The rational abelianization of the Chillingworth subgroup of the mapping class group of a surface

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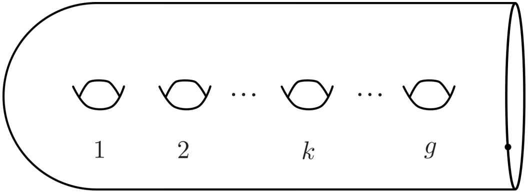
Contents

$\Sigma_{g,1} :=$ 

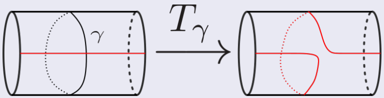
$\cdot : H_1(\Sigma_{g,1}, \mathbb{Z}) \otimes H_1(\Sigma_{g,1}; \mathbb{Z}) \to \mathbb{Z}$: the intersection form

$\mathrm{Aut}(H_1(\Sigma_{g,1}; \mathbb{Z}), \cdot) \cong \mathrm{Sp}(2g, \mathbb{Z})$

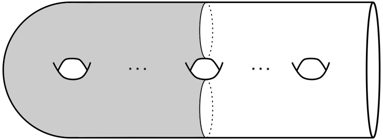
Definition (the mapping class group of $\Sigma_{g,1}$)

$$\mathcal{M}_{g,1} := \mathrm{Diff}^+(\Sigma_{g,1}, \partial\Sigma_{g,1})/\mathrm{Diff}_0(\Sigma_{g,1}, \partial\Sigma_{g,1})$$

Definition (Dehn twist)



- Bounding Pair map (BP-map)



$$BP(\gamma_1, \gamma_2) = T_{\gamma_1} T_{\gamma_2}^{-1}$$

$H = H_1(\Sigma_{g,1}, \mathbb{Z})$, $\pi = \pi_1(\Sigma_{g,1})$

Definition (the Torelli group)

The action $\mathcal{M}_{g,1} \to \mathrm{Aut}(H)$ preserves the intersection form.
$\rho\colon \mathcal{M}_{g,1} \to \mathrm{Aut}(H, \cdot) \cong \mathrm{Sp}(H, \cdot) \cong \mathrm{Sp}(2g; \mathbb{Z})$: the symplectic representation
$\mathcal{I}_{g,1} := \mathrm{Ker}(\rho)$: the **Torelli group**

BP-map $\in \mathcal{I}_{g,1}$

Definition (Johnson kernel)

$\mathcal{K}_{g,1} := \langle$Dehn twists along separating closed curves$\rangle$: the **Johnson kernel**

$\mathcal{K}_{g,1} \lhd \mathcal{I}_{g,1} \lhd \mathcal{M}_{g,1}$

$\Gamma_1 = \pi$, $\Gamma_n = [\Gamma_{n-1}, \pi]$: The lower central series of the fundamental group

Definition (the (first) Johnson homomorphism)

$1 \to \Gamma_2/\Gamma_3 \to \pi/\Gamma_3 \to \pi/\Gamma_2 \cong H \to 1$: a central extension
$\Gamma_2/\Gamma_3 \cong \bigwedge^2 H$ as $\mathcal{M}_{g,1}$-modules $\left(\overline{[x,y]} \leftrightarrow x \wedge y\right)$

$$\tau_{g,1}(1) \colon \mathcal{I}_{g,1} \to \mathrm{Hom}(H, \Gamma_2/\Gamma_3) \cong H^* \otimes \bigwedge^2 H \cong H \otimes \bigwedge^2 H$$
$$f \mapsto ([\gamma] \mapsto \overline{f_*(\gamma)\gamma^{-1}})$$

$\bigwedge^3 H \cong \{x \otimes (y \wedge z) + y \otimes (z \wedge x) + z \otimes (x \wedge y) | x, y, z \in H\} \subset H \otimes \bigwedge^2 H$
$\mathrm{Sp}(2g;\mathbb{Z})$-submodule
$\tau_{g,1}(1) \colon \mathcal{I}_{g,1} \to \bigwedge^3 H$: the (first) **Johnson homomorphism**

$\mathcal{K}_{g,1} := \langle$*DDehn twists along separating closed curves*$\rangle$

