Upsilon and secondary Upsilon invariants of L–space knots

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Contents

UPSILON INVARIANT AND L–SPACE KNOT

Ozsváth–Stipsicz–Szabó (2017)

For a knot K, a piecewise linear, continuous function $\Upsilon_K(t)\colon [0,2] \to \mathbb{R}$ is assigned.

- concordance invariant
- $\Upsilon'_K(0) = -\tau(K)$
- symmetric along $t = 1$
- additive for connected sum
- $\Upsilon_{-K} = -\Upsilon_K$
- if K is smoothly slice, then $\Upsilon_K(t) = 0$
- a lower bound for genus, 4-genus, concordance genus

Upsilon invariant

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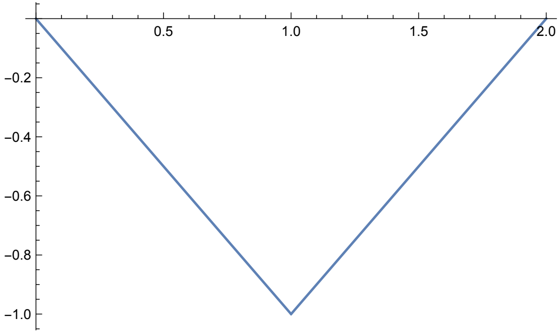
Originally, [OSS] used t-modified knot Floer complex. Later, Livingston gave an interpretation on knot Floer complex $\mathrm{CFK}^\infty(K)$.

Example: Trefoil knot $T(2,3)$

$$\Upsilon_{T(2,3)}(t) = -t \quad (0 \le t \le 1)$$



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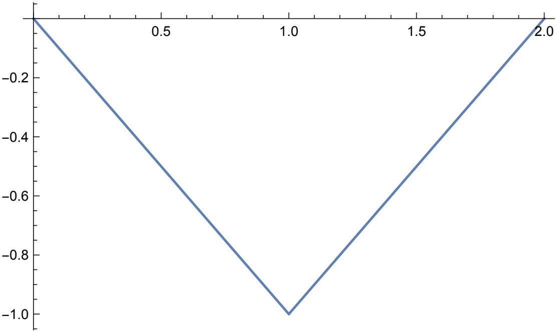

- There is an inductive formula for torus knots.
- For any alternating (or, quasi-alternating) knot, $\Upsilon_K(t) = (1 - |t-1|)\sigma/2$.

L–SPACE KNOT

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OSS

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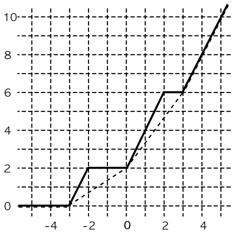
Borodzik–Hedden (2018)

For an L–space knot, the Upsilon invariant is the Legendre–Fenchel transform of a gap function, which has the same information as the Alexander polynomial.

EXAMPLE: $T(3,4)$

$$\Delta(t) = 1 - t + t^3 - t^5 + t^6 \to [1, 2, 2, 1]$$



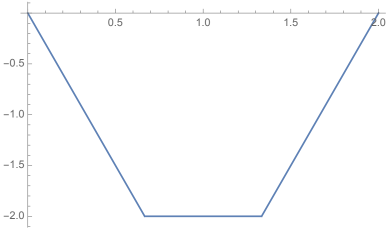


Figure: gap function and Upsilon invariant

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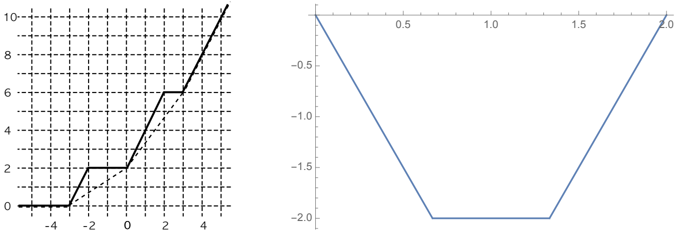

