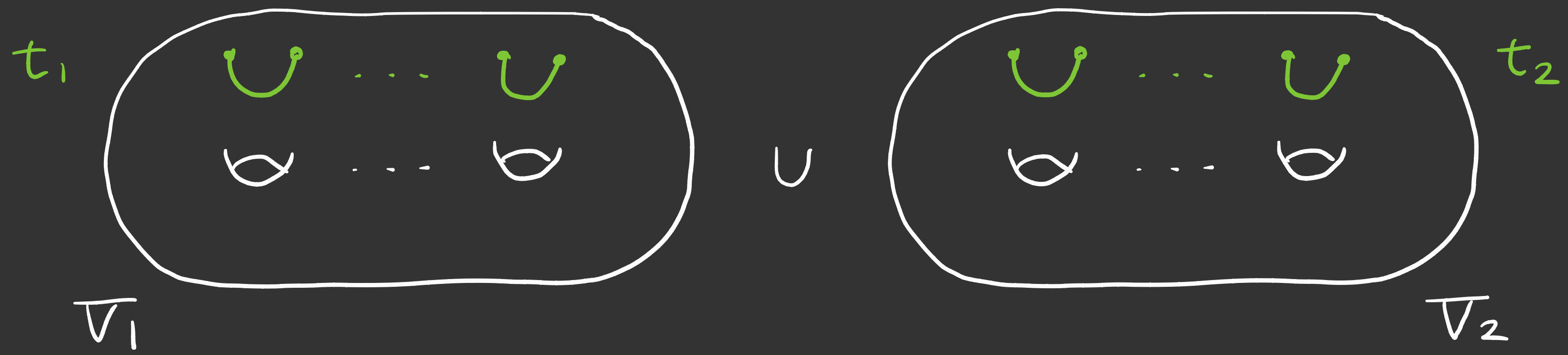
Bridge splittings

- M : closed orientable 3-manifold
  L : link in M

$(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$: (g, b)-splitting of (M, L)

$\overset{\text{def}}{\Longleftrightarrow}$
- $V_1 \cup_F V_2$: genus-g Heegaard splitting of M
- $P = L \cap F$
- $t_i = L \cap V_i$: b arcs parallel to ∂V_i $(i = 1, 2)$

* F is called a bridg surface.

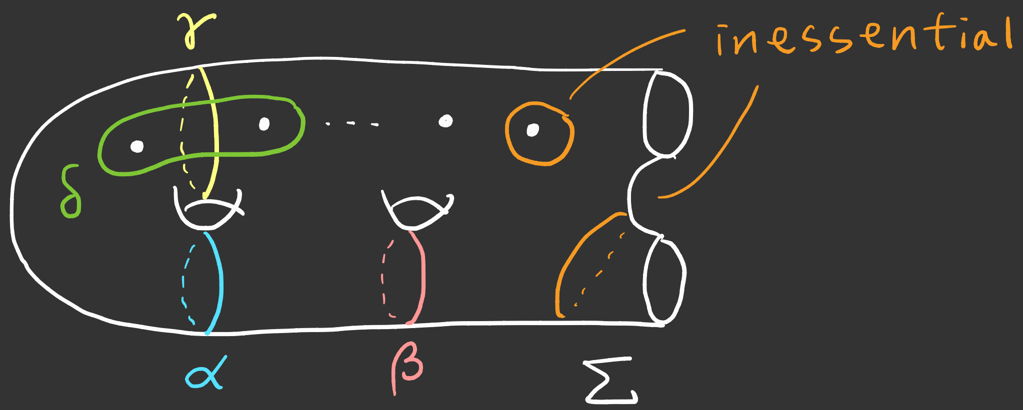


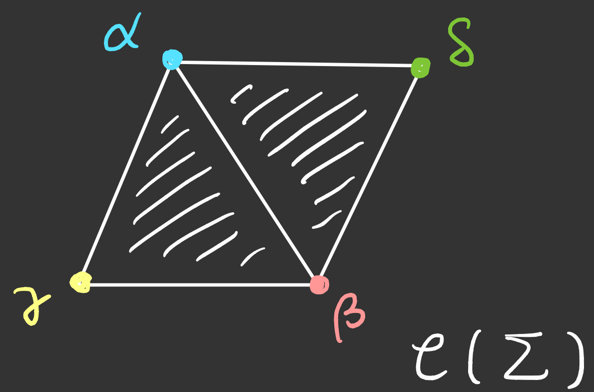
Distance of bridge splitting

- Σ : orientable surface of genus g
  with c boundary components & p punctures.
  (Assume : $3g+c+p > 4$.)

The curve complex $\mathcal{C}(\Sigma)$ of Σ
is the simplicial complex s.t.

$$\begin{cases} \text{0-simplex} \longleftrightarrow \text{(isotopy class of) an essential simple closed curve on } \Sigma \\ \text{n-simplex } (n \geq 1) \longleftrightarrow (n+1) \text{ s.c.c.s on } \Sigma \text{ which are mutually disjoint} \end{cases}$$





- $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$: (g,b)-splitting

$\mathcal{C}(F \setminus P)$: curve complex of $F \setminus P$

$\mathcal{D}(V_i \setminus t_i)$ $(\subset \mathcal{C}(F \setminus P))$: disk complex of $V_i \setminus t_i$

(i.e., $\alpha \in \mathcal{D}(V_i \setminus t_i)^{(0)} \Longleftrightarrow \alpha$ bounds a disk in $V_i \setminus t_i$)

$d((V_1, t_1) \cup_{(F,P)} (V_2, t_2)) := d_{\mathcal{C}(F \setminus P)}(\mathcal{D}(V_1 \setminus t_1), \mathcal{D}(V_2, t_2))$

: the distance of $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$

<u>Fact</u> (1) $\exists$ bridge splittings with high distance

(Saito '04, Campisi-Rathbun '12,
Blair-Tomova-Yoshizawa '13, Ichihara-Saito '13)

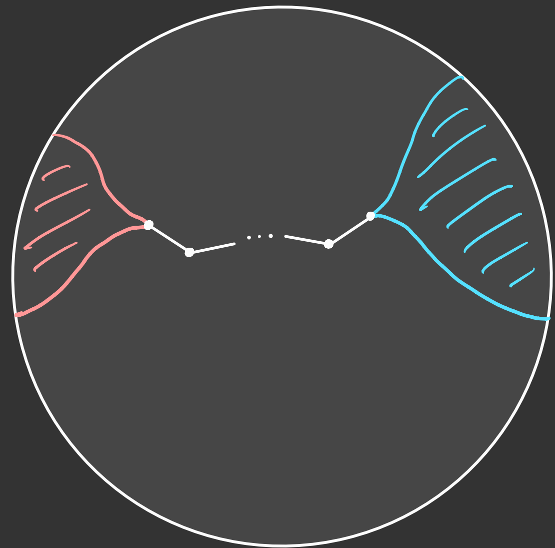
(2) Some upper bounds for distance
in terms of $\left[\begin{array}{l}\text{alternative splittings}\\ \text{essential surfaces}\end{array}\right.$

(Bachman-Schleimer '05, Tomova '07, J. '14, Ido '15)

(Strongly) keen bridge splittings

- $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$: keen

  $\overset{\text{def}}{\Longleftrightarrow}$ $d((V_1,t_1) \cup_{(F,P)} (V_2,t_2))$ is realized by a unique pair of elements of $\mathcal{D}(V_1, t_1)^{(0)}$ and $\mathcal{D}(V_2, t_2)^{(0)}$.



- $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$: strongly keen

  $\overset{\text{def}}{\Longleftrightarrow}$ $(V_1,t_1) \cup_{(F,P)} (V_2,t_2)$ is keen

  & $d((V_1,t_1) \cup_{(F,P)} (V_2,t_2))$ is realized by a unique geodesic in $\mathcal{C}(F, P)$.

Main Results

Thm 1 $\forall g \geq 0, \forall b \geq 1, \forall n \geq 1$
except for $(g,b) = (0,1)$, $(g,b,n) = (0,3,1)$,
$\exists$ strongly keen (g,b)-splitting with distance n.

