On the Potential Function of the Colored Jones Polynomial with Arbitrary Colors

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Introduction

- Kashaev constructed the Kashaev invariant and observed that a certain limit of the invariant for some hyperbolic knots is equal to the hyperbolic volume of their complements.
- Murakami-Murakami proved that the Kashaev invariant coincides with the colored Jones polynomial evaluated at the root of unity, and generalized the conjecture (= the volume conjecture).
- Yokota considered a "potential function" of the Kashaev invariant, and established the relationship between a saddle point equation and triangulation of a hyperbolic knot complement.
- Cho-Murakami considered a potential function of the colored Jones polynomial $J_N(L; q = e^{\frac{2\pi\sqrt{-1}}{N}})$ for a hyperbolic link L.

Introduction

Upshot of Yokota and Cho-Murakami's theory

The saddle point equation of the potential function coincides with the "gluing equation" of the triangulation.

- In this talk, we will consider the potential function of $J_i(L; q = e^{\frac{2\pi\sqrt{-1}}{N}})$.
- The potential function has parameters derived from colors.

This talk is based on

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- arXiv:2212.09294

Contents

The Colored Jones Polynomial

- The colored Jones polynomial is defined either skein-theoretically (Figure 1) or as an operator invariant.

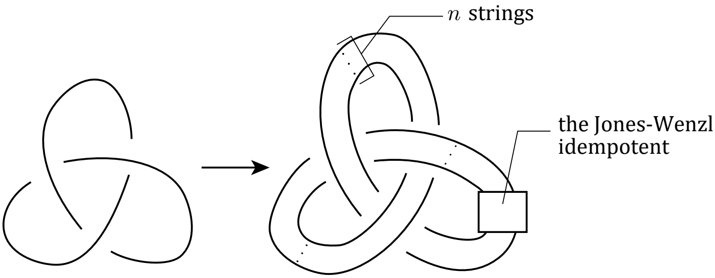


Figure 1: The skein-theoretical definition of the colored Jones polynomial.

- In this talk, we use the definition as an operator invariant.

The Operator Invariant

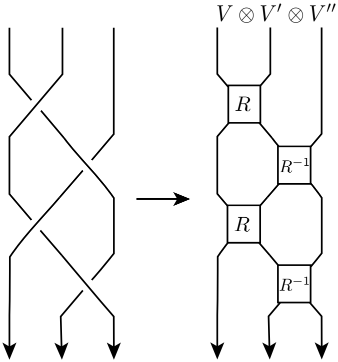


Figure 2: R-matrices and crossings.

- An R-matrix R is the operator that satisfies

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23},$$

where

$$R_{12} = R \otimes \mathrm{id},$$
$$R_{23} = \mathrm{id} \otimes R.$$

- We assign R-matrices to crossings as shown in Figure 2
- We can construct R-matrices from representaions of quantum groups.

The R-matrix for the Colored Jones Polynomial

- Let $r > 1$ be an integer, and let $s = e^{\frac{\pi\sqrt{-1}}{r}}$.
- Let $\mathcal{A}_r$ be the algebra generated by X, Y, K, $\overline{K}$ with the following relations:

$$\overline{K} = K^{-1}, \quad KX = sXK, \quad KY = s^{-1}YK,$$
$$XY - YX = \frac{K^2 - \overline{K}^2}{s - s^{-1}},$$
$$X^r = Y^r = 0, \quad K^{4r} = 1.$$

- For an integer k, we put

$$\{k\}_s = s^k - s^{-k}, \quad \{k\}_s! = \{k\}_s \cdots \{1\}_s, \quad \{0\}_s! = 1,$$
$$[k]_s = \frac{\{k\}_s}{\{1\}_s}, \qquad [k]_s! = [k]_s \cdots [1]_s, \qquad [0]_s! = 1.$$

The R-matrix for the Colored Jones Polynomial

- Let N be a positive integer and m be the half-integer satisfying $N = 2m + 1$.
- Let V be an N-dimensional vector space with a basis $\{e_{-m}, e_{-m+1}, \ldots, e_m\}$.
- We can define an N-dimensional irreducible representaion of $\mathcal{A}_r$ by

$$Xe_i = [m - i + 1]_s e_{i-1}, \; Ye_i = [m + i + 1]_s e_{i+1}, \; Ke_i = s^{-i} e_i.$$

The R-matrix for the Colored Jones Polynomial

- Let N' be a positive integer and m' be the half-integer satisfying $N' = 2m' + 1$.
- Let V' be an N'-dimensional vector space with a basis $\{e'_{-m'}, e'_{-m'+1}, \ldots, e'_{m'}\}$.
- The R-matrix obtained from the irreducible representation is

$$R_{VV'}(e_i \otimes e'_j) = \sum_{k=0}^{\min\{m+i,\ m'-j\}} \frac{\{m-i+k\}_s!\{m'+j+k\}_s!}{\{k\}_s!\{m-i\}_s!\{m'+j\}_s!} \times s^{2ij+k(i-j)-\frac{k(k+1)}{2}} e'_{j+k} \otimes e_{i-k}.$$