Figure: gap function and Upsilon invariant

$0 \leq t \leq 2/3$	$y = t(x+3)$	$\Upsilon(t) = -3t$
$2/3 < t < 4/3$	$y = tx + 2$	$\Upsilon(t) = -2$
$4/3 \leq t \leq 2$	$y = t(x-3)+6$	$\Upsilon(t) = 3t - 6$

Table: Legendre–Fenchel transformation

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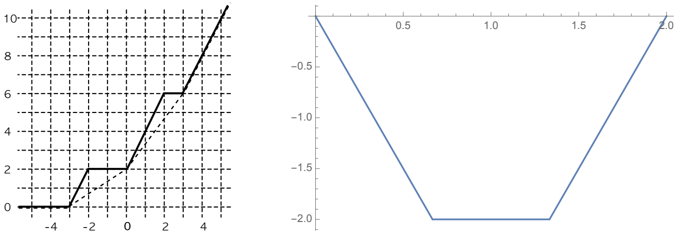

Figure: gap function and Upsilon invariant

L–space knot

The Upsilon invariant is determined only by the convex hull of gap function.

Problems and Results

Raising Problems

- If two L–space knots have distinct Alexander polynomials, but the gap functions share the same convex hull, then their Upsilon invariants coincide.
  - No duplication among torus knots.
  - Find among hyperbolic L–space knots.

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- If two L–space knots have distinct Alexander polynomials, but the gap functions share the same convex hull, then their Upsilon invariants coincide.
  - No duplication among torus knots.
  - Find among hyperbolic L–space knots.

- Since the gap function is not convex, another Legendre–Fenchel transformation of the Upsilon invariant does not return the gap function.
  So, we cannot restore the Alexander polynomial, in general.
  - However, find examples for which the Alexander polynomial can be restored from the Upsilon invariant.

Theorem 1

There exist infinitely many pairs of hyperbolic L–space knots, which have distinct Alexander polynomials, but share the same Upsilon invariant.

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Concretely, consider the closures of 4-braids ($n \geq 1$).

$$K_1 : (\sigma_2\sigma_1\sigma_3\sigma_2)(\sigma_1\sigma_2\sigma_3)^{4n}\sigma_2^{-1}(\sigma_2\sigma_3)^6,$$

$$K_2 : (\sigma_2\sigma_1\sigma_3\sigma_2)(\sigma_1\sigma_2\sigma_3)^{4n}\sigma_3^{-1}(\sigma_2\sigma_3)^6.$$

Perform $(-1/n)$–surgery on C_1 and $(-1/2)$–surgery on C_2.

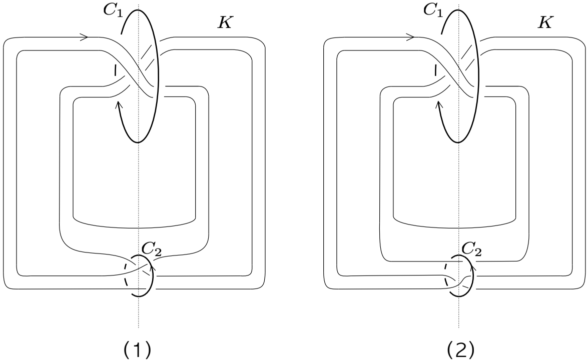

Figure: K_1 and K_2

Result 2

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Let K be hyperbolic L–space knot t09847 or v2871.
Then the Alexander polynomial is restored from the Upsilon invariant.
That is, if an L–space knot K' satisfies $\Upsilon_K(t) = \Upsilon_{K'}(t)$, then $\Delta_K(t) \doteq \Delta_{K'}(t)$.