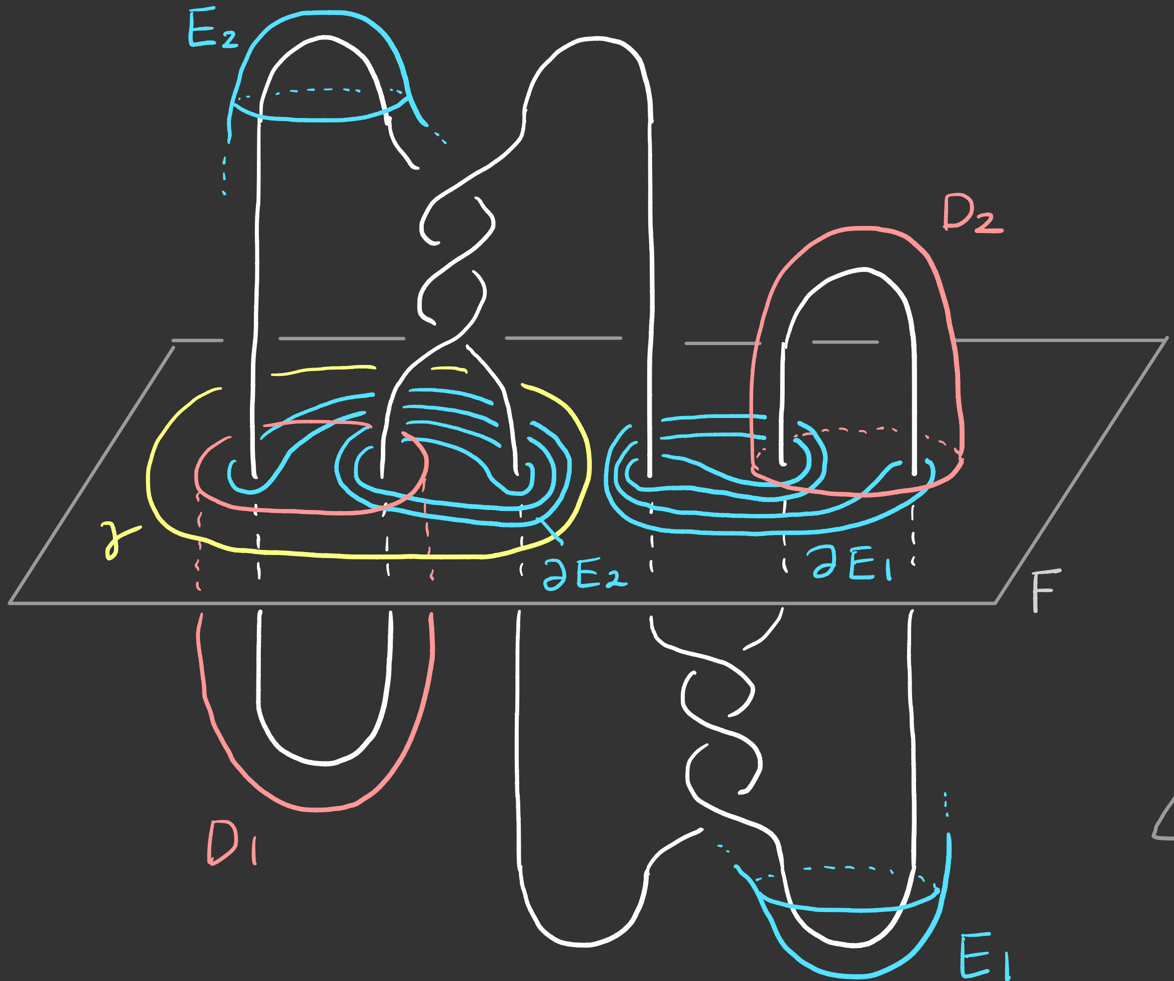
Remark

- $(g,b) = (0,1) \Rightarrow \mathcal{C}(F \setminus P) = \phi$
- $(g,b,n) = (1,1,1)$
  $\Rightarrow$ the ambient manifold is $S^2 \times S^1$
  & the link is a core knot $(\{*\} \times S^1)$
  (Saito '04)

$\rightsquigarrow$

Any $(1,1)$-splitting with distance 1
is strongly keen.

Thm 2 Any (0,3)-splitting with distance 1 cannot be keen.

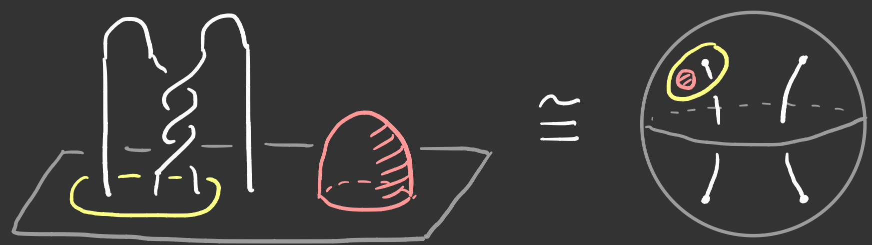


(D_1, D_2) : pair of disks realizing distance

$\rightsquigarrow$

$\exists\, \gamma \subset F \setminus P$

s.t. each component of $(F \setminus P) \setminus \gamma$ contains one of ∂D_1 & ∂D_2 & three punctures



$\rightsquigarrow$

can find another pair (E_1, E_2) realizing distance

Remark

In

Ido - J. - Kobayashi,
Bridge splittings of links with distance exactly n,
Topology Appl. 196 (2015), 608-617

we "showed" that

$\forall n \geq 2$, $\forall g \geq 0$, $\forall b \geq 1$ except for $(g,b) = (0,1), (0,2)$,
$\exists$ (g,b)-splitting with distance exactly n.

However, we realized that there is a gap in the proof.

We note that Thm 1 recovers the above result.

§3. Outline of proof of Thm 1.

<u>Thm 1</u> $\forall g \geq 0$, $\forall b \geq 1$, $\forall n \geq 1$

except for $(g,b) = (0,1)$, $(g,b,n) = (0,3,1)$,

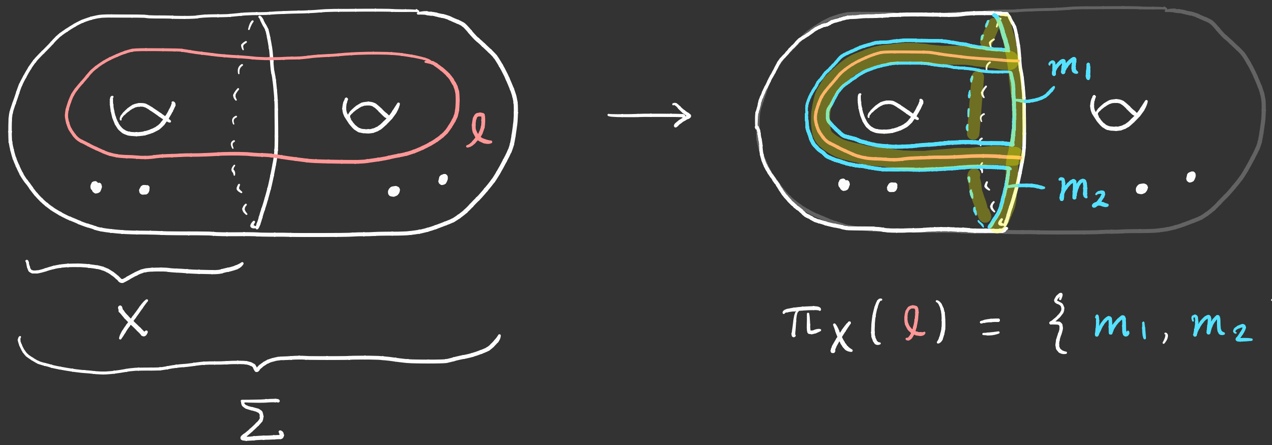
$\exists$ strongly keen (g,b)-splitting with distance n.

