Quantum invariants based on ideal triangulations

Sakie Suzuki (Tokyo Tech)

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(partly j.w.w. S.M. Mihalache, Y. Terashima)

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Toward to understanding quantum invariants uniformly via ideal triangulations

quantum invs of links : Jones poly ('84), colored Jones poly, Reshetikhin-Turaev inv ('90)

universal inv (Lawrence '89 Ohtsuki '93), (Kontsevich inv '93)

—— " —— 3-mfds : Witten-Reshetikhin-Turaev inv ('91),

unified WRT inv for ZHS (Habiro '07, Habiro-Le '16),

Hennings-Kauffman-Radford inv ('95 '96), (LMO inv '98)

Turaev-Viro inv ('92), Kuperberg inv ('91, '96).

Motivation & Background

H : findim Hopf alg

(Drinfeld '87)

(Kashaev '96)

Drinfeld double $D(H)$

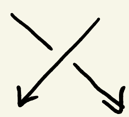
$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Heisenberg double $\mathcal{H}(H)$

$$S_{12}S_{13}S_{23} = S_{23}S_{12}$$

$$\iota : D(H) \hookrightarrow \mathcal{H}(H)\otimes\mathcal{H}(H)^{op}$$

$$R \mapsto S''_{14}S_{13}\tilde{S}_{24}S'_{23}$$



knot diagram

Jones poly.

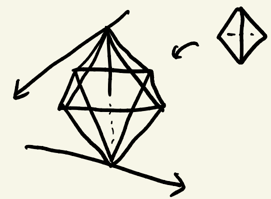
Reshetikhin-Turaev inv.

universal inv.

WRT inv.

HKR inv

$\leftrightarrow S$



$\times 4$

knot complement

Kashaev inv.

Volume conj.

q-Teichmüller TQFT

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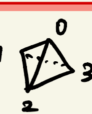
R $\longleftrightarrow$

6j-symbol + triangulation

Hopf alg, integral + Heegaard diag.

Understand them uniformly using ideal triangulations ' 

$\longleftrightarrow$ S !

Family tree $\mathfrak{g}$: simple Lie alg

Kontsevich inv $\rightsquigarrow$ LMO inv

$\rightsquigarrow \mathfrak{g}$

$\boxed{\text{universal } \mathfrak{g} \text{ invariant for framed links}}$

$\mathrm{Tr}^V \rightsquigarrow \cdots \rightsquigarrow$ V : fin dim rep of $U_q(\mathfrak{g})$

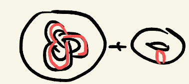
ω $\{$ Reshetikhin – Turaev $(\mathfrak{g}, V)$ inv $\}$

for framed links

 in S^3

q : root of unity $\rightsquigarrow$ sum up

$\rightsquigarrow$ surgery



unified WRT $\rightsquigarrow$ WRT inv

$\|$

integral   HKR inv

for 3-mfds ($\mathbb{Z}$HS)

$\boxed{\text{Let's start with the univ inv !}}$ $\rightarrow$

by heuristic approach

Universal invariant (regular isotopy ver. {knots diagrams}/RII, RIII) for D(H)

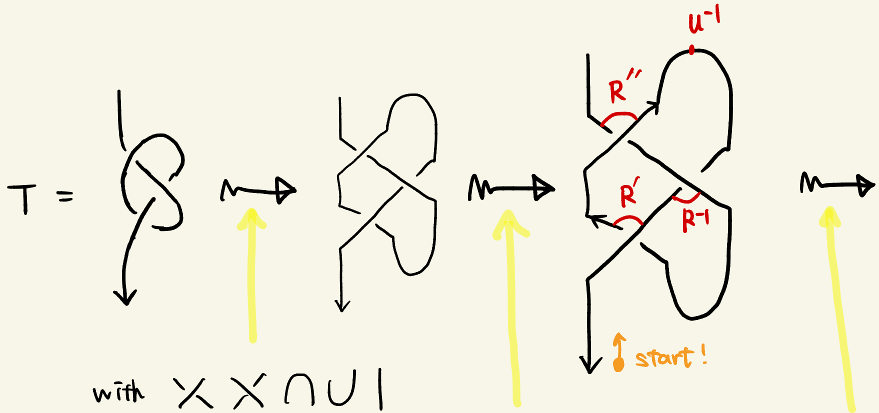
- Ingredient : findim Hopf alg H $\rightsquigarrow$ $D(H)$: Drinfeld double ,

