On complexified tetrahedron for double twist knot

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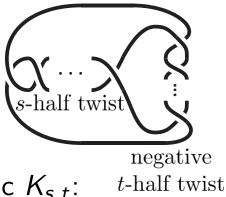
May 23, 2024

Outline

- Double twist knot $K_{s,t}$:



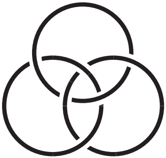
- Volume conjecture for hyperbolic $K_{s,t}$:

Theorem. $2\pi \lim_{N\to\infty} \frac{\log\left(V_{N-1}(K_{s,t})\right)}{N} = \mathrm{Vol}(K_{s,t}) + i\,\mathrm{CS}(K_{s,t}).$

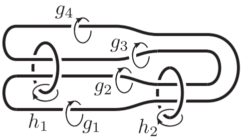
- **Strategy of proof:**
  1. $\mathrm{SL}(2,\mathbb{C})$ repr. of $\pi_1(S^3 \setminus K_{s,t}) \to$ **complexified tetrahedron**
  2. $V_{N-1}(K_{s,t}) = \mathrm{ADO}_N(K_{s,t}^{\frac{N-1}{2}}) \leftarrow$ quantum $6j$ symbol
  3. The **saddle point** of the potential function corresponds to the **eigenvalues** of some representation matrices (mer. & long.)
  4. Compare with Neumann-Zagier-Yoshida func. $\to$ **Volume**

  ★ **Technical key: Big cancellation**
  (vanishing of the largest term of the Poisson sum)
  proved through **l'Hopital's rule** and **integral by part**.

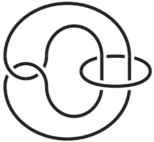
Borromean rings to double twist knots



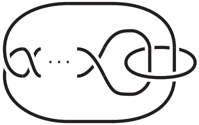
Borromean rings B



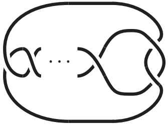
Generators of $\pi_1(S^3 \setminus B)$



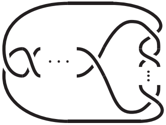
Whitehead link W



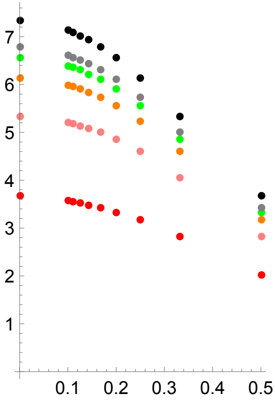
Twisted Whitehead link W_s



Twist knot K_s



Double twist knot $K_{s,t}$

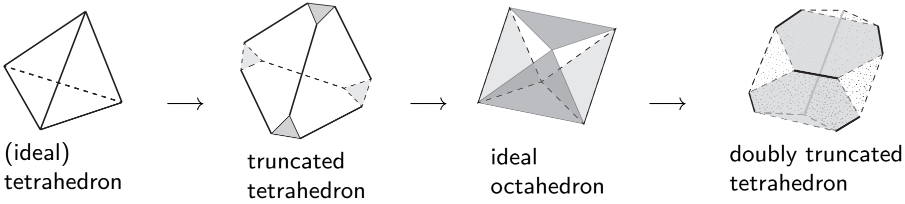


Volumes of $K_{s,t}$

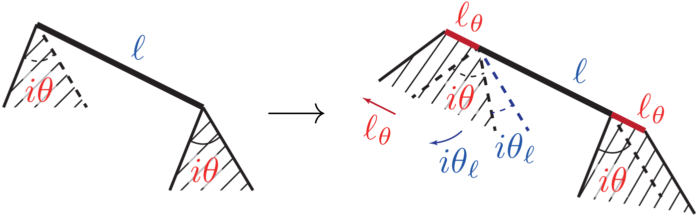
Complexified tetrahedron

Complexify the parameter at edges (angles and lengths) of a generalized tetrahedron.

- **Generalized tetrahedron**

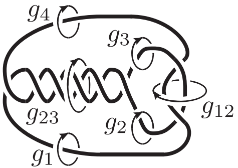


- **Complexification:** $i\,\theta \to \ell_\theta + i\,\theta$, $\ell \to \ell + i\,\theta_\ell$.