Subsurface projection

- X : essential non-simple subsurface of Σ

  The subsurface projection $\pi_X : \mathcal{C}(\Sigma)^{(0)} \longrightarrow \mathcal{P}(\mathcal{C}(X)^{(0)})$ is defined as follows :

  $\uparrow$ the power set



$\pi_X(\ell) = \{ m_1, m_2 \}$

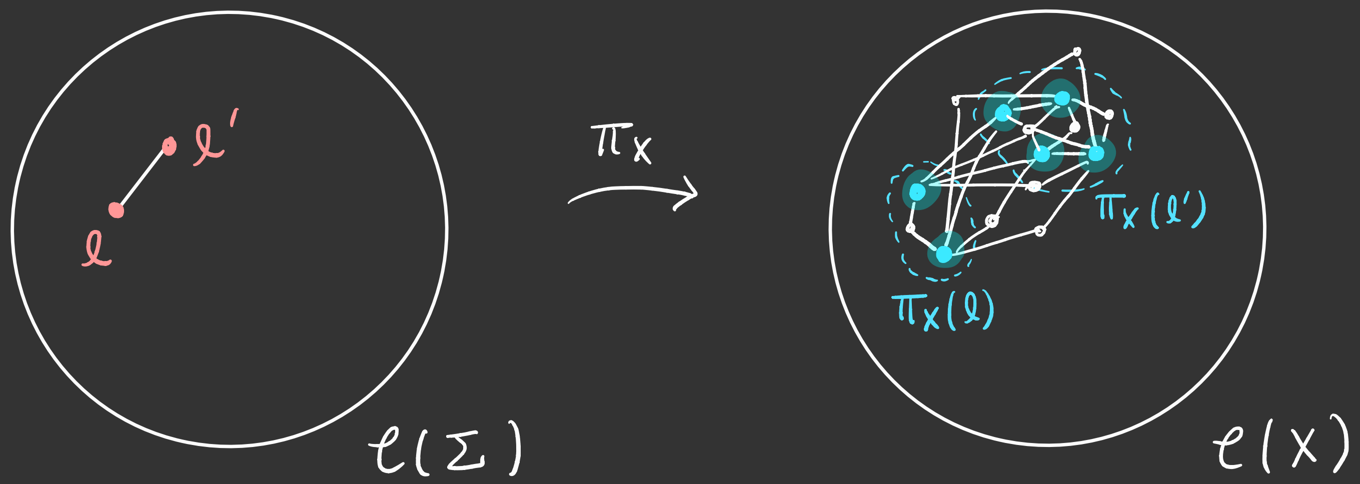
$\pi_X(\ell)$ = the union of

the set of the isotopy classes of the components of $\partial N_X(\alpha \cup \partial X)$ which are essential in X for every component α of $\ell \cap X$.

Lemma 1 (Masur - Minsky '00)

$$d_{\mathcal{C}(\Sigma)}(\ell, \ell') \leq 1 \quad \Rightarrow \quad \operatorname{diam}_{\mathcal{C}(X)}\left(\pi_X(\ell) \cup \pi_X(\ell')\right) \leq 2 .$$

$\ell \cap X \neq \phi ,\ \ell' \cap X \neq \phi$



Lemma 1' ("π_X is 2-Lipschitz.")

$[\ell_0, \ell_1, \cdots, \ell_m]$: path in $\mathcal{C}(\Sigma)$

s.t. $\ell_i \cap X \neq \phi \ (\forall i)$.

$$\Rightarrow \quad \operatorname{diam}_{\mathcal{C}(X)}\left(\pi_X(\ell_0) \cup \cdots \cup \pi_X(\ell_m)\right) \leq 2m .$$

Proof of Thm 1 for the case of $n \geq 2$

Let ℓ : essential s.c.c. on Σ

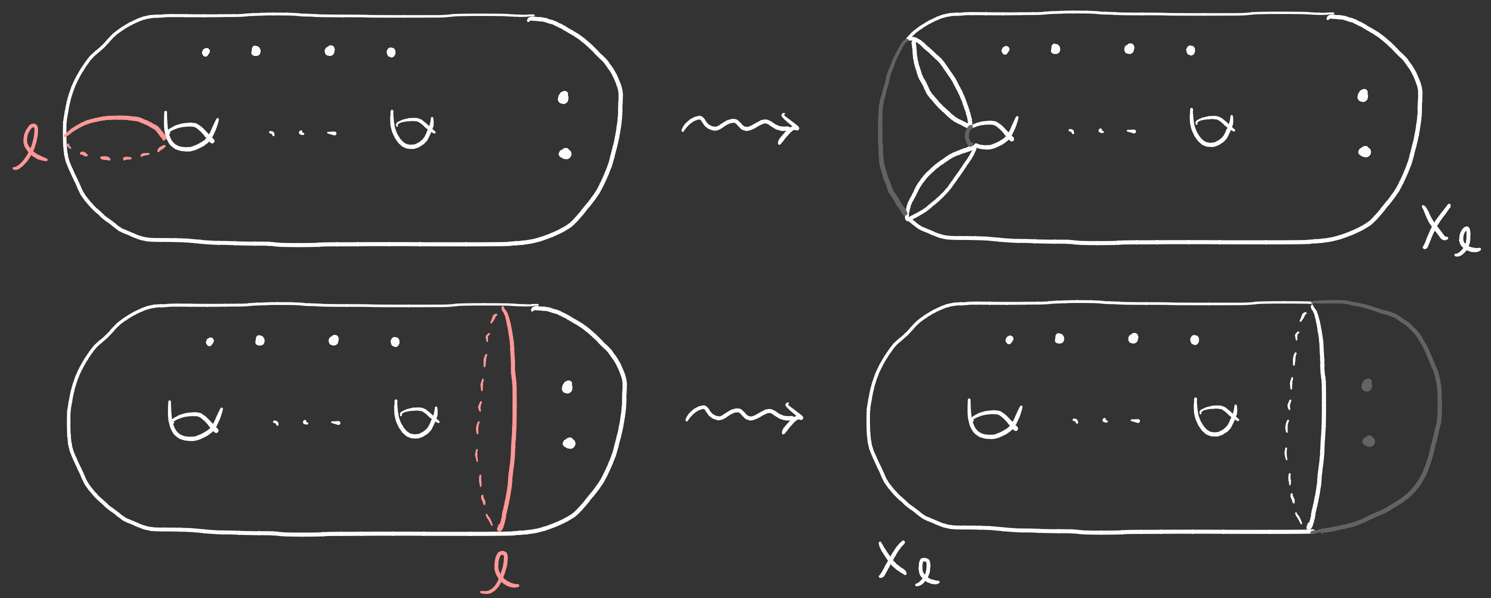
s.t. (i) ℓ : non-separating in Σ, or

(ii) ℓ cuts off a twice-punctured disk from Σ.

The subsurface associated with ℓ (denoted by X_ℓ)
is defined as follows :

(i)

(ii)



Remark $\quad m \cap X_\ell = \phi \Rightarrow m = \ell$.

$(m \in \mathcal{C}(\Sigma)^{(0)})$

Proposition (Ido - J. - Kobayashi)

Let $[\ell_0, \ell_1, \cdots, \ell_{n-1}, \ell_n]$ be a path in $\mathcal{C}(\Sigma)$ s.t.

(1) $[\ell_0, \ell_1, \cdots, \ell_{n-1}]$: unique geodesic connecting ℓ_0 & ℓ_{n-1},

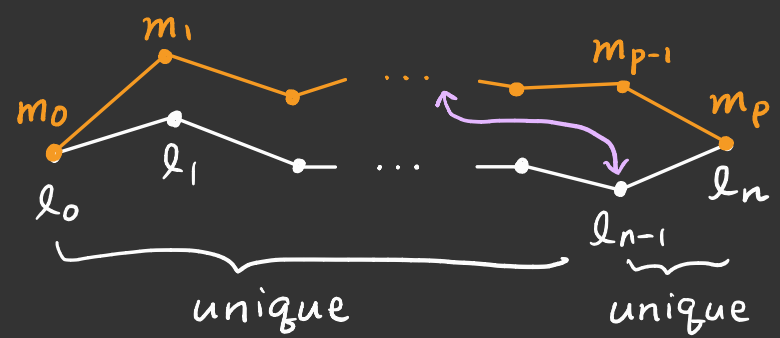
(2) ℓ_{n-1} is non-separating in Σ, or
ℓ_{n-1} cuts off a twice-punctured disk from Σ,

(3) $\mathrm{diam}_{\mathcal{C}(X_{\ell_{n-1}})}\left(\pi_{X_{\ell_{n-1}}}(\ell_0) \cup \pi_{X_{\ell_{n-1}}}(\ell_n)\right) > 2n$.