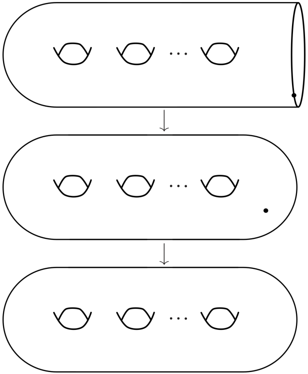
$1 \to \mathcal{K}_{g,1} \to \mathcal{I}_{g,1} \xrightarrow{\tau_{g,1}(1)} \bigwedge^3 H \to 0$ (Johnson)

The mapping class group and the Johnson homomorphism for $\Sigma_{g,*}$, Σ_g

$$\mathcal{M}_{g,*} := \mathrm{Diff}^+(\Sigma_{g,*}, \{*\})/\mathrm{Diff}_0(\Sigma_{g,*}, \{*\})$$
$$\mathcal{M}_g := \mathrm{Diff}^+(\Sigma_g)/\mathrm{Diff}_0(\Sigma_g)$$

$\mathcal{I}_{g,*}$ and $\mathcal{I}_g$ are defined similarly. These act trivially on H.

$$\mathcal{M}_{g,1} \to \mathcal{M}_{g,*} \to \mathcal{M}_g$$
$$\mathcal{I}_{g,1} \to \mathcal{I}_{g,*} \to \mathcal{I}_g$$



The (first) Johnson homomorphism $\tau_{g,*}(1)\colon \mathcal{I}_{g,*} \to \bigwedge^3 H$ for $\Sigma_{g,*}$ is defined similarly.
But, $\mathcal{M}_g$ does not act on $\pi_1(\Sigma_g)$. It acts as an outer automorphism on $\pi_1(\Sigma_g)$.

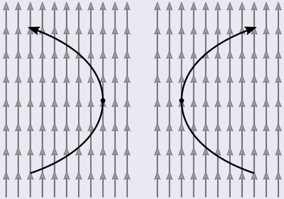
$$\begin{array}{ccc}
\mathcal{I}_{g,1} & \xrightarrow{\tau_{g,1}(1)} & \bigwedge^3 H \\
\downarrow & & \| \\
\mathcal{I}_{g,*} & \xrightarrow{\tau_{g,*}(1)} & \bigwedge^3 H \\
\downarrow & & \downarrow \\
\mathcal{I}_{g} & \xrightarrow{\tau_{g}(1)} & \bigwedge^3 H/H
\end{array}$$

$$\tau_g(1)\colon \mathcal{I}_g \to \bigwedge^3 H/H := \bigwedge^3 H/\tau_{g,*}(1)(\mathrm{Ker}(\mathcal{I}_{g,*} \to \mathcal{I}_g))$$

Let $\Xi(\Sigma_{g,1})$ be the set of homotopy classes of nonsingular vector fields on $\Sigma_{g,1}$. $\mathcal{M}_{g,1}$ acts on $\Xi(\Sigma_{g,1})$.
$H^1(\Sigma_{g,1})$ acts freely and transitively on $\Xi(\Sigma_{g,1})$.

Definition

$X \in \Xi(\Sigma_{g,1})$. Let γ be an oriented regular closed curve on $\Sigma_{g,1}$.
The **winding number** $\omega_X(\gamma)$ is defined by the number of times its tangent transversely intersects with the section of the unit tangent bundle $UT\Sigma_{g,1} \to \Sigma_{g,1}$ induced by X.



$+1$ $\qquad$ -1

Definition

For $X \in \Xi(\Sigma_{g,1})$,

$$e_X : \mathcal{M}_{g,1} \to H^1(\Sigma_{g,1};\mathbb{Z}),$$
$$f \mapsto ([\gamma] \mapsto \omega_X(f \circ \gamma) - \omega_X(\gamma))$$

is called the **Chillingworth homomorphism**.

The Chllingworth homomorphism e_X is NOT a homomorphism but a crossed homomorphism.
i.e., $e_X(fg) = e_X(g) + (g^{-1})^* e_X(f)$
$\mathrm{Ker}(e_X) := e_X^{-1}(0)$ is the subgroup whose elements preserve $X \in \Xi(\Sigma_{g,1})$. (the framed mapping class group)
e_X depends on $X \in \Xi(\Sigma_{g,1})$ as a map from $\mathcal{M}_{g,1}$.

Definition

The restriction of the Chillingworth homomorphism on the Torelli group $e_X|_{\mathcal{I}_{g,1}}$ is no longer independent of the choice of $X \in \Xi(\Sigma_{g,1})$, and it is a homomorphism. The kernel of this homomorphism

$$Ch_{g,1} := \mathrm{Ker}(e_X|_{\mathcal{I}_{g,1}})$$

is called the **Chillingworth subgroup**.