The R-matrix for the Colored Jones Polynomial

- The operator invariant obtained from the above 2-dimensional representation coincides with the Jones polynomial under substitution $s = -q^{-\frac{1}{2}}$.
- For an n-component link $L \subset S^3$, the colored Jones polynomial $J_{\boldsymbol{i}}(L; q)$ with a multi-integer $\boldsymbol{i} = (i_1, \dots, i_n)$ is determined by the quantum R-matrix $R : V \otimes V' \to V' \otimes V$

$$R_{VV'}(e_i \otimes e'_j) = \sum_{k=0}^{\min\{m+i,\ m'-j\}} (-1)^{k+k(m+m')+2ij} q^{-ij-\frac{k(i-j)}{2}+\frac{k(k+1)}{4}} \times \frac{\{m-i+k\}!\{m'+j+k\}!}{\{k\}!\{m-i\}!\{m'+j\}!} e'_{j+k} \otimes e_{i-k},$$

where

$$\{k\} = q^{\frac{k}{2}} - q^{-\frac{k}{2}},\ \{k\}! = \{k\}\{k-1\}\cdots\{1\},\ \{0\}! = 1.$$

The Colored Jones Polynomial

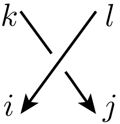
- We put

$$R_{VV'}(e_i \otimes e'_j) = \sum_{k,l} (R^+)^{kl}_{ij} e'_k \otimes e_l,$$

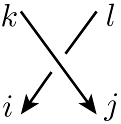
$$R^{-1}_{VV'}(e'_i \otimes e_j) = \sum_{k,l} (R^-)^{kl}_{ij} e_k \otimes e'_l.$$

- We assign $(R^{\pm})^{kl}_{ij}$ to each crossing of the diagram:

$(R^+)^{kl}_{ij}$:



$(R^-)^{kl}_{ij}$:



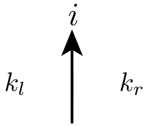
- We assign $(-1)^{N-1} q^{\pm i}$ to each maximum point of the diagram:

$(-1)^{N-1} q^{i}$:

$(-1)^{N-1} q^{-i}$:

The Colored Jones Polynomial

- In this talk, we change the indices i, j, k, l to the ones labeled to each regions around the crossing:



$$: \quad i = k_l - k_r$$

- We obtain the R-matrix $R^{\pm}(m, m', k_{j_1}, k_{j_2}, k_{j_3}, k_{j_4})$, where k_{j_1}, k_{j_2}, k_{j_3}, k_{j_4} are indeces as shown in Figure 3.

Figure 3: Indices around a crossing.

The Colored Jones Polynomial

- $i,\ j,\ k$ and $k_{j_1},\ldots,k_{j_4}$ satisfy

$$\begin{aligned} i &= k_{j_2} - k_{j_1}, \\ j &= k_{j_3} - k_{j_2}, \\ k &= k_{j_2} + k_{j_4} - k_{j_1} - k_{j_3}. \end{aligned}$$

- The colored Jones polynomial $J_{\boldsymbol{i}}(L;q)$ is the multiplication of all these factors with modification for the Reidemeister move I.
- We normalize the colored Jones polynomial so that

$$J_{\boldsymbol{i}}(\bigcirc\cdots\bigcirc;q) = 1,$$

where $\bigcirc\cdots\bigcirc$ is a trivial link.

The Volume Conjecture

For an integer N, we put $\xi_N = e^{\frac{2\pi\sqrt{-1}}{N}}$.

Conjecture 1 (the Volume Conjecture)

For any knot K,

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(K; q = \xi_N)|}{N} = v_3||K||,$$

where v_3 is the volume of the ideal regular tetrahedron in the three-dimensional hyperbolic space and $||\cdot||$ is the simplicial volume for the complement of K.

Remark 1

If K is hyperbolic, $v_3||K||$ is equal to the hyperbolic volume of K.

The Volume Conjecture

Let u be a primitive r-th root of unity.

Conjecture 2 (the Chen-Yang Conjecture)

For any 3-manifold M with a complete hyperbolic structure of the finite volume,

$$2\pi \lim_{r\to\infty} \frac{\log TV_r(M, u = \xi_r)}{r} = \mathrm{Vol}(M),$$

where r runs over all odd integers, $TV(M)$ is a Turaev-Viro invariant of M and $\mathrm{Vol}(M)$ is a hyperbolic volume of M.

Remark 2 (Detcherry-Kalfragianni-Yang '18)

For an odd integer $r \geq 3$, $TV_r(S^3 \setminus L, u)$ can be written as a sum of $|J_{\boldsymbol{i}}(L; u^2)|^2$ w.r.t $\boldsymbol{i}$.