Then $[\ell_0, \ell_1, \cdots, \ell_{n-1}, \ell_n]$ is the unique geodesic connecting ℓ_0 & ℓ_n.

Proof (Idea) Let $[m_0, \cdots, m_p]$ be a geodesic with $m_0 = \ell_0$, $m_p = \ell_n$, $p \leq n$.

Claim : $\exists\, i$ s.t. $m_i = \ell_{n-1}$.



$\because$ If $m_i \neq \ell_{n-1}$ for $\forall i$,
then $m_i \cap X_{\ell_{n-1}} \neq \phi$ for $\forall i$ by (2).
By Lemma 1′, this implies
$\mathrm{diam}_{\mathcal{C}(X_{\ell_{n-1}})}\left(\pi_{X_{\ell_{n-1}}}(\ell_0) \cup \pi_{X_{\ell_{n-1}}}(\ell_n)\right) \leq 2p \leq 2n$,
contradicting (3).

By Claim & (1), we have $[m_0, \cdots, m_p] = [\ell_0, \cdots, \ell_n]$.

Construction of strongly keen bridge splittings

F : closed orientable surface of genus g

P : union of $2b$ points on F

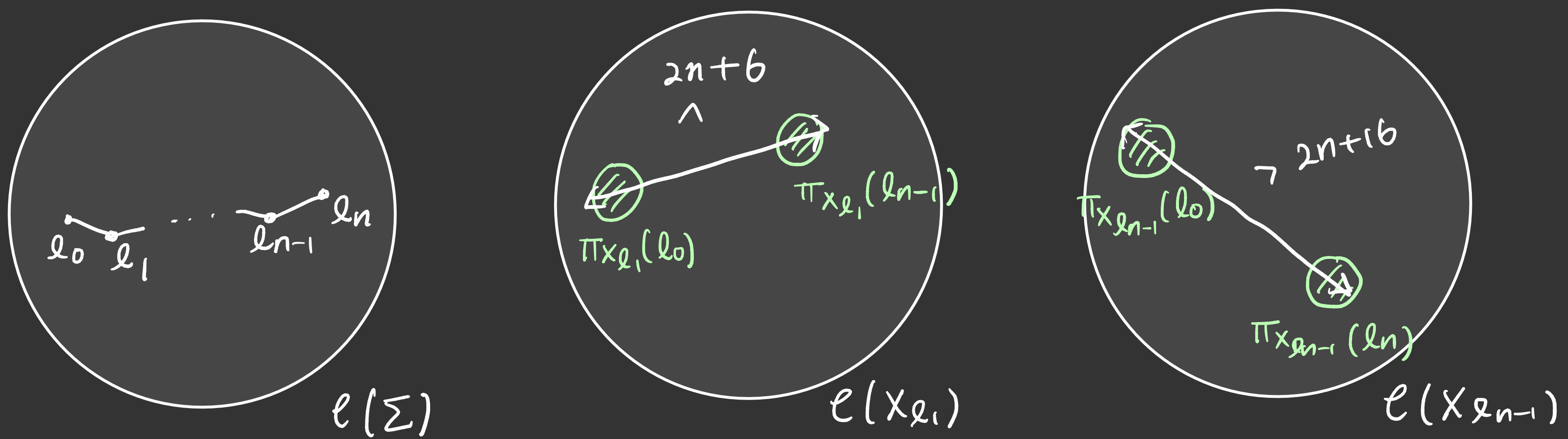
$[l_0, l_1, \cdots, l_n]$: geodesic in $\mathcal{C}(F \setminus P)$

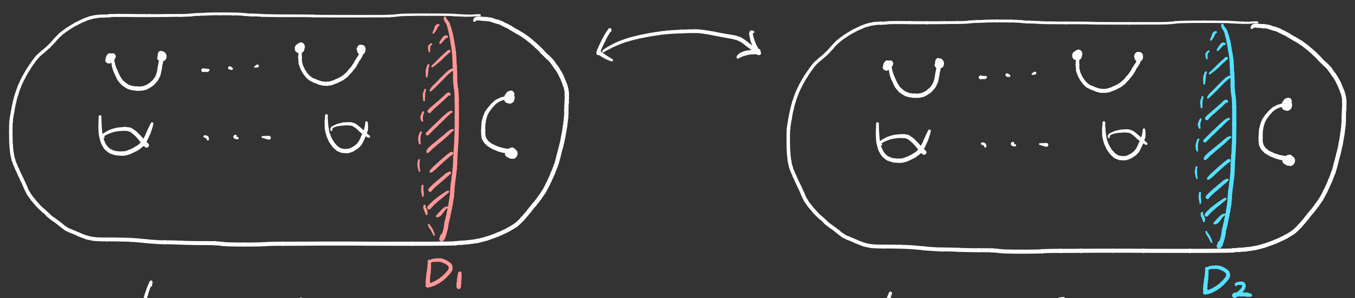
s.t.

(1) $[l_0, l_1, \cdots, l_n]$: unique geodesic connecting l_0 & l_n,

(2) $\mathrm{diam}_{\mathcal{C}(X_{l_1})}(\pi_{X_{l_1}}(l_0) \cup \pi_{X_{l_1}}(l_{n-1})) > 2n+6$,

(3) $\mathrm{diam}_{\mathcal{C}(X_{l_{n-1}})}(\pi_{X_{l_{n-1}}}(l_0) \cup \pi_{X_{l_{n-1}}}(l_n)) > 2n+16$.



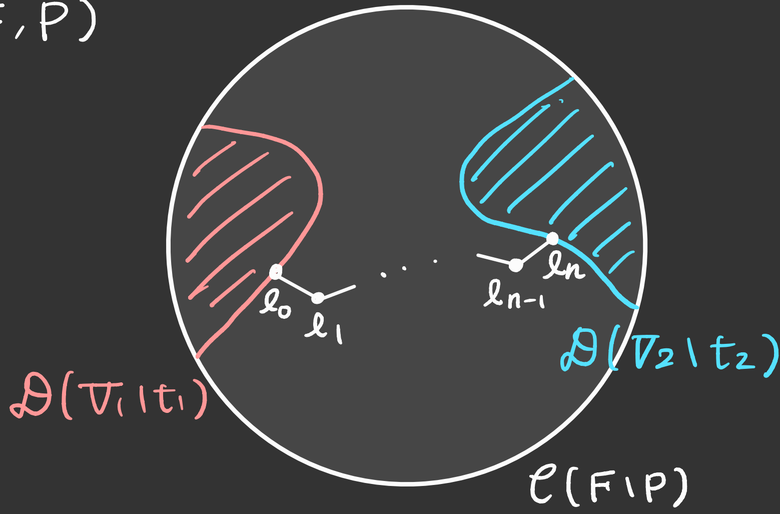


(V_1, t_1) $\qquad$ (V_2, t_2)

Identify $(\partial V_1, \partial t_1)$, $(\partial V_2, \partial t_2)$ with (F, P)
so that $\partial D_1 = \ell_0$, $\partial D_2 = \ell_n$.

$\rightsquigarrow$ $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$: (g, b)-splitting

& $d\big((V_1, t_1) \cup_{(F,P)} (V_2, t_2)\big) \leq n$.



Modify (V_i, t_i) as follows :

glue back by h_i s.t.