We have $Ch_{g,1} = \mathrm{Ker}(\mathcal{M}_{g,1} \curvearrowright \Xi(\Sigma_{g,1}))$.
$[e_X] \in H^1(\mathcal{M}_{g,1}, H^{(*)}) \cong \mathbb{Z}$ is a generator.
$k\colon \mathcal{M}_{g,1} \to H$ is trivial on the Chillingworth subgroup.

Let $t_f \in H$ be the Poincaré dual of $e_X(f) \in H^1(\Sigma_{g,1}, \mathbb{Z})$. t_f is called the Chillingworth class.

$$\begin{tikzcd} \mathcal{I}_{g,1} \arrow[r, "\tau_{g,1}(1)"] \arrow[dr, "t"'] & \bigwedge^3 H \arrow[d, "2C_3"] \\ & H \end{tikzcd}$$

$C_3\colon \bigwedge^3 H \to H$ is defined by $C_3(x \wedge y \wedge z) = (x \cdot y)z + (y \cdot z)x + (z \cdot x)y$. C_3 is $\mathrm{Sp}(2g;\mathbb{Z})$-equivariant and called the contraction.
$U := \mathrm{Ker}(C_3)$

$$\tau_{g,1}(1) = \tau_{g,1}(1)|_{Ch_{g,1}} : Ch_{g,1} \to U \subset \bigwedge^3 H$$

$\mathcal{K}_{g,1} = \mathrm{Ker}(\tau_{g,1}(1))$ (Johnson)

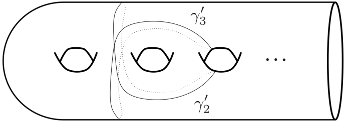
$$Ch_{g,1} \lhd \mathcal{M}_{g,1}$$

$$\mathcal{K}_{g,1} \lhd Ch_{g,1} \lhd \mathcal{I}_{g,1}$$

$$\mathcal{K}_{g,*} \lhd Ch_{g,*} \lhd \mathcal{I}_{g,*}$$

$$\mathcal{K}_g \lhd Ch_g \overset{\textit{finite}}{\lhd} \mathcal{I}_g$$

- $\mathcal{I}_g/Ch_g \cong (\mathbb{Z}/(g-1)\mathbb{Z})^{2g}$ -(K.)
- For $g = 2$, $\mathcal{K}_{2,1} = Ch_{2,1}$, $\mathcal{K}_{2,*} = Ch_{2,*}$, $\mathcal{K}_2 = Ch_2 = \mathcal{I}_2$
- $Ch_{g,1} = \langle\langle B_0 = T_{\gamma_2'} T_{\gamma_3'}{}^{-1} \rangle\rangle \mathcal{K}_{g,1}$ - (K.)

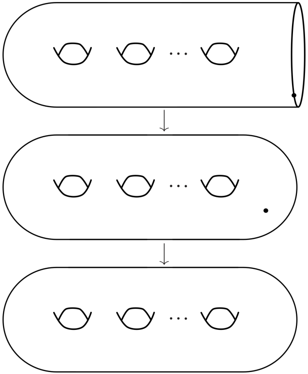


Collapsing the boundary of $\Sigma_{g,1}$ to the based point

$0 \to \mathbb{Z} \to \mathcal{I}_{g,1} \to \mathcal{I}_{g,*} \to 1$
$0 \to \mathbb{Z} \to Ch_{g,1} \to Ch_{g,*} \to 1$
$0 \to \mathbb{Z} \to \mathcal{K}_{g,1} \to \mathcal{K}_{g,*} \to 1$

Forgetting the based point of $\Sigma_{g,*}$

$1 \to \pi_1(\Sigma_g) \to \mathcal{I}_{g,*} \to \mathcal{I}_g \to 1$
$1 \to [\pi_1(\Sigma_g), \pi_1(\Sigma_g)] \to Ch_{g,*} \to Ch_g \to 1$ -(K.)
$1 \to [\pi_1(\Sigma_g), \pi_1(\Sigma_g)] \to \mathcal{K}_{g,*} \to \mathcal{K}_g \to 1$



The Casson-Morita "map" is defined by a boundary of a certain 2-cocycle of the mapping class group.