Definition of the potential function

Definition 3 (the potential function)

Suppose that a certain quantity Q_N can be written as

$$Q_N \underset{N\to\infty}{\sim} \int \cdots \int_\Omega P_N e^{\frac{N}{2\pi\sqrt{-1}}\Phi(z_1,\dots,z_\nu)} dz_1 \cdots dz_\nu,$$

where P_N grows at most polynomially and Ω is a region in $\mathbb{C}^\nu$. We call this function $\Phi(z_1,\dots,z_\nu)$ a potential function of Q_N.

- The saddle point of the potential function contributes to the limit of such integral (=the saddle point method).
- In the case of the colored Jones polynomial for a hyperbolic link, the saddle point equation relates to the geometry of the link complement.

The potential function of $J_i(L; \xi_N^p)$

- We fix a diagram D of the n-component hyperbolic link L.
- For each crossing c of D, we can obtain the local potential function $\Phi^{\pm}_{c,p}(a, b, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})$, where $p = 1$ or 2 and $w_{j_i} = \xi_N^{k_{j_i}}$, by approximating the R-matrix with $q = \xi_N^p$ by continuous functions:

$$\text{quantum factorial } \{k\}! \rightsquigarrow \mathrm{Li}_2(z)$$

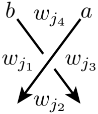
Here, $\mathrm{Li}_2(z)$ is the dilogarithm function

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-x)}{x} dx.$$

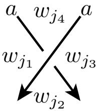
- The potential function $\Phi_{D,p}(\boldsymbol{a}, w_1, \ldots, w_\nu)$ is the sum of all local potential functions, where $\boldsymbol{a} = (a_1, \ldots, a_n)$ is an n-tuple of real numbers $a_j = \lim_{N\to\infty} \frac{i_j}{N}$.

$\Phi^{\pm}_{c,p}$

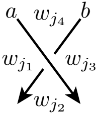
The functions $\Phi^{\pm}_{c,p}$ are of the form:



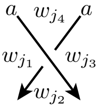
$$: \quad \Phi^{+}_{c,p} = f^{+}_{p}(a, b, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})$$



$$: \quad \Phi^{+}_{c,p} = p(\pi\sqrt{-1}a)^2 + f^{+}_{p}(a, a, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})$$



$$: \quad \Phi^{-}_{c,p} = f^{-}_{p}(a, b, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})$$



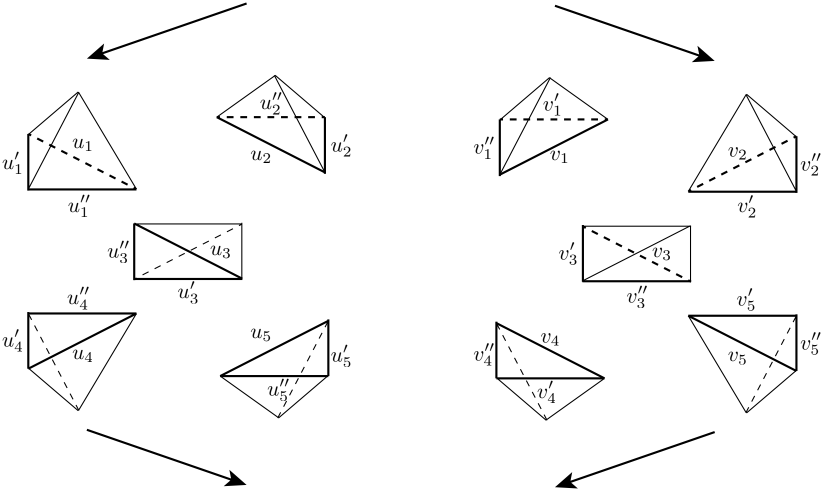
$$: \quad \Phi^{-}_{c,p} = -p(\pi\sqrt{-1}a)^2 + f^{-}_{p}(a, a, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})$$

$f_p^{\pm}(a, b, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})$

Let $e_a = e^{\pi\sqrt{-1}a}$. $f_p^{\pm}(a, b, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})$ are:

$$
\begin{aligned}
f_p^{+}(a, b, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4}) &= \frac{1}{p}\Bigg\{\pi\sqrt{-1}p^2\frac{a+b}{2}\log\frac{w_{j_1}w_{j_3}}{w_{j_2}w_{j_4}} \\
&\quad - p^2\log\frac{w_{j_2}}{w_{j_1}}\log\frac{w_{j_3}}{w_{j_2}} - \mathrm{Li}_2\left(e_a^p\frac{w_{j_4}^p}{w_{j_3}^p}\right) - \mathrm{Li}_2\left(e_b^p\frac{w_{j_4}^p}{w_{j_1}^p}\right) \\
&\quad + \mathrm{Li}_2\left(\frac{w_{j_2}^p w_{j_4}^p}{w_{j_1}^p w_{j_3}^p}\right) + \mathrm{Li}_2\left(e_a^p\frac{w_{j_1}^p}{w_{j_2}^p}\right) + \mathrm{Li}_2\left(e_b^p\frac{w_{j_3}^p}{w_{j_2}^p}\right) - \frac{\pi^2}{6}\Bigg\}, \\
f_p^{-}(a, b, w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4}) &= \frac{1}{p}\Bigg\{-\pi\sqrt{-1}p^2\frac{a+b}{2}\log\frac{w_{j_1}w_{j_3}}{w_{j_2}w_{j_4}} \\
&\quad + p^2\log\frac{w_{j_3}}{w_{j_4}}\log\frac{w_{j_4}}{w_{j_1}} - \mathrm{Li}_2\left(e_a^p\frac{w_{j_1}^p}{w_{j_4}^p}\right) - \mathrm{Li}_2\left(e_b^p\frac{w_{j_3}^p}{w_{j_4}^p}\right) \\
&\quad - \mathrm{Li}_2\left(\frac{w_{j_2}^p w_{j_4}^p}{w_{j_1}^p w_{j_3}^p}\right) + \mathrm{Li}_2\left(e_a^p\frac{w_{j_2}^p}{w_{j_3}^p}\right) + \mathrm{Li}_2\left(e_b^p\frac{w_{j_2}^p}{w_{j_1}^p}\right) + \frac{\pi^2}{6}\Bigg\}.
\end{aligned}
$$

D. Thurston's triangulation



The geometry of the link complement

- If all these tetrahedra are glued well, the link complement admits a hyperbolic structure.
- Let M be the hyperbolic link complement. In general, $\partial\overline{M}$ has a similarity structure, i.e. a curve γ in $\partial\overline{M}$ induces the action of the form

$$\mathbb{C} \ni z \mapsto az + b \in \mathbb{C}, \quad a, b \in \mathbb{C}.$$

  We call the coefficient a the dilation component of γ and write $\delta(\gamma)$.
- M is complete iff. $\partial\overline{M}$ admits an Euclidean structure.

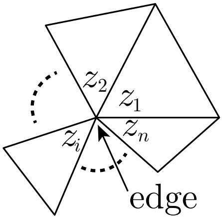
The case where a is fixed

Let L be an n-component hyperbolic link. Since $\Phi_{D,p}$ is easily obtained from $\Phi_{D,1}$, we mainly consider the case where $p = 1$ and write $\Phi_D = \Phi_{D,1}$

- When we assume that $a_j \in [1-\varepsilon, 1]$ for all $j = 1, \ldots, n$, where ε is a sufficiently small positive real number, the system of equations

$$\frac{\partial \Phi_D}{\partial w_i} = 0, \quad i = 1, \ldots, \nu$$

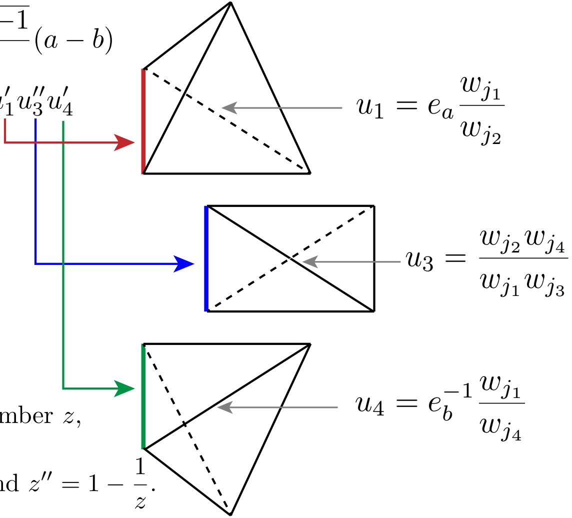
coincides with the "gluing equation".



$$z_1 z_2 \cdots z_n = 1$$

The case where a is fixed

$$w_{j_1}\frac{\partial \Phi_c^+}{\partial w_{j_1}} = \frac{\pi\sqrt{-1}}{2}(a-b) + \log u_1' u_3'' u_4'$$



For a complex number z,

$$z' = \frac{1}{1-z}, \text{ and } z'' = 1 - \frac{1}{z}.$$

The case where a is fixed

- Let G_i be a product of moduli of ideal tetrahedra around the region R_i with w_i labeled. Then, we have

$$w_i \frac{\partial \Phi_D}{\partial w_i} = \frac{\pi\sqrt{-1}}{2} r(a_1, \ldots, a_n) + \log G_i,$$

where $r(a_1, \ldots, a_n)$ is a linear polynomial w.r.t $a_1, \ldots, a_n$.

- We can verify that $r(a_1, \ldots, a_n) = 0$.
- A saddle point $(\sigma_1(\boldsymbol{a}), \ldots, \sigma_\nu(\boldsymbol{a}))$ determines a hyperbolic structure of the link complement, which is not necessarily complete.
    - Choose the saddle point such that $(\sigma_1(\mathbf{1}), \ldots, \sigma_\nu(\mathbf{1}))$ gives a hyperbolic structure with the volume $\mathrm{Vol}(M)$.
- Let $M_{\boldsymbol{a}}$ be a manifold with this hyperbolic structure.