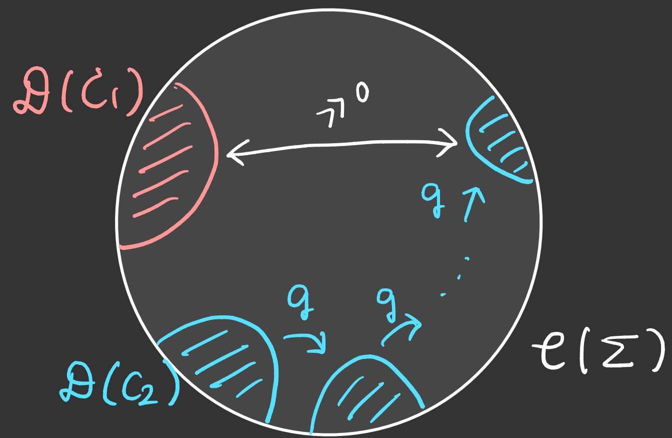
$$\begin{cases} d_{\mathcal{E}(\partial_- W_1, s_1)}\left(\ell_1, h_1(\mathcal{D}(V_1^\circ, t_1^\circ))\right) > 2, & \cdots (4) \\ d_{\mathcal{E}(\partial_- W_2, s_2)}\left(\ell_{n-1}, h_2(\mathcal{D}(V_2^\circ, t_2^\circ))\right) > 2. & \cdots (5) \end{cases}$$

✾ The existence of such h_i is guaranteed by [Ichihara-Saito '13] , a variation of Hempel's argument.

Thm (Hempel)

$\exists\, g : \Sigma \to \Sigma$ s.t.

$d(C_1 \cup_{g^n} C_2) \to \infty \quad (n \to \infty)$

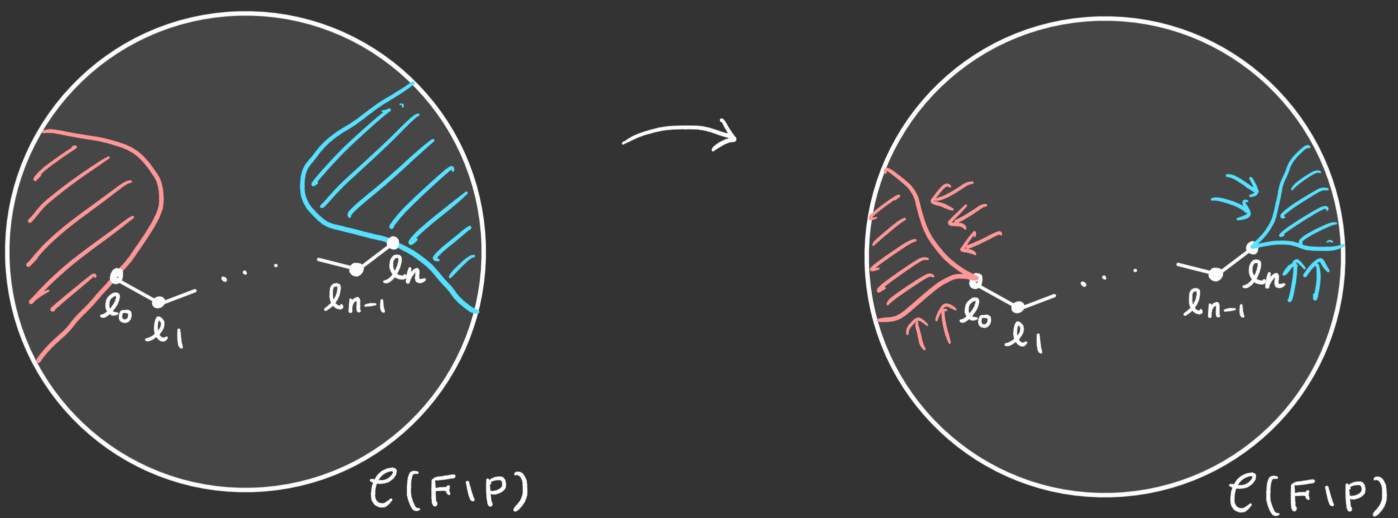


Modify $(\overline{V}_i, t_i)$ as follows :

by h_i s.t.

$$\begin{cases} d_{e(\partial_- \overline{W}_1 \setminus s_1)}\left(l_1, h_1(\mathcal{D}(\overline{V}_1^\circ \setminus t_1^\circ))\right) > 2, \\ d_{e(\partial_- \overline{W}_2 \setminus s_2)}\left(l_{n-1}, h_2(\mathcal{D}(\overline{V}_2^\circ \setminus t_2^\circ))\right) > 2. \end{cases}$$

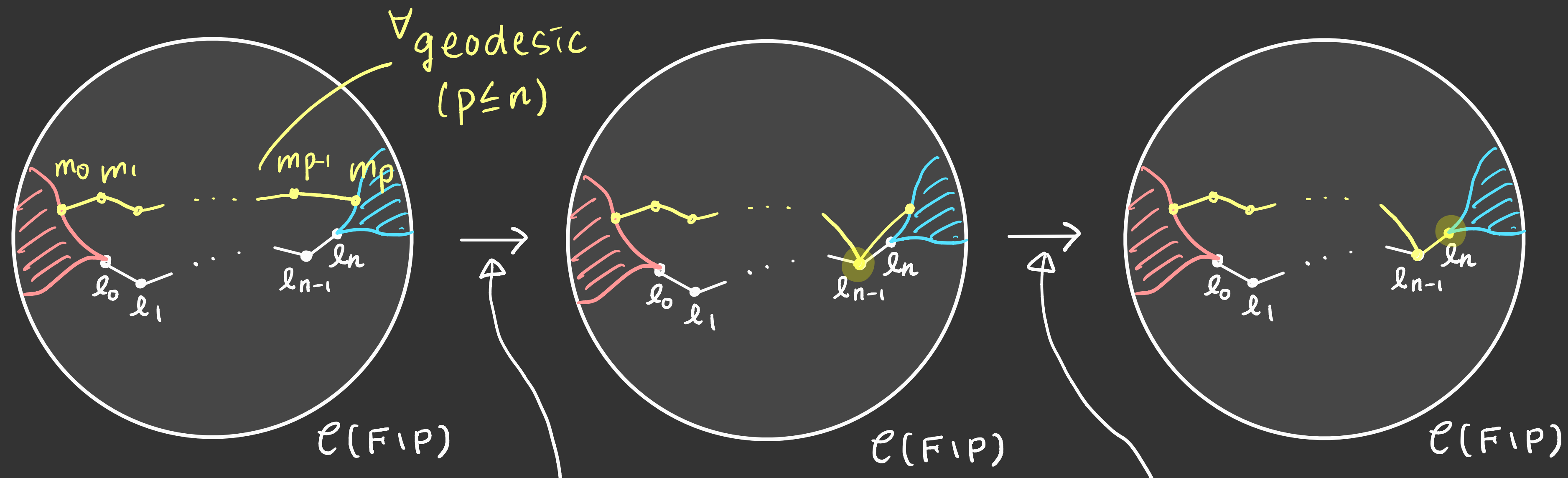
This modification changes the disk complexes like :

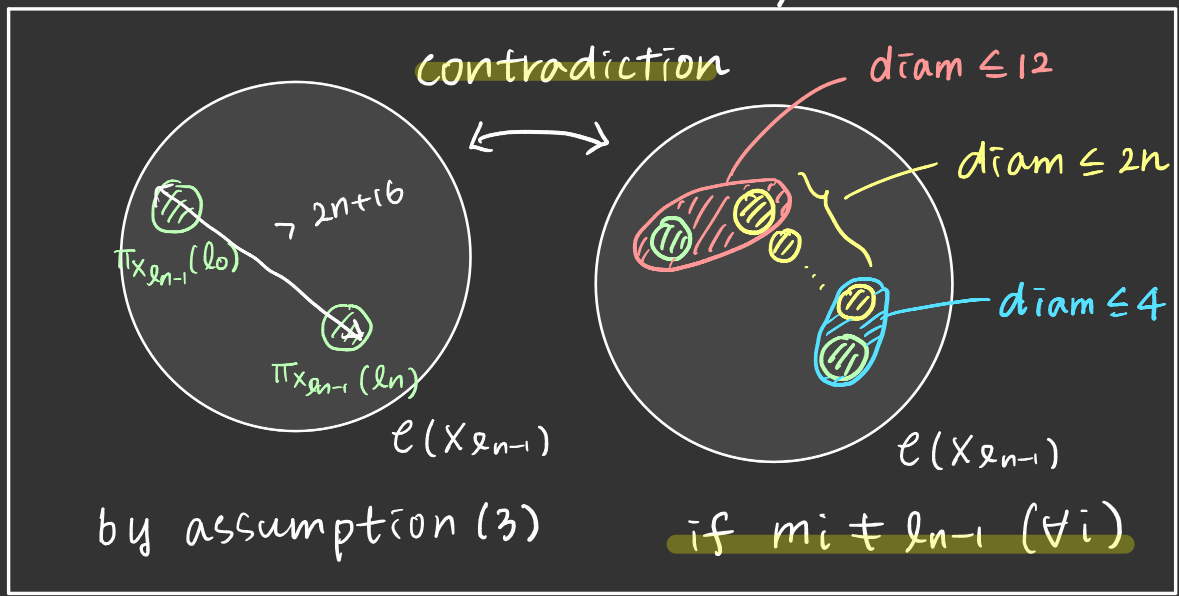


& we obtain a strongly keen (g, b)-splitting with distance n.