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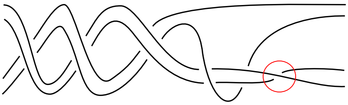

Outline of Result 1

Proposition

For $n \geq 1$, K_1 and K_2 satisfy the following:

- hyperbolic.
- $(16n+21)$-surgery on K_1, $(16n+20)$-surgery on K_2 yield L–spaces. [Montesinos trick]
- Alexander polynomials are distinct. [Torres formula]
- Upsilon invariants coincide.

$n = 1$

$K_1 = m240$, $K_2 = t10496$ in SnapPy's census

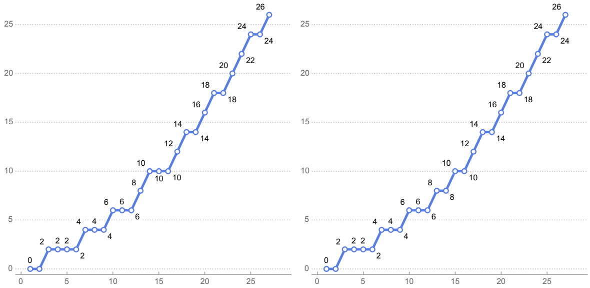

Figure: Gap functions of K_1 and K_2

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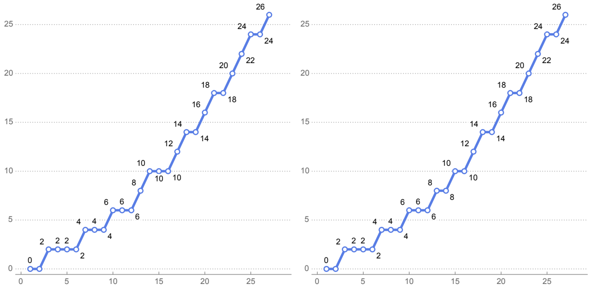

Figure: Gap functions of K_1 and K_2

These share the same convex hull $\Longrightarrow \Upsilon_{K_1}(t) = \Upsilon_{K_2}(t)$

ALEXANDER POLYNOMIAL

$$\Delta_{K_1}(t) = \sum_{i=0}^{n}(t^{8n+12+4i} - t^{8n+11+4i}) + (t^{8n+9} - t^{8n+8})$$
$$+ \sum_{i=0}^{n}(t^{4n+6+4i} - t^{4n+4+4i}) + (t^{4n+3} - t^{4n+1})$$
$$+ \sum_{i=0}^{n-1}(t^{4+4i} - t^{1+4i}) + 1.$$

$$\Delta_{K_2}(t) = \sum_{i=0}^{n}(t^{8n+12+4i} - t^{8n+11+4i}) + (t^{8n+9} - t^{8n+8})$$
$$+ \sum_{i=0}^{2n-1}(t^{4n+8+2i} - t^{4n+7+2i}) + (t^{4n+6} - t^{4n+4})$$
$$+ (t^{4n+3} - t^{4n+1}) + \sum_{i=0}^{n-1}(t^{4+4i} - t^{1+4i}) + 1.$$

Formal semigroup

For an L–space knot K,

$$\Delta_K(t) = 1 - t^{a_1} + t^{a_2} + \cdots - t^{a_{k-1}} + t^{a_k},$$

where $1 = a_1 < a_2 < \cdots < a_k = 2g(K)$.

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Expand $\Delta_K(t)/(1-t)$ as a formal power series:

$$\frac{\Delta_K(t)}{1-t} = \sum_{s \in \mathcal{S}_K} t^s.$$

The set $\mathcal{S}_K$ is a subset of non-negative integers, called the formal semigroup of K.

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For example, for $T(p,q)$,

$$\mathcal{S} = \langle p, q \rangle = \{ap + bq \mid a, b \geq 0\}$$

is a semigroup.