The dilation components

Let L_j be a component of L with a parameter a_j.

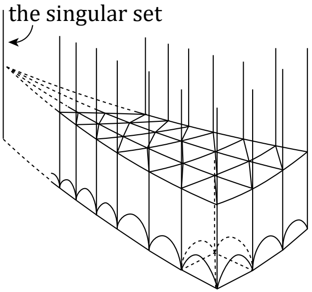


Figure 4: The developing image of $M_{\boldsymbol{a}}$ in $\mathbb{H}^3$.

- For the meridian m_j of the component L_j,

$$\delta(m_j) = e^{-2\pi\sqrt{-1}a_j}.$$

  Figure 4 shows the developing image of $M_{\boldsymbol{a}}$ in $\mathbb{H}^3$.

- $M_{\boldsymbol{a}}$ is a cone-manifold with cone-angles $2\pi(1-a_j)$ around $L_j, \quad j = 1, \ldots, n$.

The case where a is variable

- When we regard a_j $(j = 1, \dots, n)$ as variables, we have

$$\exp\left(\frac{1}{\pi\sqrt{-1}}\frac{\partial \Phi_D}{\partial a_j}\right) = \delta(\tilde{l}_j), \qquad (1)$$

where $\tilde{l}_j$ is the longitude of the component L_j with $\mathrm{lk}(\tilde{l}_j, L_j) = 0$.

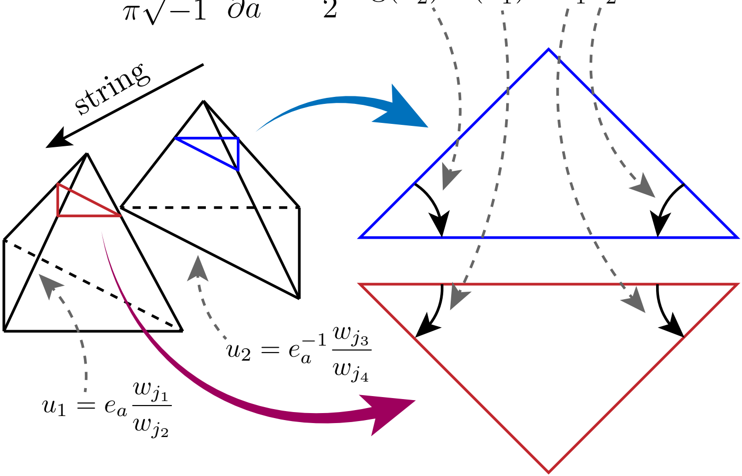
- The saddle point equation w.r.t a_j coincides with the "completeness equation".

Theorem 4 (S.)

Let D be a diagram of a hyperbolic link with n components, and $\mathbf{1}$ be $(1, \dots, 1) \in \mathbb{Z}^n$. The point $(\mathbf{1}, \sigma_1(\mathbf{1}), \dots, \sigma_\nu(\mathbf{1}))$ is a saddle point of the function $\Phi_D(a_1, \dots, a_n, w_1, \dots, w_\nu)$ and gives a complete hyperbolic structure to the link complement.

Idea of the proof of (1)

$$\frac{1}{\pi\sqrt{-1}}\frac{\partial \Phi_c^+}{\partial a} = \frac{1}{2}\log(u_2')^{-1}(u_1'')^{-1}u_1'u_2''$$



The Witten-Reshetikhin-Turaev invariant

- Let $M_{f_1,\ldots,f_n}$ be the hyperbolic manifold obtained by Dehn surgery on a link $L = L_1 \cup \cdots \cup L_n$ with a framing f_j on L_j $(j = 1, \ldots, n)$.
- Let $\alpha_j = e^{\pi\sqrt{-1}a_j}$, and $\Phi(\alpha_1, \ldots, \alpha_n, w_1, \ldots, w_\nu)$ be the potential function of the Witten-Reshetikhin-Turaev invariant of $M_{f_1,\ldots,f_n}$.
- We regard each α_j as a complex parameter which is not necessarily in the unit circle.

The Witten-Reshetikhin-Turaev invariant

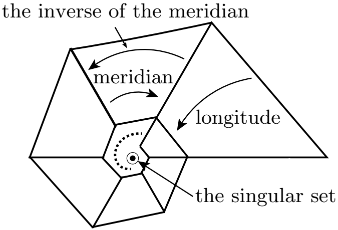


Figure 5: The schematic diagram of the developing image in the case of $f_j = 6$.

- The derivative with respect to α_j is

$$\exp\left(\alpha_j \frac{\partial \Phi}{\partial \alpha_j}\right) = \alpha_j^{-2f_j} \delta(\tilde{l}_j),$$

  where $j = 1, \ldots, n$.
- Recall that $\delta(m_j) = \alpha_j^{-2}$.
- The saddle point equation implies that $\delta(m_j)^{-f_j} = \delta(\tilde{l}_j)$.
- Assuming that $f_j > 0$ and $|\alpha_j| < 1$, the developing image would be as shown in Figure 5.