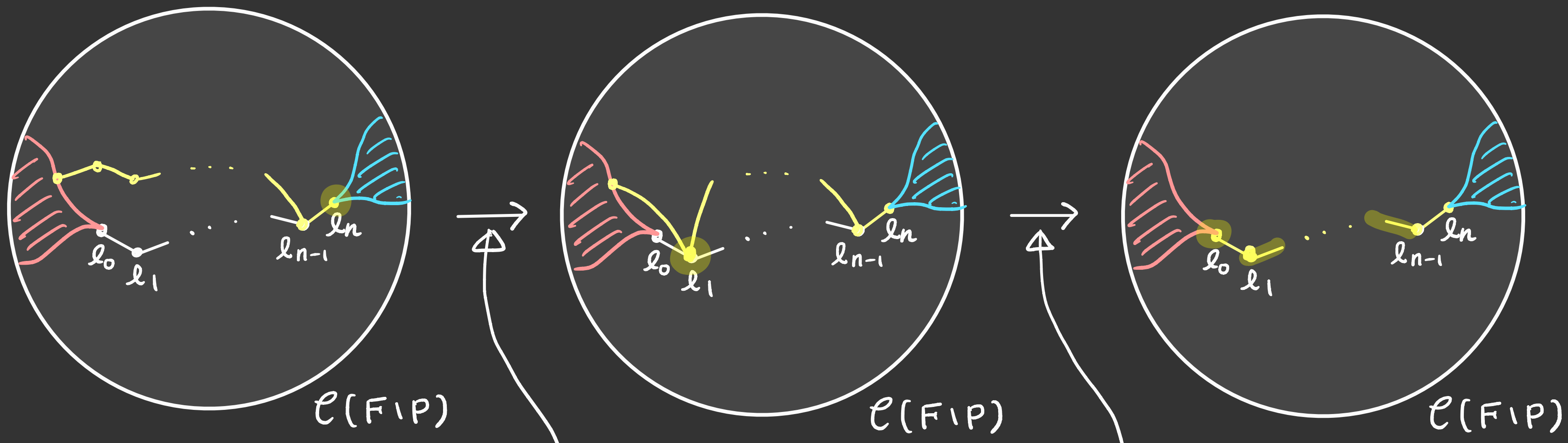
Proof of the uniqueness of the geodesic realizing the distance

(rough idea)

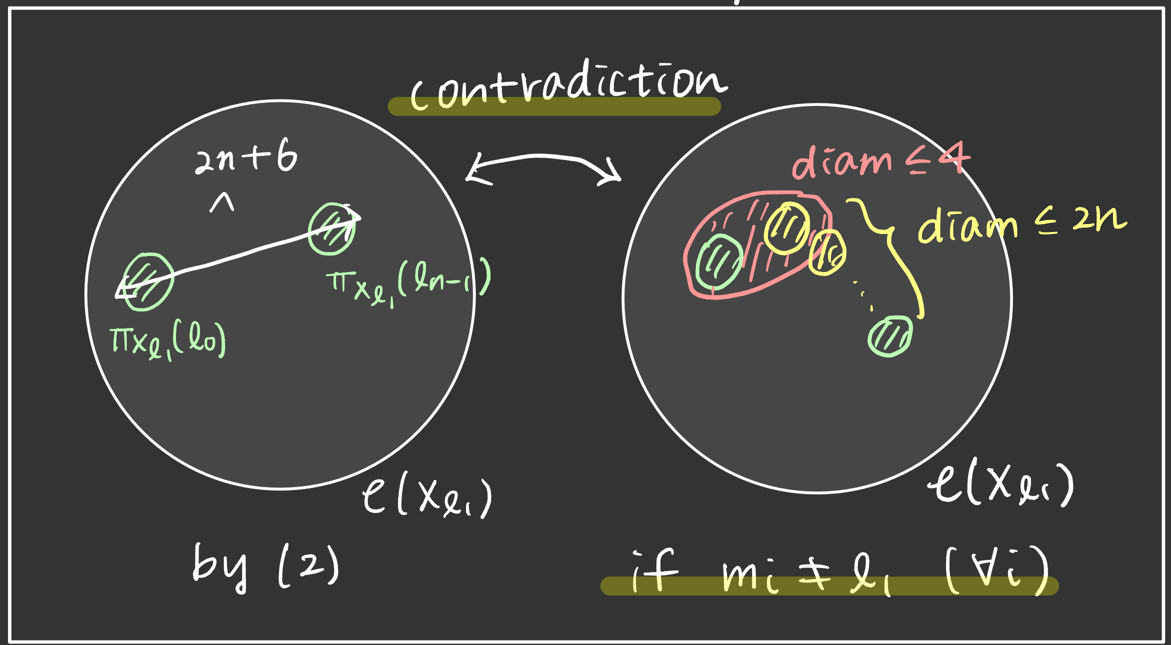




$$d_{\mathcal{C}(\partial_- W_2 \setminus S_2)}(l_{n-1}, h_2(\mathcal{D}(V_2^{\circ}, t_2^{\circ}))) > 2 \quad \cdots (5)$$



$$d_{\mathcal{C}(\partial W_1 \setminus S_1)}\left(\ell_1, h_1\left(\mathcal{D}(\nabla_i^0 \setminus t_i^0)\right)\right) > 2 \quad \cdots (4)$$



Proof of Thm 1 for the case of $n=1$

We have the following three cases:

1. Case of $g \geq 2$,
2. Case of $g = 1$,
3. Case of $g = 0$ ($b \geq 4$).

In each case, the proof has a different flavor.

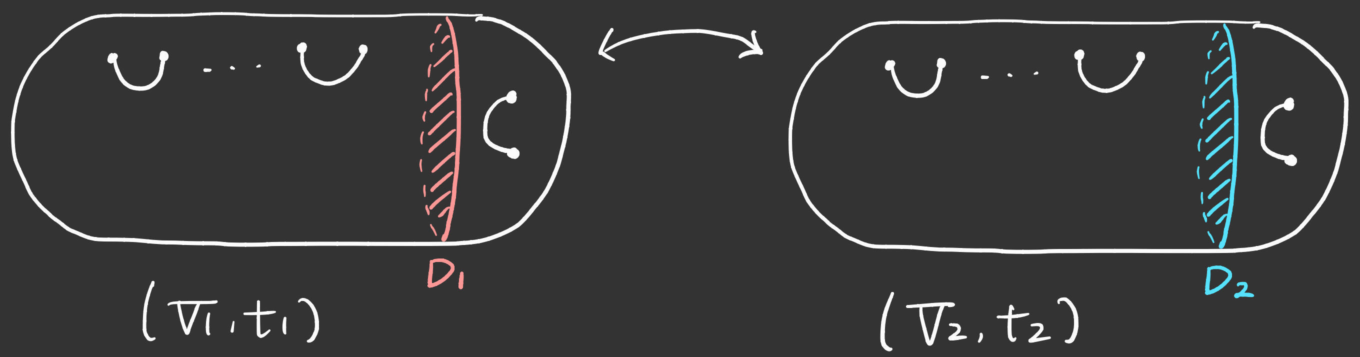
Today, I will explain about the 3rd case.