K_1 $(n = 1)$

$$\begin{aligned}\Delta_{K_1}(t) &= 1 - t + t^4 - t^5 + t^7 - t^8 + t^{10} - t^{12} + t^{14} - t^{16} + t^{17} - t^{19} + t^{20} - t^{23} + t^{24} \\ &= (1-t) + t^4(1-t) + t^7(1-t) + t^{10}(1-t^2) + t^{14}(1-t^2) + t^{17}(1-t^2) \\ &\quad + t^{20}(1-t^3) + t^{24}\end{aligned}$$

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- $\mathcal{S} = \{0, 4, 7, 10, 11, 14, 15, 17, 18, 20, 21, 22\} \cup \{24, 25, 26, \dots\}$
- $4 \in \mathcal{S}$, but $8 \notin \mathcal{S} \implies \mathcal{S}$ is not a semigroup.

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For K_1 and K_2,

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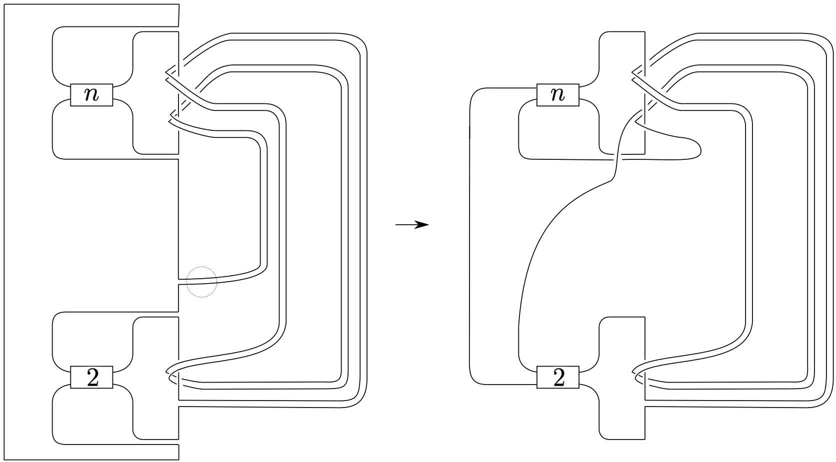
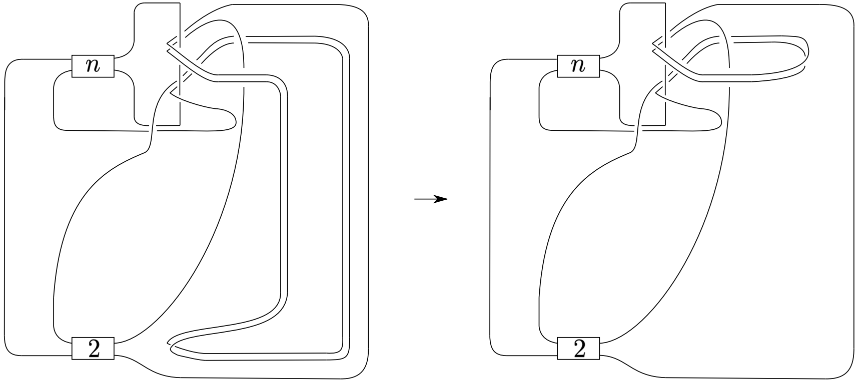
- Assume that K_i is satellite for a contradiction.
  - The bridge number is 4, so its companion is 2-bridge and the pattern knot has wrapping number 2.
  - We know that the companion and pattern are L–space knots.
  - In addition, the pattern is braided by Baker–Motegi.
  - Hence, the companion is a 2-bridge torus knot, and K_i is its 2-cable.
  - However, the formal semigroup of an iterated torus L–space knot is a semigroup by Shida Wang.

$(16n+21)$-surgery yields a Seifert fibered L–space.