The A-polynomial

Let K be a hyperbolic knot.

- A factor of the A-polynomial $A_K(l,m)$ is conjectured to be obtained from the system of equations

$$\begin{cases} \exp\left(w_i \dfrac{\partial \Phi_D}{\partial w_i}\right) = 1, \quad (i = 1, \dots, \nu) \\ \exp\left(\alpha \dfrac{\partial \Phi_D}{\partial \alpha}\right) = l^2 \end{cases} \tag{2}$$

  by eliminating $w_1, \dots, w_\nu$.
- The other factor of $A_K(l,m)$ is $l - 1$ that corresponds to abelian representations.

Example: figure-eight knot

- The colored Jones polynomial of the figure-eight knot is

$$J(n) = J_n(4_1; q) = \frac{1}{\{n\}} \sum_{i=0}^{n-1} \frac{\{n+i\}!}{\{n-i-1\}!}.$$

- The potential function $\Phi(\alpha, x)$ of $J_i(4_1, \xi_N)$ is

$$\Phi(\alpha, x) = -2 \log \alpha \log x - \mathrm{Li}_2(\alpha^2 x) + \mathrm{Li}_2(\alpha^2 x^{-1}).$$

- The derivatives of Φ with x and α are

$$x \frac{\partial \Phi}{\partial x} = \log \alpha^{-2} (1 - \alpha^2 x)(1 - \alpha^2 x^{-1}),$$
$$\alpha \frac{\partial \Phi}{\partial \alpha} = 2 \log (1 - \alpha^2 x)(x - \alpha^2)^{-1}.$$

Example: figure-eight knot

- From

$$\begin{cases} \alpha^{-2}(1-\alpha^2 x)(1-\alpha^2 x^{-1}) = 1, \\ (1-\alpha^2 x)(x-\alpha^2)^{-1} = l, \end{cases}$$

  we obtain

$$A'_{4_1}(l,\alpha) = \alpha^4 - l + \alpha^2 l + 2\alpha^4 l + \alpha^6 l - \alpha^8 l + \alpha^4 l^2$$

  by eliminating x.
- In fact, the A-polynomial for 4_1 is

$$(l-1)(m^4 - l + m^2 l + 2m^4 l + m^6 l - m^8 l + m^4 l^2) \qquad (3)$$

The A_q-polynomial

- The A_q-polynomial $A_q(K)$ for a knot K is the polynomial defined as an annihilator of $J_K(n) = J_n(K;q)$.

$$\sum_{i=0}^{d} c_i(q,q^n)J_K(n+i) = 0 \to \left(\sum_{i=0}^{d} c_i(q,Q)E^i\right) J_K(n) = 0.$$

Here, $(EJ_K)(n) = J_K(n+1)$ and $(QJ_K)(n) = q^n J_K(n)$.

- $I_K = \{P \in \mathcal{A} \mid PJ_K(n) = 0\}$ is a left ideal in $\mathcal{A}$, where

$$\mathcal{A} = \left\{ \sum_{i=0}^{d} c_i(q,Q)E^i \;\middle|\; \begin{array}{l} d \in \mathbb{Z}_{\geq 0} \\ c_i(q,Q) \in \mathbb{Z}[q,Q] \\ EQ = qQE \end{array} \right\}.$$

The AJ conjecture

Definition 5 (Garoufalidis '04)

The A_q-polynomial $A_q(K)(E,Q)$ for a knot K is an generator with the smallest E-degree and coprime coefficients of the annihilating ideal I_K in a certain localization of $\mathcal{A}$.

Conjecture 3 (the AJ conjecture)

For any knot K, $A_K(l,m)$ is equal to $\varepsilon A_q(K)(l,m^2)$ up to multiplication by an element in $\mathbb{Q}(m)$, where ε is the evaluation map at $q=1$.

Creative telescoping

- Let $F(n, k_1, \dots, k_\nu)$ be a multi-$\mathbb{Z}$-variable discrete function. We define the operators $Q,\ E,\ Q_i,\ E_i\ (i = 1, \dots, \nu)$ by

$$\begin{aligned}(QF)(n, k_1, \dots, k_\nu) &= q^n F(n, k_1, \dots, k_\nu),\\ (EF)(n, k_1, \dots, k_\nu) &= F(n+1, k_1, \dots, k_\nu),\\ (Q_iF)(n, k_1, \dots, k_\nu) &= q^{k_i} F(n, k_1, \dots, k_\nu),\\ (E_iF)(n, k_1, \dots, k_\nu) &= F(n, k_1, \dots, k_i + 1, \dots, k_\nu).\end{aligned}$$

- These operators generate the noncommutative algebra $\mathbb{Q}[q, Q, Q_{\boldsymbol{k}}]\langle E, E_{\boldsymbol{k}}\rangle$ with following relations:

$$Q_iQ_j = Q_jQ_i,\ E_iE_j = E_jE_i,\ E_iQ_j = q^{\delta_{ij}}Q_jE_i,$$

  where $i, j \in \{0, \dots, \nu\}$ and $E_0 = E,\ Q_0 = Q$.
- $F : \mathbb{Z}^{\nu+1} \to \mathbb{Q}(q)$ is called q-hypergeometric if $E_iF/F \in \mathbb{Q}(q, q^n, q^{k_1}, \dots, q^{k_\nu})$ holds for all $i = 0, \dots, \nu$.