  $(H, 1, m, \varepsilon, \Delta, \gamma)$ $\rightsquigarrow$ $R = \sum_i \alpha_i \otimes \beta_i \in D(H)^{\otimes 2}$: universal R-matrix

  $\gamma : H \to H$ antipode

  $u = \sum_i \gamma(\beta_i)\alpha_i \in D(H)$: Drinfeld element

- Construction for (1,1)-Tangle



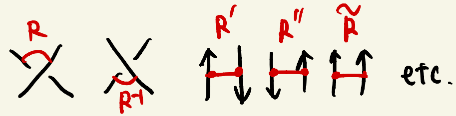
Read labels from ↑

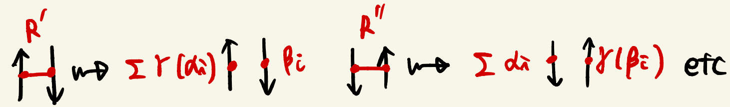
$$\sum_{i,j,k} \beta_i \overline{\alpha_j}\, u^{-1} \gamma(\beta_k)\gamma(\alpha_i)\, \overline{\beta_j}\, \alpha_k$$

$$= u^{-1} \sum_{i,j,k} \gamma^2(\beta_i \overline{\alpha_j})\gamma(\beta_k)\gamma(\alpha_i)\, \overline{\beta_j}\, \alpha_k$$

$$\Downarrow$$

$$J^r(T) = \sum_{i,j,k} \gamma^2(\beta_i \overline{\alpha_j})\gamma(\beta_k)\gamma(\alpha_i)\, \overline{\beta_j}\, \alpha_k$$

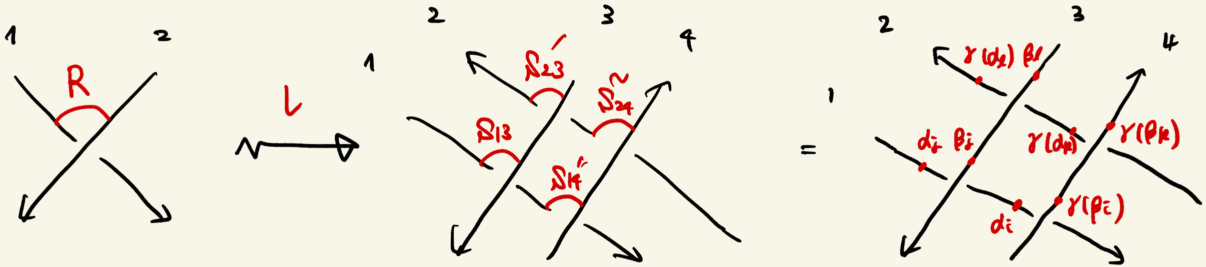




<u>Kashaev's embedding</u> $\iota : D(H) \hookrightarrow \mathcal{H}(H) \otimes \mathcal{H}(H)^{op}$,

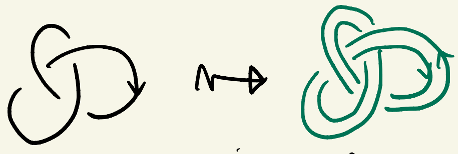
$$\iota^{\otimes 2} : R \mapsto S''_{14} S_{13} \tilde{S}_{24} S'_{23} \in \mathcal{H} \otimes \mathcal{H}^{op} \otimes \mathcal{H} \otimes \mathcal{H}^{op}$$

$$S''_{14} S_{13} \tilde{S}_{24} S'_{23} = \sum \alpha_i \alpha_j \otimes \gamma(\alpha_k)\gamma(\alpha_l) \otimes \beta_j \beta_l \otimes \gamma(\beta_i)\gamma(\beta_k)$$



What is this diagram ??