MONTESINOS TRICK FOR K_1



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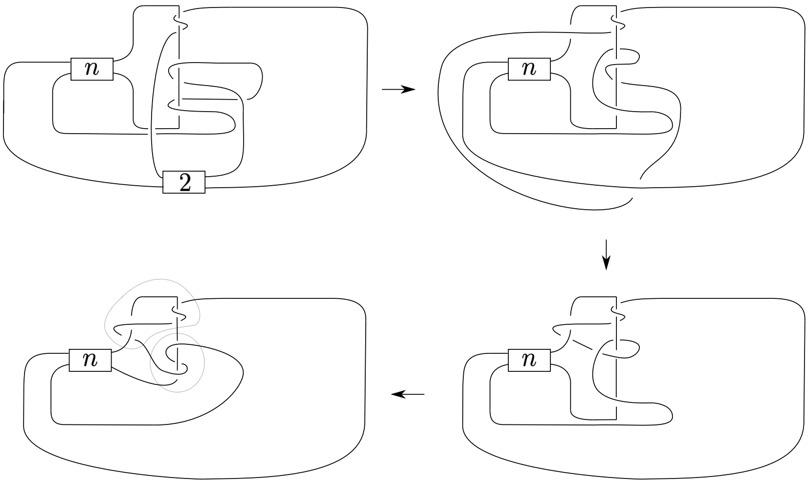
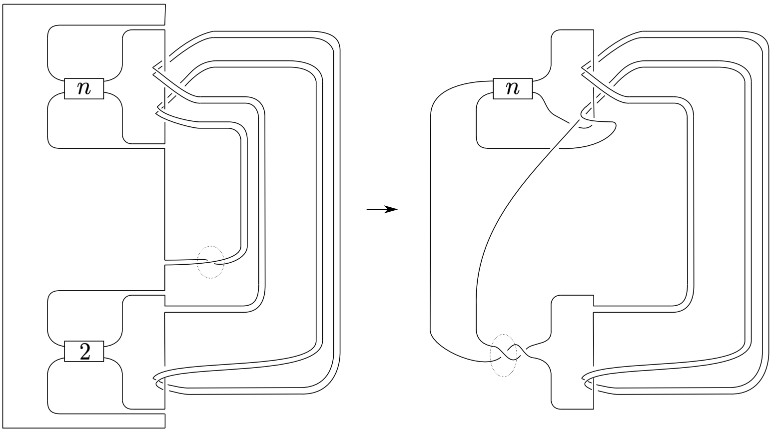


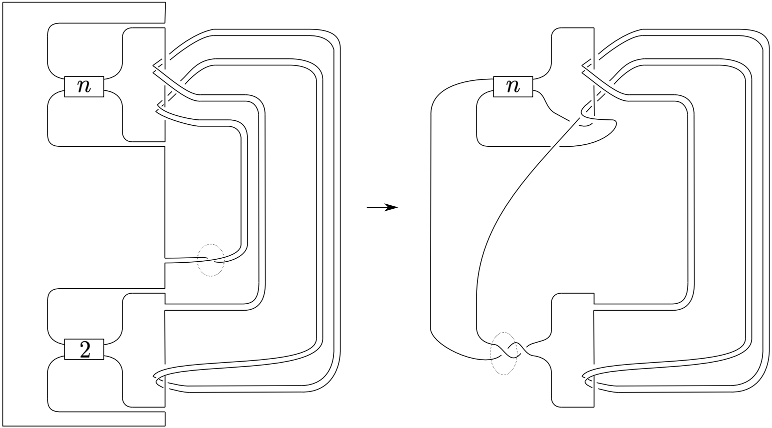
The last is the Montesinos knot $M(-3/7, -1/3, -1/n)$.
By the criterion, its double cover is an L–space.

$(16n + 20)$-surgery yields an L–space.



Montesinos trick for K_2

$(16n+20)$-surgery yields an L–space.



Perform two resolutions to obtain ℓ_∞ and ℓ_0.