Creative telescoping

Theorem 6 (Wilf-Zeilberger '92)

Every "proper" q-hypergeometric function $F(n, \boldsymbol{k})$ has a $\boldsymbol{k}$-free recurrence

$$\sum_{(i,\boldsymbol{j})\in S} \sigma_{i,\boldsymbol{j}}(q^n) F(n+i, \boldsymbol{k}+\boldsymbol{j}) = 0,$$

where S is a finite set, and $\sigma_{i,\boldsymbol{j}}$ are $\mathbb{Q}(q)$-coefficient polynomials.

- i.e. $\exists P(E, Q, E_1, \ldots, E_\nu) \in \mathbb{Q}[q, Q]\langle E, E_{\boldsymbol{k}}\rangle$ s.t. $PF = 0$.
- Expanding P at $(E_1, \ldots, E_\nu) = \mathbf{1}^\nu = (1, \ldots, 1)$, we have

$$P_0(E, Q) + \sum_{i=1}^{\nu} (E_i - 1) R_i(E, Q, E_1, \ldots, E_\nu),$$

where $P_0(E, Q) = P(E, Q, \mathbf{1}^\nu)$, and $R_i \in \mathbb{Q}[q, Q]\langle E, E_{\boldsymbol{k}}\rangle$.

Creative telescoping

- Putting $G_i = R_i F$, we have

$$P_0(E,Q)F(n,\boldsymbol{k}) \\ = -\sum_{i=1}^{\nu}(G_i(n,k_1,\dots,k_i+1,\dots,k_\nu) - G_i(n,k_1,\dots,k_\nu)).$$

- Summing up this equality, we verify that $P_0(E,Q)G(n)$ is a sum of multisums of proper q-hypergeometric functions with one variable less, where $G(n) = \sum_{\boldsymbol{k}} F(n,\boldsymbol{k})$.
- Repeating this process, we obtain $P_1(E,Q)P_0(E,Q)G(n) = 0$ for a polynomial $P_1(E,Q)$.

Creative telescoping

- Note that

$$P(E, Q, E_1, \dots, E_\nu) \in \mathrm{Ann}(F) \cap \mathbb{Q}[q, Q]\langle E, E_{\boldsymbol{k}}\rangle,$$

where $\mathrm{Ann}(F) = \{P \in \mathbb{Q}[q, Q, Q_{\boldsymbol{k}}]\langle E, E_{\boldsymbol{k}}\rangle \mid PF = 0\}$ is an annihilating ideal of F.

- If we put

$$\frac{E_i F}{F} = \frac{R_i}{S_i}\Big|_{Q=q^n,\ Q_j=q^{k_j}}$$

for $R_i, S_i \in \mathbb{Z}[q, Q, Q_{\boldsymbol{k}}]$, then, $\mathrm{Ann}(F)$ is generated by $\{S_i E_i - R_i \mid i = 0, \dots, \nu\} \subset \mathbb{Q}[q, Q, Q_{\boldsymbol{k}}]\langle E, E_{\boldsymbol{k}}\rangle$.

- We would be able to obtain $P(E, Q, E_1, \dots, e_\nu)$ from

$$S_i E_i - R_i = 0, \quad i = 0, \dots, \nu$$

by eliminating $Q_1, \dots, Q_\nu$.

Example: figure-eight knot revisited

- Put

$$F(n,i) = \frac{1}{\{n\}} \frac{\{n+i\}!}{\{n-i-1\}!}.$$

- Calculating EF/F and E_1F/F with $Q = q^n$ and $Q_1 = q^i$,

$$(E + qQ)Q_1(Q-1) = (1 + QE)(Q-1), \qquad (4)$$

$$q^2Q_1^2Q + qQ_1(-Q^2 + QE_1 - 1) + Q = 0. \qquad (5)$$

- From (4), we have

$$(1 + QE)Q_1^{-1}(Q-1) = (E + qQ)(Q-1) \qquad (6)$$

Example: figure-eight knot revisited

- Multiplying (5) by $q^{-1}Q_1^{-1}Q^{-1}(Q-1)$ from the left,

$$\{qQ_1 + Q^{-1}(-Q^2 + QE_1 - 1) + q^{-1}Q_1^{-1}\}(Q-1) = 0. \quad (7)$$

- Multiplying (7) by

$$X(q,E,Q) = \frac{qQ}{1-q^3Q^2}E^2 + \left(\frac{1}{1-q^3Q^2} + \frac{1}{1-qQ^2} - 1\right)E + \frac{qQ}{1-qQ^2}$$