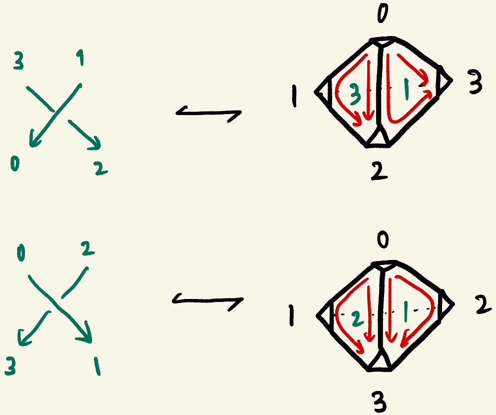
3-dim meaning of duplicated diagram

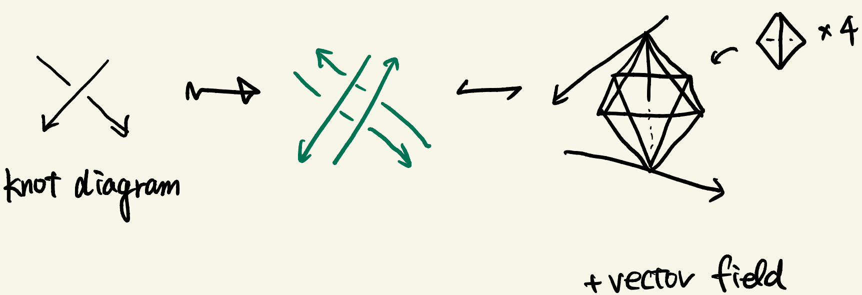
$K =$ 

"normal o-graph"

: ideal triangulation of $S^3 \setminus (K \cup 2pt)$ with a non-vanishing vector field

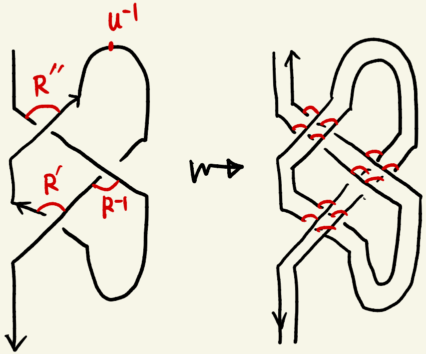
(we can obtain " of $S^3 \setminus K$)



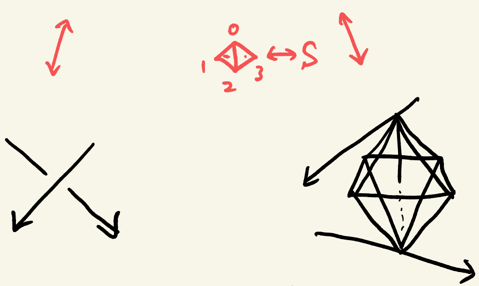


We are now on :

$J^r(T) =$



$$R \mapsto S''_{14} S_{13} \tilde{S}_{24} S'_{23}$$



knot diagram   knot complement

Problem 1

Can we reconstruct $\iota(J^r(T))$ in this setting ?

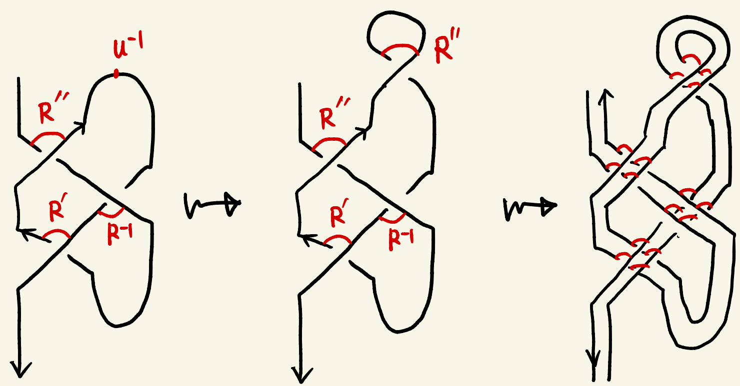
Problem 2

Can we extend this construction to 3-mfds ?