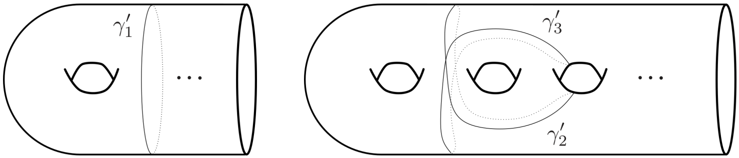
・the **Meyer cocycle** $\tau\colon \mathcal{M}_{g,1}\times\mathcal{M}_{g,1}\to\mathbb{Z}$ is characterized by the signature of the surface bundle over a pair of pants with given monodromies.

・the **intersection cocycle** $c\colon \mathcal{M}_{g,1}\times\mathcal{M}_{g,1}\to\mathbb{Z}$ is defined by $c(\phi,\psi):=k(\phi)\cdot k(\psi^{-1})$ where $[k]\in H^1(\mathcal{M}_{g,1},H^{(*)})\cong\mathbb{Z}$ is a generator.

$[c+3\tau]=0$ in $H^2(\mathcal{M}_{g,1};\mathbb{Z})$ and $H^1(\mathcal{M}_{g,1};\mathbb{Z})$ $\exists!\ d\colon \mathcal{M}_{g,1}\to\mathbb{Z}$ s.t. $\delta d=c+3\tau$

$$d(\varphi\psi)=d(\varphi)+d(\psi)-k(\varphi)\cdot k(\psi^{-1})-3\tau(\varphi,\psi)$$

- $d|_{Ch_{g,1}} : Ch_{g,1} \to \mathbb{Z}$ is a $\mathcal{M}_{g,1}$-inavariant homomorphism. i.e., $d(f^{-1}hf) = d(h)$
- d(Dehn twist along the boundary of a genus h subsurface) $= 4h(h-1)$
- $d(Ch_{g,1}) = d(\mathcal{K}_{g,1}) = 8\mathbb{Z}$
- $\mathrm{Ker}(d|_{\mathcal{K}_{g,1}} : \mathcal{K}_{g,1} \to \mathbb{Z}) = \langle T_{\gamma_1'} \rangle [\mathcal{K}_{g,1}, \mathcal{M}_{g,1}]$ - (Faes)
- $\mathrm{Ker}(d\colon Ch_{g,1} \to \mathbb{Z}) = \langle\langle B_0 = T_{\gamma_2'} T_{\gamma_3'}{}^{-1} \rangle\rangle \langle T_{\gamma_1'} \rangle [\mathcal{K}_{g,1}, \mathcal{M}_{g,1}]$ - (K.)



The rational abelianization of the Torelli group is induced by the Johnson homomorphism.

$$\tau_{g,1}(1)\colon \mathcal{I}_{g,1} \to (\textstyle\bigwedge^3 H) \otimes \mathbb{Q}$$

$$(\mathcal{I}_{g,1})^{ab} \xrightarrow{\cong} (\textstyle\bigwedge^3 H) \oplus (\text{2-torsions})$$

$$(\tau_{g,1}(1), d)\colon Ch_{g,1} \to (U \oplus \mathbb{Z}) \otimes \mathbb{Q}$$

$$1 \to \mathcal{K}_{g,1} \to Ch_{g,1} \xrightarrow{\tau_{g,1}(1)} U \to 1$$

- the inflation-restriction exact sequence:

$$\to H_2(Ch_{g,1};\mathbb{Q}) \xrightarrow{(\tau_{g,1}(1))_*} \bigwedge^2 U \otimes \mathbb{Q} \to H_1(\mathcal{K}_{g,1};\mathbb{Q})_U \to H_1(Ch_{g,1};\mathbb{Q}) \xrightarrow{(\tau_{g,1}(1))_*} U \otimes \mathbb{Q} \to 0$$

$$1 \to \mathcal{K}_{g,1} \to Ch_{g,1} \xrightarrow{\tau_{g,1}(1)} U \to 1$$

・the inflation-restriction exact sequence:

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By taking the tensor product $-\otimes\mathbb{Q}$, we can use representation theory of $\mathrm{Sp}(2g;\mathbb{Q})$.
Every finite dimensional polynomial representation of $\mathrm{Sp}(2g;\mathbb{Q})$ is parametrized by Young diagrams, and these representations are naturally isomorphic to their dual representation.
e.g., $[1] = H_{\mathbb{Q}} \cong (H_{\mathbb{Q}})^*$ as representations of $\mathrm{Sp}(2g,\mathbb{Q})$

Example (irreducible decompositions of $\mathrm{Sp}(2g;\mathbb{Q})$)