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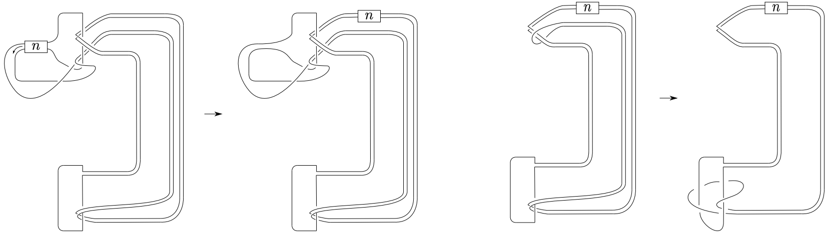
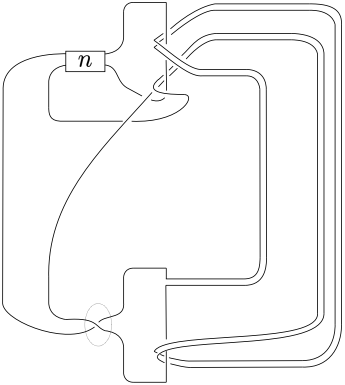


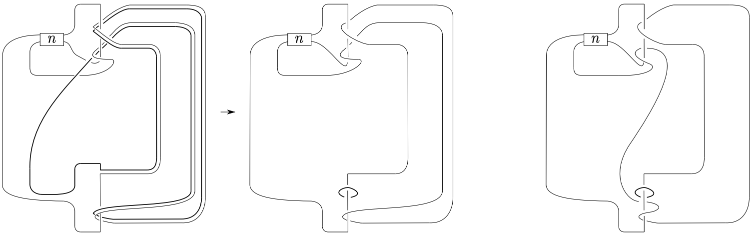
The last is the $(-3, 3, n-1)$–pretzel knot, whose double cover is an L–space.

For ℓ_0,



Perform two further resolutions to obtain $\ell_{0\infty}$ ($= \ell_\infty$) and ℓ_{00}.

MONTESINOS TRICK FOR K_2



ℓ_{00} is the connected sum of the Hopf link and the Montesinos knot $M(1/2, -1/3, n/(2n+1))$.
Then its double cover is an L–space.

Outline of Result 2

Result 2

Theorem 2

Let K be hyperbolic L–space knot t09847 or v2871.
Then the Alexander polynomial is restored from the Upsilon invariant.
That is, if an L–space knot K' satisfies $\Upsilon_K(t) = \Upsilon_{K'}(t)$, then $\Delta_K(t) \doteq \Delta_{K'}(t)$.

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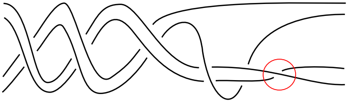

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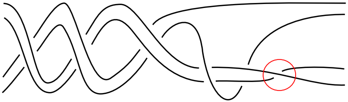


- Are there infinitely many such hyperbolic L–space knots?

$T(3,4)$	$\Delta(t) = 1 - t + t^3 - t^5 + t^6$
$T(3,5)$	$\Delta(t) = 1 - t + t^3 - t^4 + t^5 - t^7 + t^8$

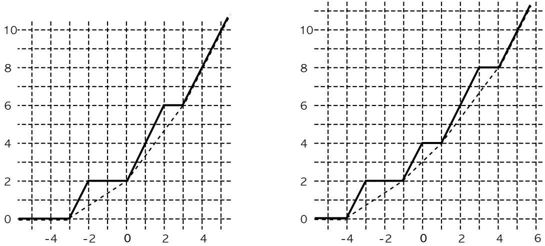


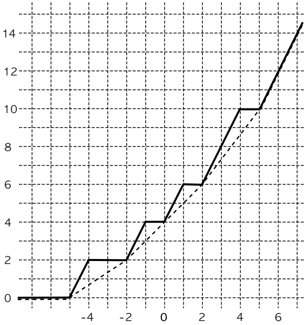
Figure: Gap functions of $T(3,4)$ and $T(3,5)$

- Each segment of the graph has slope 0 or 2.
- These $\Delta(t)$ are restorable from $\Upsilon(t)$.