Problem 1

Can we reconstruct $\iota(J^r(T))$ in this setting ?

→ Yes

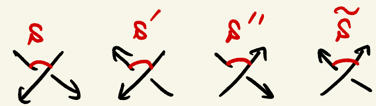


+ adjustment [S'18]

→ A little unnatural … What is u in 3-dim?

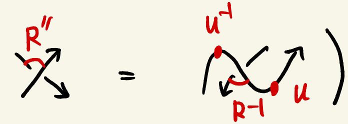
Problem 2

Can we extend this construction to 3-mfds ?

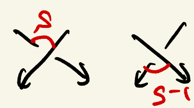
Difficulty ⌣̈ : the construction 

is Not planar isotopy inv.

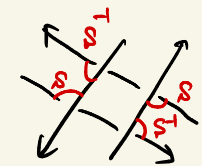
↓

(For knot : 

)

Idea 💡 : How about 

in any direction.



↓

Difficulty ⌣̈ : The result is not 3-mfd inv in general

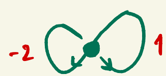
↓

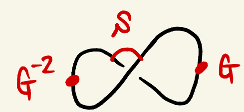
Idea 💡 : How about to use framing structures on 3-mfd ? (w/ Terashima)

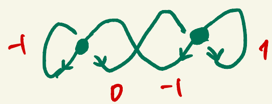
Reconstruction & extension of universal invariant

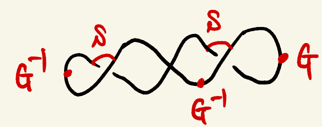
[S. M. Mihalache, S, Y. Terashima, '21 '22]

Invariant $Z(M, f; H)$ of closed framed 3-mfd (M, f) with $b_1(M) = 0$

(S^3, f_{S^3}) : 

$\longmapsto$ 

$(L(2,1), f_{L(2,1)})$: 

$\longmapsto$ 

$\sim$ SO(3) WRT inv for $H = u_q(sl_2^+)$

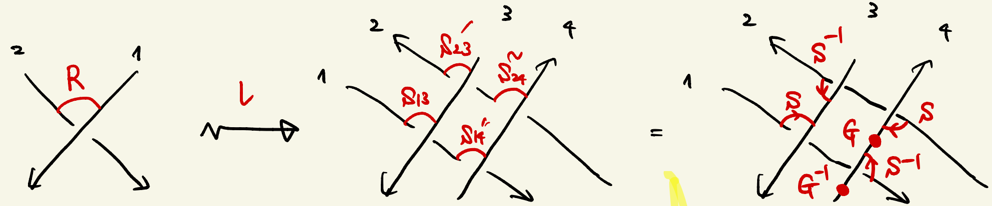
framing : a trivialization $f = (v_1, v_2, v_3)$ of TM.

- We already have v_1.
- We specify a section of $(\mathbb{R}v_1)^\perp$ by integer weights as rotation numbers.

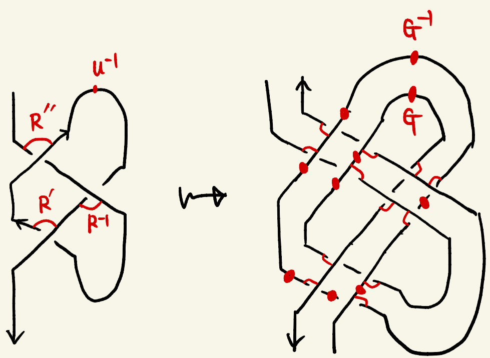
Problem 1 (again)

Can we reconstruct $\iota(J^r(T))$ in this setting?