$(\bigwedge^3 H)_{\mathbb{Q}} = \bigwedge^3 H_{\mathbb{Q}} = [1^3]_{\mathrm{Sp}} + [1]_{\mathrm{Sp}} \quad (g \geq 3)$
$(\bigwedge^3 H/H)_{\mathbb{Q}} = \bigwedge^3 H_{\mathbb{Q}}/H_{\mathbb{Q}} = [1^3]_{\mathrm{Sp}} \quad (g \geq 3)$
$U_{\mathbb{Q}} = [1^3]_{\mathrm{Sp}} \quad (g \geq 3) \qquad (U := \mathrm{Ker}(C_3 \colon \bigwedge^3 H \to H))$
Especially, $U_{\mathbb{Q}} \to \bigwedge^3 H_{\mathbb{Q}} \to \bigwedge^3 H_{\mathbb{Q}}/H_{\mathbb{Q}}$ is an isomorphism as a representation of $\mathrm{Sp}(2g;\mathbb{Q})$.

$$H^2(U;\mathbb{Q}) \cong \bigwedge^2 U_{\mathbb{Q}} = \bigwedge^2 [1^3]_{\mathrm{Sp}} =$$
$$\begin{cases} [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} + [1^2]_{\mathrm{Sp}} + [2^2 1^2]_{\mathrm{Sp}} + [1^4]_{\mathrm{Sp}} + [1^6]_{\mathrm{Sp}} & (g \geq 6) \\ [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} + [1^2]_{\mathrm{Sp}} + [2^2 1^2]_{\mathrm{Sp}} + [1^4]_{\mathrm{Sp}} & (g = 5) \\ [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} + [1^2]_{\mathrm{Sp}} + [2^2 1^2]_{\mathrm{Sp}} & (g = 4) \\ [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} & (g = 3) \end{cases} \quad \text{(Hain)}$$

Example (irreducible decompositions of $\mathrm{Sp}(2g;\mathbb{Q})$)

$(\bigwedge^3 H)_{\mathbb{Q}} = \bigwedge^3 H_{\mathbb{Q}} = [1^3]_{\mathrm{Sp}} + [1]_{\mathrm{Sp}} \quad (g \geq 3)$
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$U_{\mathbb{Q}} = [1^3]_{\mathrm{Sp}} \quad (g \geq 3) \qquad (U := \mathrm{Ker}(C_3 \colon \bigwedge^3 H \to H))$
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$$H^2(U;\mathbb{Q}) \cong \bigwedge^2 U_{\mathbb{Q}} = \bigwedge^2 [1^3]_{\mathrm{Sp}} =$$
$$\begin{cases} [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} + [1^2]_{\mathrm{Sp}} + [2^2 1^2]_{\mathrm{Sp}} + [1^4]_{\mathrm{Sp}} + [1^6]_{\mathrm{Sp}} & (g \geq 6) \\ [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} + [1^2]_{\mathrm{Sp}} + [2^2 1^2]_{\mathrm{Sp}} + [1^4]_{\mathrm{Sp}} & (g = 5) \\ [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} + [1^2]_{\mathrm{Sp}} + [2^2 1^2]_{\mathrm{Sp}} & (g = 4) \\ [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} & (g = 3) \end{cases} \quad \text{(Hain)}$$

Theorem (Hain)

$$\mathrm{Ker}\left((\tau_g(1))^* \colon H^2(\bigwedge^3 H/H;\mathbb{Q}) \to H^2(\mathcal{I}_g;\mathbb{Q}) \right) = [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} \quad (g \geq 3)$$

Theorem (K.)

$$\mathrm{Ker}\left((\tau_{g,1}(1))^*\colon H^2(U;\mathbb{Q}) \to H^2(Ch_{g,1};\mathbb{Q})\right) = \begin{cases} [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} + [1^2]_{\mathrm{Sp}} & (g \geq 4) \\ [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} & (g = 3) \end{cases}$$