Bad example

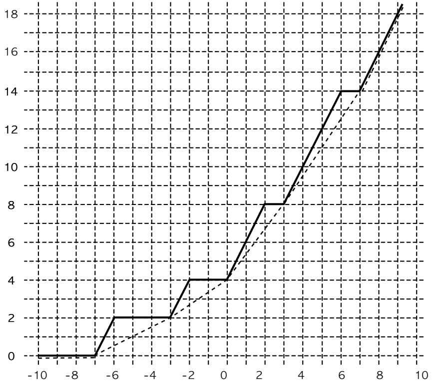
K: $(-2, 3, 7)$-pretzel knot

$$\Delta_K(t) = 1 - t + t^3 - t^4 + t^5 - t^6 + t^7 - t^9 + t^{10}$$



$\Delta(t) = 1 - t + t^3 - t^5 + t^7 - t^9 + t^{10}$ shares the same convex hull.

$$\Delta_K(t) = 1 - t + t^4 - t^5 + t^7 - t^9 + t^{10} - t^{13} + t^{14}$$



Proposition

Let $m \geq 3$ be an integer, and let $\Delta(t) = 1 - t + t^m - t^{m+1} + t^{m+2} - t^{2m+1} + t^{2m+2}$. Then its gap function, defined formally, is restorable from the convex hull.

Proposition

Let $m \geq 3$ be an integer, and let $\Delta(t) = 1 - t + t^m - t^{m+1} + t^{m+2} - t^{2m+1} + t^{2m+2}$. Then its gap function, defined formally, is restorable from the convex hull.

- There exists a hyperbolic knot whose Alexander polynomial is $\Delta(t)$.
- Δ satisfies Krcatovich's condition.
- If $m = 3$, then $\Delta(t)$ is the Alexander polynomial of $T(3, 5)$.
- Similar polynomials can be given more.

Secondary Upsilon invariant and Result 3

Knot Floer complex of L–space knot

For $K = T(3,7)$,

$$\Delta_K(t) = 1 - t + t^3 - t^4 + t^6 - t^8 + t^9 - t^{11} + t^{12}.$$

Hence, the staircase diagram $\mathrm{St}(K)$ of $\mathrm{CFK}^\infty(K)$ is specified by

$$[1, 2, 1, 2, 2, 1, 2, 1]$$

on the $(\mathrm{alg}, \mathrm{Alex})$-plane.
Black vertices have Maslov grading 0, but white ones have 1.

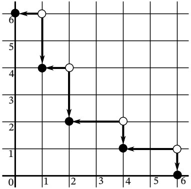
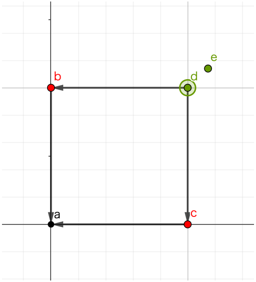
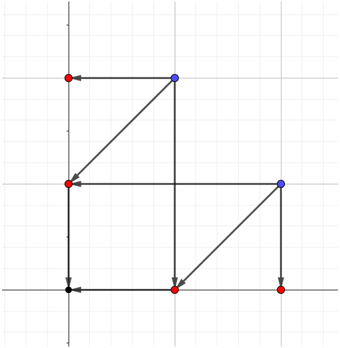


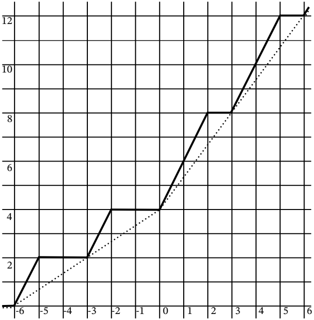
figure eight knot

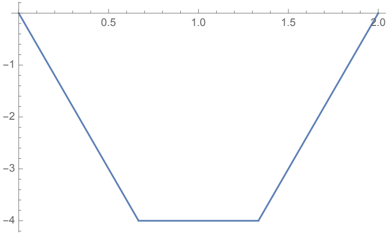


$(2, 1)$-cable of $T(2, 3)$



$T(3,7)$: Gap function and Upsilon





- $\Upsilon'(t)$ is singular at $t = 2/3$ and $4/3$.