from the left and using (4) and (6), we have

$$P(E,Q,E_1) = \left\{\frac{qQ}{1-q^3Q^2}E_1E^2 + \left(\frac{1}{1-q^3Q^2}E_1 + \frac{1}{1-qQ^2}E_1 + qQ - E_1 - \frac{1}{qQ}\right)E + \frac{qQ}{1-qQ^2}E_1\right\} \times (Q-1).$$

Example: figure-eight knot revisited

Remark 7

$X(q,E,Q)$ is factorized in two ways:

$$X(q,E,Q) = \left(\frac{qQ}{1-q^3Q^2}E + \frac{1}{1-qQ^2}\right)(E+qQ)$$
$$= \left(\frac{1}{1-q^3Q^2}E + \frac{qQ}{1-qQ^2}\right)(1+QE).$$

- $P_0(E,Q) = P(E,Q,1)$ satisfies

$$P_0(E,Q)J(n) + q^{n+1} + 1 = 0.$$

- Since $q^{n+1}+1$ is annihilated by $P_1(E,Q) = (E-1)\cdot\frac{1}{1+qQ}$, we have the third order homogeneous recursion relation

$$P_1(E,Q)P_0(E,Q)J(n) = 0.$$

Example: figure-eight knot revisited

- The annihilating polynomial with $q = 1$ is

$$\varepsilon P_1(E,Q)P_0(E,Q) = \frac{(E-1)(Q^2 - E + QE + 2Q^2E + Q^3E - Q^4E + Q^2E^2)}{Q(1-Q^2)}.$$

- This is equal to (3)

$$(l-1)(m^4 - l + m^2 l + 2m^4 l + m^6 l - m^8 l + m^4 l^2)$$

in the sense of the statement of the AJ conjecture.

Comparison with the potential function

- We would be able to obtain $\varepsilon P_0(E,Q)$ by eliminating $Q_1,\dots,Q_\nu$ from

$$\begin{cases} \varepsilon(S_iE_i - R_i)\,|_{E_i=1} = 0 \quad (i=1,\dots,\nu), \\ \varepsilon(SE-R)=0, \end{cases} \tag{8}$$

  where $S=S_0,\ R=R_0$.

- In the case of the colored Jones polynomial, the system of equations (8) is equivalent to the system of the equations (2)

$$\begin{cases} \exp\left(w_i \dfrac{\partial \Phi_D}{\partial w_i}\right) = 1, \quad (i=1,\dots,\nu) \\ \exp\left(\alpha \dfrac{\partial \Phi_D}{\partial \alpha}\right) = l^2. \end{cases}$$

Comparison with the potential function

Proposition 8 (S.)

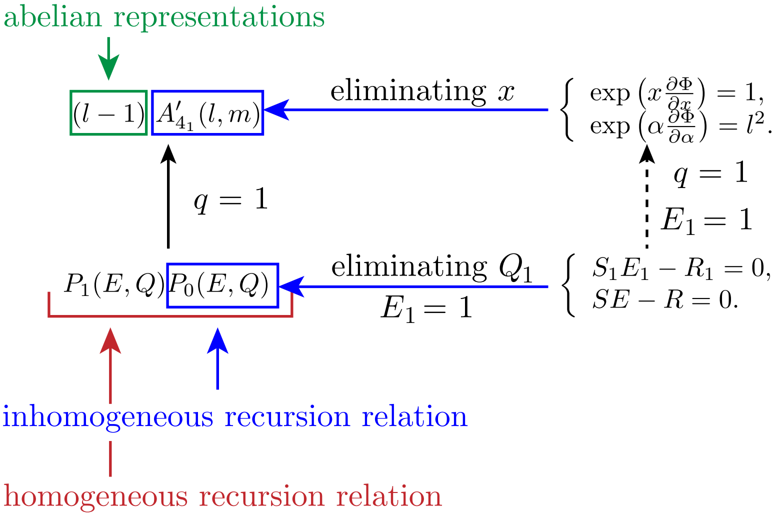
Following equalities hold:

$$\exp\left(w_i \frac{\partial \Phi}{\partial w_i}\right) = \varepsilon \left.\frac{E_i F}{F}\right|_{\substack{q^{k_j}=w_j \\ q^m=\alpha}},$$

$$\exp\left(\alpha \frac{\partial \Phi}{\partial \alpha}\right) = \varepsilon \left.\frac{E_m F}{F}\right|_{\substack{q^{k_i}=w_i \\ q^m=\alpha}},$$

where E_m is an operator that shifts m to $m+1$.

Figure-eight knot re-revisited



Thank you for your attention.