$Ch_{g,1} \hookrightarrow \mathcal{I}_{g,1} \to \mathcal{I}_g$ $\qquad [1^3] = (\bigwedge^3 H/H)_{\mathbb{Q}} \cong U_{\mathbb{Q}}$

$$\begin{tikzcd}
\bigwedge^2[1^3]_{\mathrm{Sp}} \cong \bigwedge^2 U_{\mathbb{Q}} \arrow[r, "\cong"] \arrow[d, hook] \arrow[dd, bend right, "\cong"'] & H^2(\bigwedge^3 H/H;\mathbb{Q}) \arrow[r, "(\tau_g(1))^*"] \arrow[d] & H^2(\mathcal{I}_g;\mathbb{Q}) \arrow[d] \\
\bigwedge^2 H_{\mathbb{Q}} \oplus (H_{\mathbb{Q}} \otimes U_{\mathbb{Q}}) \oplus \bigwedge^2 U_{\mathbb{Q}} \arrow[r, "\cong"] \arrow[d, two heads] & H^2(\bigwedge^3 H;\mathbb{Q}) \arrow[r, "(\tau_{g,1}(1))^*"] \arrow[d] & H^2(\mathcal{I}_{g,1};\mathbb{Q}) \arrow[d] \\
\bigwedge^2[1^3]_{\mathrm{Sp}} \cong \bigwedge^2 U_{\mathbb{Q}} \arrow[r, "\cong"] & H^2(U;\mathbb{Q}) \arrow[r, "(\tau_{g,1}(1))^*"] & H^2(Ch_{g,1};\mathbb{Q})
\end{tikzcd}$$

$[0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} \subset \mathrm{Ker}((\tau_{g,1}(1))^*\colon H^2(U;\mathbb{Q}) \to H^2(Ch_{g,1};\mathbb{Q}))$

By taking the dual of $[0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} \subset \mathrm{Ker}((\tau_{g,1}(1))^*\colon H^2(U;\mathbb{Q}) \to H^2(Ch_{g,1};\mathbb{Q}))$

$$\mathrm{Im}\,((\tau_{g,1}(1))_*\colon H_2(Ch_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})) \subset \begin{cases} [1^2]_{\mathrm{Sp}} + [2^21^2]_{\mathrm{Sp}} + [1^4]_{\mathrm{Sp}} + [1^6]_{\mathrm{Sp}} & (g \geq 6) \\ [1^2]_{\mathrm{Sp}} + [2^21^2]_{\mathrm{Sp}} + [1^4]_{\mathrm{Sp}} & (g = 5) \\ [1^2]_{\mathrm{Sp}} + [2^21^2]_{\mathrm{Sp}} & (g = 4) \\ \{0\} & (g = 3) \end{cases}$$

- $[2^21^2]_{\mathrm{Sp}}$ $(g \geq 4)$
- $[1^4]_{\mathrm{Sp}}$ $(g \geq 5)$
- $[1^6]_{\mathrm{Sp}}$ $(g \geq 6)$

are contained in Im. (i)

- $[1^2]_{\mathrm{Sp}}$ $(g \geq 4)$

is NOT contained in Im. (ii)

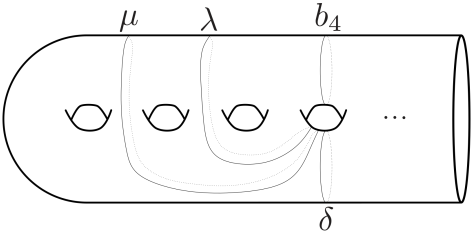
(i)
Using **abelian cycles**
G : a group, A : a free abelian group, $T : G \to A$: a homomorphism

$$\mathrm{Im}(H_2(G;\mathbb{Z}) \xrightarrow{T_*} H_2(A;\mathbb{Z}))$$

For an arbitrary homomorphism $c : \mathbb{Z}^2 \to G$ induces a homomorphism

$$H_2(\mathbb{Z}^2;\mathbb{Z}) \xrightarrow{c_*} H_2(G;\mathbb{Z}) \xrightarrow{T_*} H_2(A;\mathbb{Z})$$

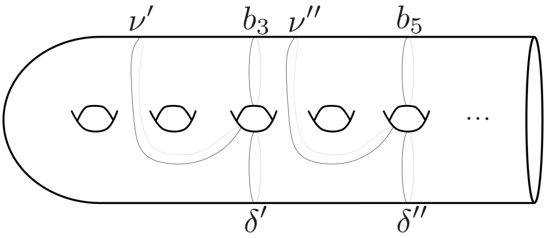
and the value of a generator of $H_2(\mathbb{Z}^2;\mathbb{Z}) \cong \mathbb{Z}$ is $T(c(e_1)) \wedge T(c(e_2)) \in H_2(A;\mathbb{Z}) \cong \bigwedge^2 A$.