Construction of strongly keen bridge splittings

$F := S^2$

P : union of $2b$ points on F

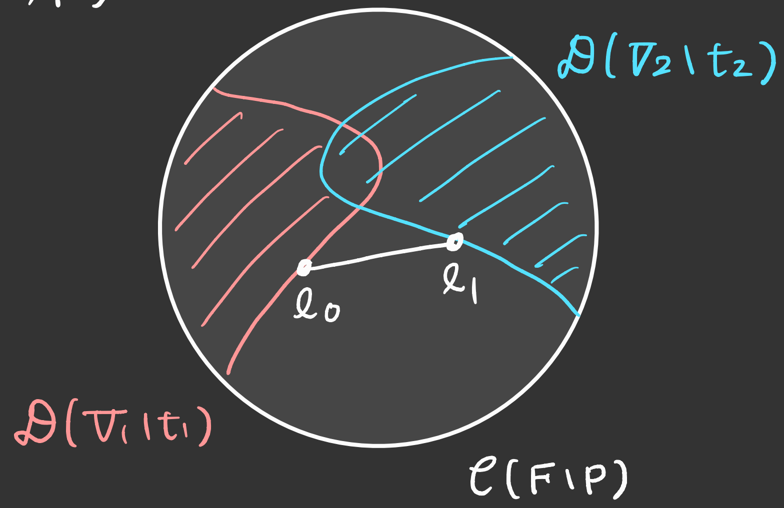
l_0, l_1 : s.c.c. on $F \setminus P$ bounding a twice-punctured disk



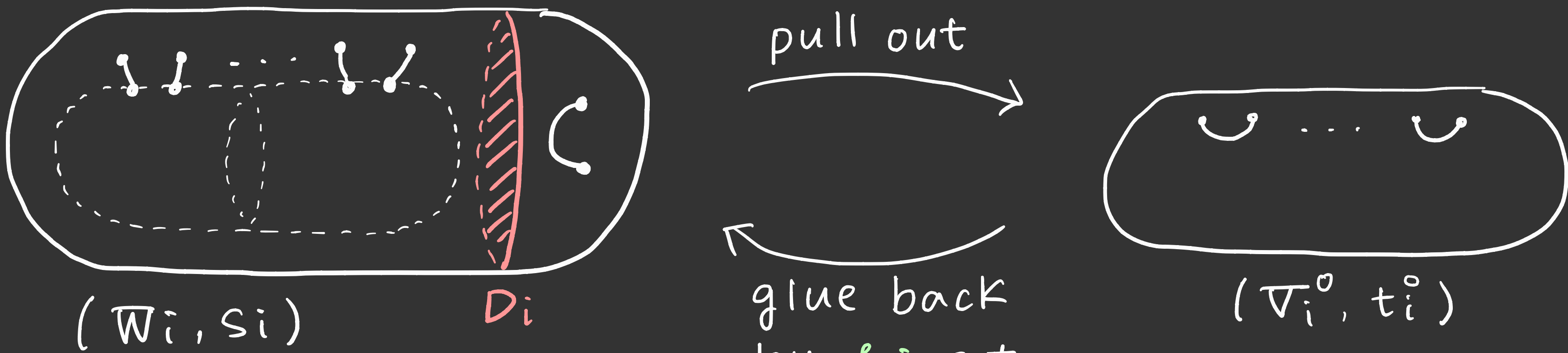
Identify $(\partial V_1, \partial t_1)$, $(\partial V_2, \partial t_2)$ with (F, P)
so that $\partial D_1 = l_0$, $\partial D_2 = l_1$.

$\rightsquigarrow$ $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$: $(0, b)$-splitting

& $d\big((V_1, t_1) \cup_{(F,P)} (V_2, t_2)\big) \le 1$.

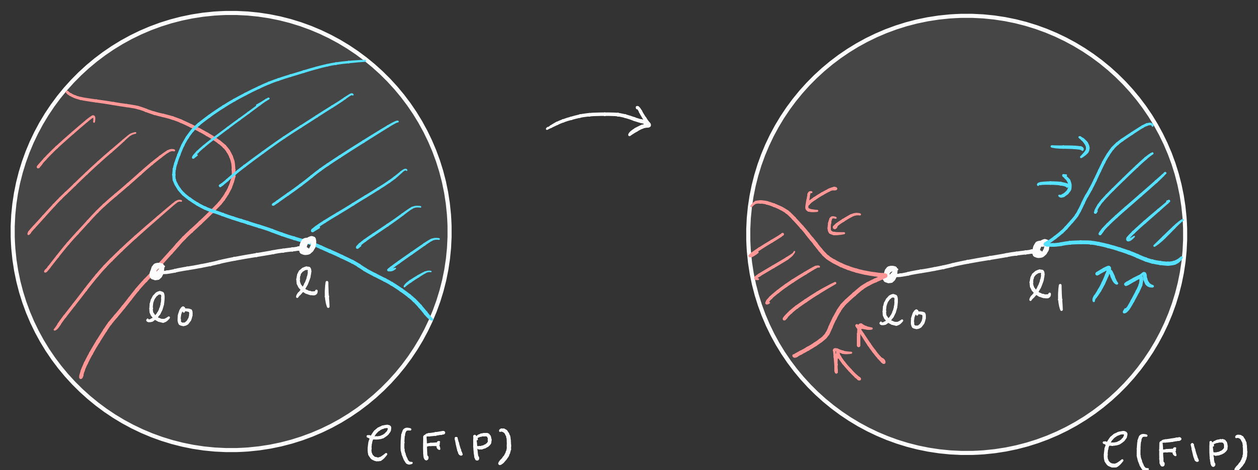


Modify (V_i, t_i) as follows :



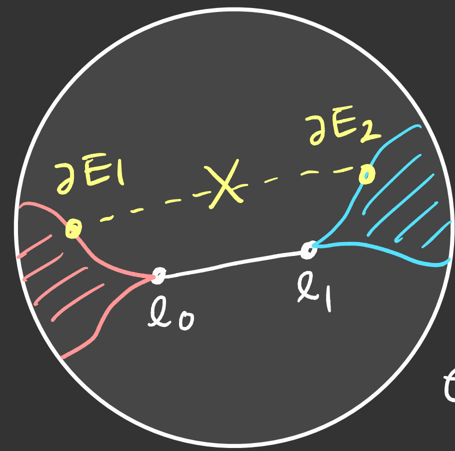
$$\begin{cases} d_{\mathcal{C}(\partial_- W_1, s_1)}(\ell_1, h_1(\mathcal{D}(V_1^0, t_1^0))) > 3, \\ d_{\mathcal{C}(\partial_- W_2, s_2)}(\ell_0, h_2(\mathcal{D}(V_2^0, t_2^0))) > 3. \end{cases}$$

This modification changes the disk complexes like :



<u>Assertion</u>: $\forall E_1 \in$  & $\forall E_2 \in$ ,

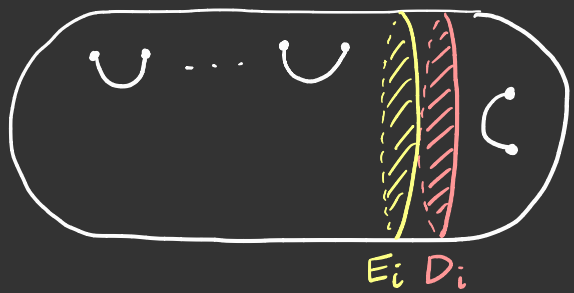
$$(E_1, E_2) \neq (D_1, D_2) \Rightarrow E_1 \cap E_2 \neq \phi.$$



$\mathcal{C}(F|P)$

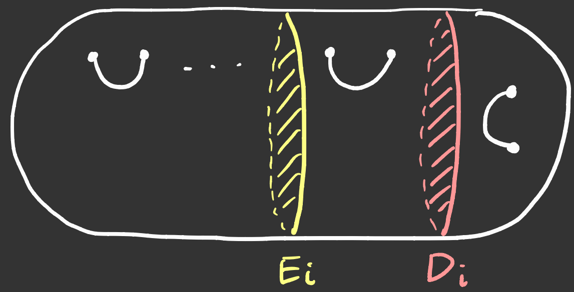
For E_i, we have the three possibilities:

Type 1



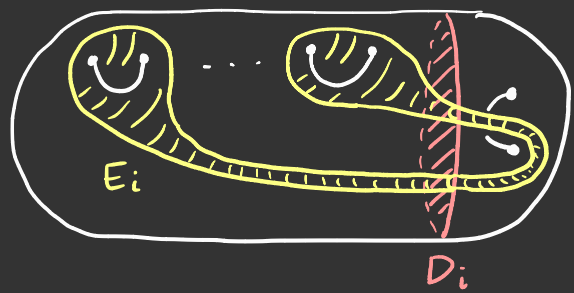
E_i is isotopic to D_i.