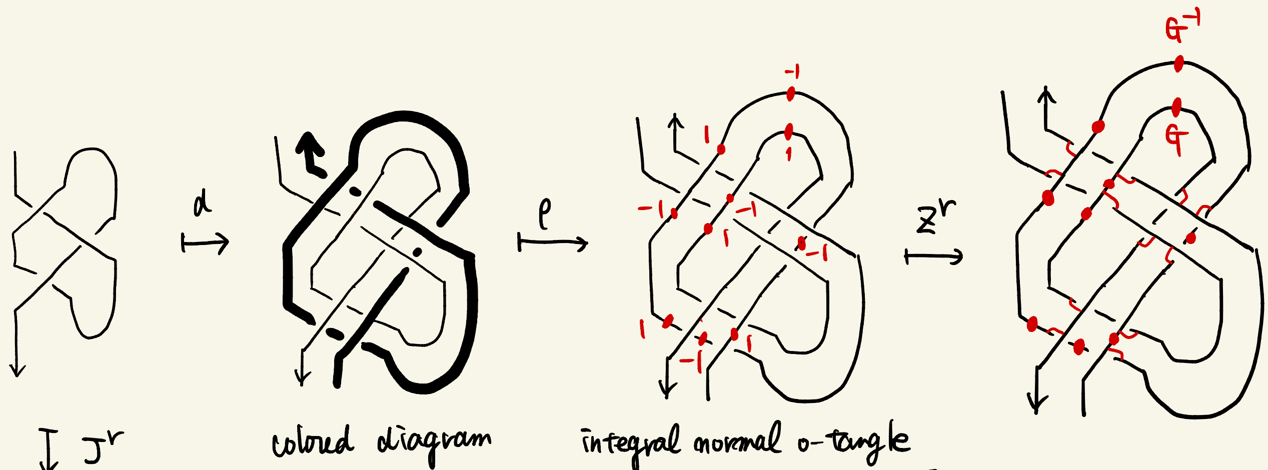
→ Yes

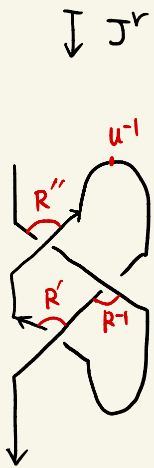


e.g.)



Reconstruction & extension of universal invariant





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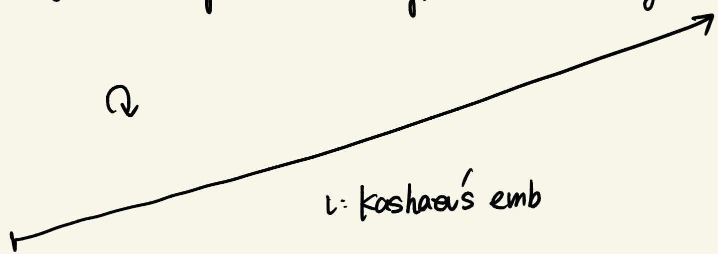


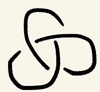
- $\rho \circ d$ is a regular isotopy inv
- we can extend ρ to diagrams not coming from knots.

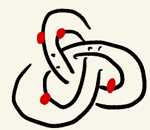
Summary

Input: Regular isotopy class of knot diagram D

$[D]_r = ($ isotopy class of K_D, winding # of D, framing of $D)$

universal invariant (regular isotopy ver.)

$J^r([$  $]_r ; D(H))$

$\xrightarrow[\iota]{\sim} Z^r($  $; H)$

$\rightsquigarrow$

$(S^3 \setminus K \cup \{2pt\}, f([D]_r))$

In particular, the Drinfeld element $u \rightsquigarrow$ twist of framing of 3-mfds.

Reconstruction & extension of universal invariant

Gloval picture