$$
\begin{aligned}
e_1 \mapsto & BP(b_4,\delta)BP(b_4,\mu)^{-1}BP(b_4,\lambda)^{-1} = {T_{b_4}}^{-1}{T_\delta}^{-1}T_\mu T_\lambda \\
e_2 \mapsto & BP(b_4,\mu)BP(b_4,\lambda)^{-2} = {T_{b_4}}^{-1}{T_\mu}^{-1}{T_\lambda}^2
\end{aligned}
$$

$$
\bigwedge^2 U_{\mathbb{Q}} \to \bigwedge^2(\bigwedge^3 H_{\mathbb{Q}}) \xrightarrow{i^2_{\bigwedge^3 H_{\mathbb{Q}}}} \bigotimes^2(\bigwedge^3 H_{\mathbb{Q}}) \xrightarrow{id_{\bigwedge^3 H_{\mathbb{Q}}} \otimes j_{H_{\mathbb{Q}}}} (\bigwedge^3 H_{\mathbb{Q}}) \otimes H_{\mathbb{Q}} \otimes (\bigwedge^2 H_{\mathbb{Q}})
$$

$$
\xrightarrow{\phi^{3,1}_{H_{\mathbb{Q}}} \otimes id_{\bigwedge^2 H_{\mathbb{Q}}}} (\bigwedge^4 H_{\mathbb{Q}}) \otimes (\bigwedge^2 H_{\mathbb{Q}}) \supset [2^2 1^2]_{\mathrm{Sp}}
$$



$$e_1 \mapsto BP(b_3,\delta')BP(b_3,\nu')^{-2} = {T_{b_3}}^{-1}{T_{\delta'}}^{-1}{T_{\nu'}}^{2}$$
$$e_2 \mapsto BP(b_5,\delta'')BP(b_5,\nu'')^{-4} = {T_{b_5}}^{-3}{T_{\delta''}}^{-1}{T_{\nu''}}^{4}$$

$$\textstyle\bigwedge^2 U_{\mathbb{Q}} \to \bigwedge^2(\bigwedge^3 H_{\mathbb{Q}}) \xrightarrow{i^2_{\bigwedge^3 H_{\mathbb{Q}}}} \bigotimes^2(\bigwedge^3 H_{\mathbb{Q}}) \xrightarrow{\phi_{H^{3,3}_{\mathbb{Q}}}} \bigwedge^6 H_{\mathbb{Q}} \xrightarrow{C_6} \bigwedge^4 H_{\mathbb{Q}} \supset [1^4]_{\mathrm{Sp}}$$
$$\textstyle\bigwedge^2 U_{\mathbb{Q}} \to \bigwedge^2(\bigwedge^3 H_{\mathbb{Q}}) \xrightarrow{i^2_{\bigwedge^3 H_{\mathbb{Q}}}} \bigotimes^2(\bigwedge^3 H_{\mathbb{Q}}) \xrightarrow{\phi_{H^{3,3}_{\mathbb{Q}}}} \bigwedge^6 H_{\mathbb{Q}} \supset [1^6]_{\mathrm{Sp}}$$

(ii)

$$\begin{tikzcd}
H_2(Ch_{g,1};\mathbb{Q}) \arrow[r, "_^{ab}_*"] \arrow[d, "(\tau_{g,1}(1))_*"] & \begin{array}{c} H_2(Ch^{ab}_{g,1};\mathbb{Q}) \\ \cong H_2(H_1(Ch_{g,1};\mathbb{Z});\mathbb{Q}) \\ \cong \bigwedge^2 H_1(Ch_{g,1};\mathbb{Q}) \end{array} \arrow[r, "\text{bracket}"] \arrow[d, two heads, "(\tau_{g,1}(1))_*"] & (\Gamma_2(Ch_{g,1})/\Gamma_3(Ch_{g,1}))\otimes\mathbb{Q} \arrow[r] \arrow[d] & 0 \\
H_2(U;\mathbb{Q}) \arrow[r, "\cong"] & \bigwedge^2 U_{\mathbb{Q}} \arrow[r, "\text{bracket}"] & \begin{array}{c} (\mathcal{K}_{g,1}/\mathcal{M}_{g,1}[3])\otimes\mathbb{Q} \\ \cong \mathrm{Im}(\tau_{g,1}(2))\otimes\mathbb{Q} \end{array} &
\end{tikzcd}$$

$$\mathrm{Im}(\tau_{g,1}(2))\otimes\mathbb{Q} = \begin{cases} [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} + [1^2]_{\mathrm{Sp}} & (g\geq 4) \\ [0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} & (g=3) \end{cases}$$