Secondary Upsilon invariant

Secondary Upsilon invariant is defined at each singularity t of $\Upsilon'(t)$.

- Put $t = 2/3$, and let L_t be the line with slope $1 - 2/t = -2$ touching $\mathrm{St}(K)$ from south-west. [Support line]
- L_t meets $\mathrm{St}(K)$ in at least 2 points.
- The top most is $p_t^- = (0, 6)$, and the bottom most is $p_t^+ = (2, 2)$. [pivot points]

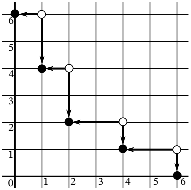


Figure: The staircase diagram $\mathrm{St}(K)$ of $T(3, 7)$.

Secondary Upsilon invariant

Consider the part of $\mathrm{St}(K)$ between two pivot points.

- For $s \in [0, 2]$, let L_s be the line with slope $1 - 2/s$ touching it from north-east. (L_0 is vertical.)
- Let ξ be the intercept of L_s when $s \neq 0$.
- $\Upsilon^2_{K,t}(s) = -s\xi - \Upsilon_K(t)$ $(s \neq 0)$, $-2\,\mathrm{alg}(p_t^+) - \Upsilon_K(t)$ $(s = 0)$.

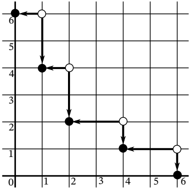


Figure: Pivot points $p_t^- = (0, 6)$, $p_t^+ = (2, 2)$ for $t = 2/3$

$$\Upsilon_K(2/3) = -4$$

$$\Upsilon^2_{K,2/3}(s) = \begin{cases} -2s & (0 \leq s \leq 2/3) \\ -5s + 2 & (2/3 \leq s \leq 2) \end{cases}$$

RESULT 3

- Define $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi(x, y) = (x - y, 2x)$.
- For $p \in \mathbb{R}^2$, $\Phi_1(p)$ denotes the first coordinate of $\Phi(p)$.

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Result 3

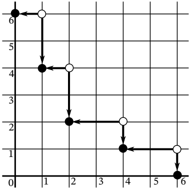
Let K be an L–space knot. Let $t_0 \in (0, 2)$ be a singularity of $\Upsilon'_K(t)$, and let $p^\pm$ be the corresponding pivot points on $\mathrm{St}(K)$. Then

$$\Upsilon^2_{K,t_0}(s) = G^*(s) - \Upsilon_K(t_0),$$

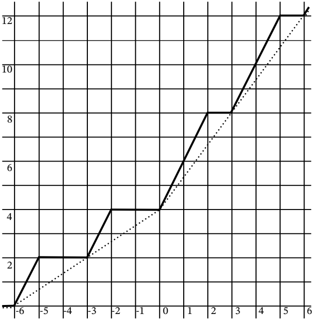
where $G^*(s)$ is the concave conjugate of the gap function $G(x)$ restricted on $[\Phi_1(p^-), \Phi_1(p^+)]$.

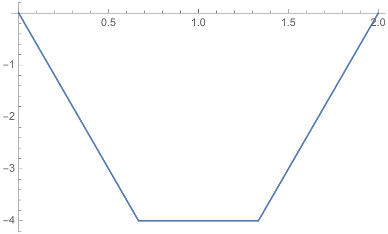
* $G^*(s) = \min_{x \in I}(sx - G(x))$.

$T(3,7)$, $t = 2/3$



$$\Upsilon^2_{K,2/3}(s) = \begin{cases} -2s & (0 \leq s \leq 2/3) \\ -5s + 2 & (2/3 \leq s \leq 2) \end{cases}$$





$$\Upsilon_K(2/3) = -4$$