Type 2



$E_i \cap D_i = \phi$ & E_i is not isotopic to D_i.

Type 3

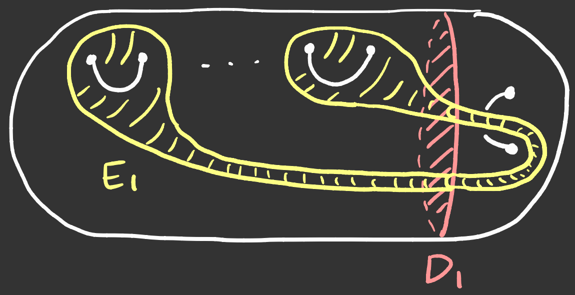


$E_i \cap D_i \neq \phi$.

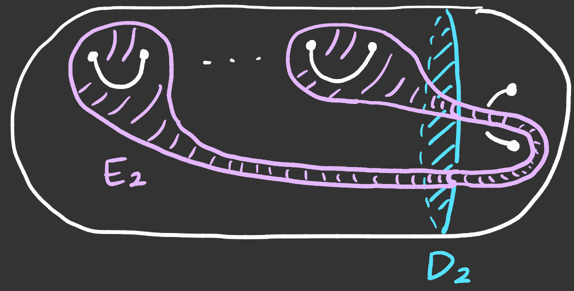
We assume $E_1 \cap E_2 = \phi$ and lead to a contradiction in each case.

In the following, we suppose both E_1 and E_2 are of Type 3.

Case of



vs



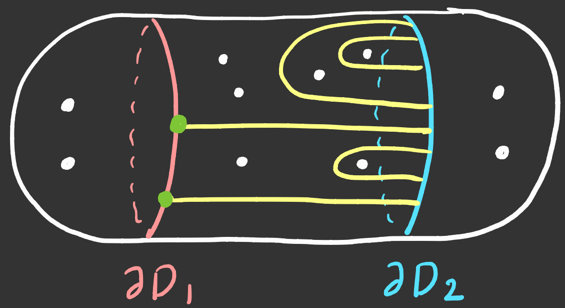
Let Δ_1 : any "outermost" disk of $E_1 \setminus D_1$,
$\quad\;\;\Delta_2$: any "outermost" disk of $E_2 \setminus D_2$.

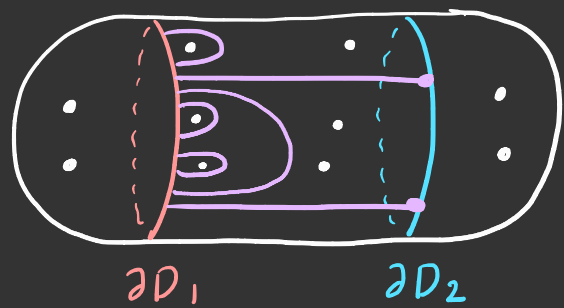
By $\begin{cases} d_{e(\partial_- W_1 \setminus S_1)}\left(l_1,\, h_1(\mathcal{D}(V_1^\circ \setminus t_1^\circ))\right) > 3, \\ d_{e(\partial_- W_2 \setminus S_2)}\left(l_0,\, h_2(\mathcal{D}(V_2^\circ \setminus t_2^\circ))\right) > 3 \end{cases}$ $\cdots$ (*),

we can see that $\Delta_1 \cap \partial D_2 \neq \phi$ & $\Delta_2 \cap \partial D_1 \neq \phi$,

& $cl(\partial\Delta_1 \setminus D_1)$ & $cl(\partial\Delta_2 \setminus D_2)$ intersect the (punctured) annulus
bounded by $\partial D_1 \cup \partial D_2$ as follows :

$\uparrow$
A

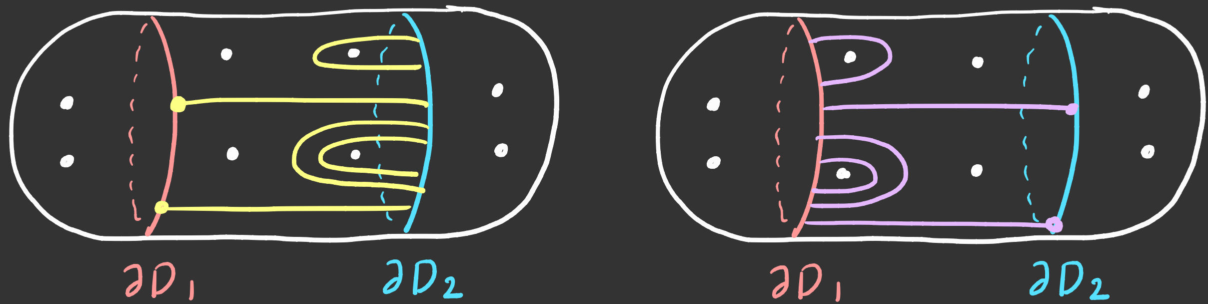




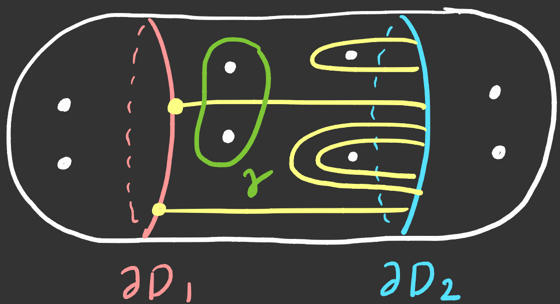
Let G_1^1, G_1^2 be the two components of $A \setminus \Delta_1$ adjacent to ∂D_1,
$\quad G_2^1, G_2^2$ be the two components of $A \setminus \Delta_2$ adjacent to ∂D_2.

By ⊛ again, we can see that each G_i^j $(i, j \in \{1,2\})$ contains at most one puncture.

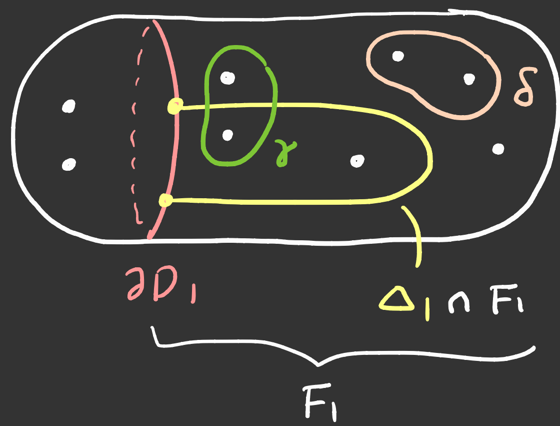
This implies that $b=4$, and $cl(\partial\Delta_1 \setminus D_1)$ & $cl(\partial\Delta_2 \setminus D_2)$ intersect A as follows:



Let γ, δ be simple closed curves as follows:



- $\gamma \subset A$
- $\gamma \cap (\Delta_1 \cap A) = $ (2 points)



Then $d_{\mathcal{C}(\partial W_1 \setminus s_1)}(\ell_1, h_1(\mathcal{D}(V_1^0 \setminus t_1^0)))$

$\leq d(\partial D_2, \gamma) + d(\gamma, \delta) + d(\delta, \partial E_1)$

$= 1 + 1 + 1 = 3$: contradiction to ⊛.