$\mathcal{M}_{g,1}[n] = \mathrm{Ker}(\mathcal{M}_{g,1}\to \mathrm{Aut}(\pi/\Gamma_{n+1}))$: the Johnson filtration

Theorem (Faes-Massuyeau)

For $g \geq 6$, $H_1(\mathcal{K}_{g,1};\mathbb{Q}) \cong (\mathcal{K}_{g,1})^{ab} \otimes \mathbb{Q}$ *is given by* $(d, r^{\theta}_{2,3}) \colon \mathcal{K}_{g,1} \to \mathbb{Q} \oplus (\mathcal{T}_2(H_{\mathbb{Q}}) \oplus \mathrm{Ker}(Tr_3))$ *as* $\mathcal{M}_{g,1}$*-modules.*

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Lemma (K.)

For $g \geq 6$, $H_1(\mathcal{K}_{g,1};\mathbb{Q})_U \cong ((\mathcal{K}_{g,1})^{ab} \otimes \mathbb{Q})_U$ *is isomorphic to*
$\mathbb{Q} \otimes \mathcal{T}_2(H_{\mathbb{Q}}) \cong [0]_{\mathrm{Sp}} + ([0]_{\mathrm{Sp}} + [2^2]_{\mathrm{Sp}} + [1^2]_{\mathrm{Sp}})$ *as* $\mathrm{Sp}(2g;\mathbb{Q})$*-modules.*

Theorem (Faes-Massuyeau)

For $g \geq 6$, $H_1(\mathcal{K}_{g,1};\mathbb{Q}) \cong (\mathcal{K}_{g,1})^{ab} \otimes \mathbb{Q}$ *is given by* $(d, r^\theta_{2,3})\colon \mathcal{K}_{g,1} \to \mathbb{Q} \oplus (\mathcal{T}_2(H_\mathbb{Q}) \oplus \mathrm{Ker}(Tr_3))$ *as* $\mathcal{M}_{g,1}$*-modules.*

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$$\underset{}{H_2(Ch_{g,1};\mathbb{Q})} \xrightarrow{(\tau_{g,1}(1))_*} \underset{\substack{([2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} \oplus [1^6]_{\mathrm{Sp}}) \\ \oplus\, ([0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}})}}{\textstyle\bigwedge^2 U_\mathbb{Q}} \to \underset{\substack{([0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}}) \\ \oplus\, [0]_{\mathrm{Sp}}}}{H_1(\mathcal{K}_{g,1};\mathbb{Q})_U} \to \underset{\substack{[0]_{\mathrm{Sp}} \\ \oplus\, [1^3]_{\mathrm{Sp}}}}{H_1(Ch_{g,1};\mathbb{Q})} \to \underset{[1^3]_{\mathrm{Sp}}}{U_\mathbb{Q}} \to 0$$

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For $g \geq 6$, $H_1(\mathcal{K}_{g,1};\mathbb{Q}) \cong (\mathcal{K}_{g,1})^{ab} \otimes \mathbb{Q}$ *is given by*
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For $g \geq 6$, $H_1(\mathcal{K}_{g,1};\mathbb{Q})_U \cong ((\mathcal{K}_{g,1})^{ab} \otimes \mathbb{Q})_U$ *is isomorphic to*
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$$H_2(Ch_{g,1};\mathbb{Q}) \xrightarrow{(\tau_{g,1}(1))_*} \underset{\substack{([2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} \oplus [1^6]_{\mathrm{Sp}}) \\ \oplus\, ([0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}})}}{\Lambda^2 U_{\mathbb{Q}}} \to \underset{\substack{([0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}}) \\ \oplus\, [0]_{\mathrm{Sp}}}}{H_1(\mathcal{K}_{g,1};\mathbb{Q})_U} \to \underset{\substack{[0]_{\mathrm{Sp}} \\ \oplus\, [1^3]_{\mathrm{Sp}}}}{H_1(Ch_{g,1};\mathbb{Q})} \to \underset{[1^3]_{\mathrm{Sp}}}{U_{\mathbb{Q}}} \to 0$$

Theorem (K.)

For $g \geq 6$, *the rational abelianization of the Chillingworth subgroup is given by*
$d \oplus \tau_{g,1}(1)\colon Ch_{g,1} \to \mathbb{Q} \oplus U_{\mathbb{Q}} \cong [0]_{\mathrm{Sp}} + [1^3]_{\mathrm{Sp}}$ *as* $\mathrm{Sp}(2g,\mathbb{Q})$*-modules.*

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Note