Double twist knot $K_{6,2}$

$K_{6,2}$ $(= 8_1)$:



$g_{12} = g_1 g_2$
$g_{23} = g_2 g_3$

Assign elements of the fundamental group

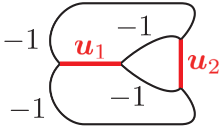
$\rho : \pi_1(S^3 \setminus K_{4,3}) \to \mathrm{SL}(2, \mathbb{C})$: the geometric representation.
$\rho(g_1), \cdots, \rho(g_4)$: parabolic elements with eigenvalues $\mathbf{-1}$.

Relations: $g_1 = g_{23}^3 g_2 g_{23}^{-3}$, $g_4 = g_{23}^3 g_3 g_{23}^{-3}$, $g_3 = g_{12} g_2 g_{12}^{-1}$, $g_4 = g_{12} g_1 g_{12}^{-1}$.

Let u_1, u_1^{-1} : the eigenvalues of $\rho(g_{23})$, u_2, u_2^{-1} : those of $\rho(g_{12})$.

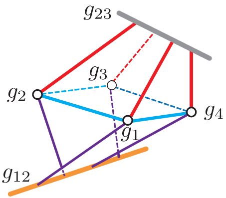
Then we have $u_1 = -0.6193 - 0.8845i$, $u_2 = 1.7256 + 2.0605i$.

Construct the **complexified tetrahedron** from these eigenvalues u_1, u_2.

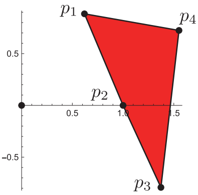


Complexified tetrahedron for $K_{6,2}$

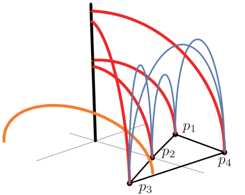
ρ : the geometric representation of π_1 with diagonal $\rho(g_{23})$.
Let $p_1, \cdots, p_4$ be the fixed (ideal) points of $\rho(g_1), \cdots, \rho(g_4)$.
Deform the regular ideal octahedron $\leftrightarrow S^3 \setminus B$.



deformed octahedron

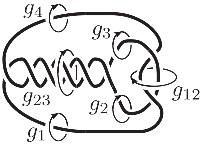


at infinity



in the upper half space

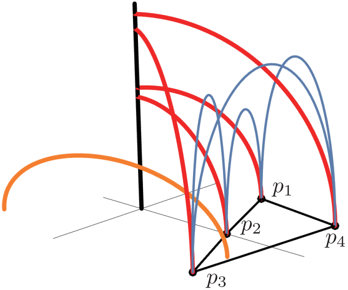
Thick black line: The axis of $\rho(g_{23})$ from 0 to ∞ corresponds to the edge parametrized by u_1.
Thick orange line: The axis of $\rho(g_{12})$ corresponds to the edge parametrized by u_2.

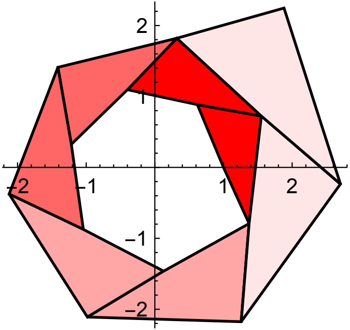


The points p_1, p_2, p_3, p_4 correspond edges parametrized by -1.

Action of g_{23}

$\mathbb{H}^3$:





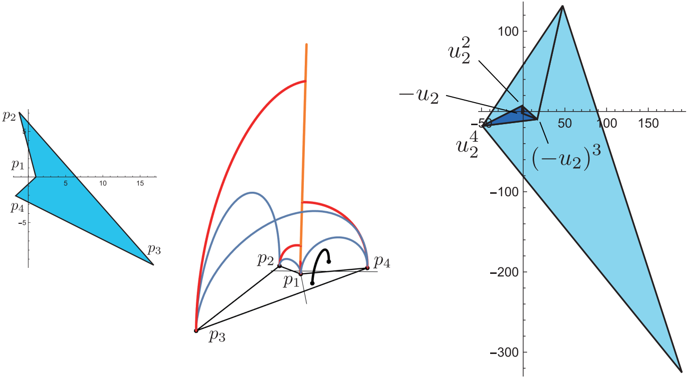
$p_2 = 1$, $p_1 = -u_1$, $p_3 =$ square of the eigenvalue of h_1,

$\rho(g_{23}) = \begin{pmatrix} u_1 & 0 \\ 0 & u_1^{-1} \end{pmatrix} \curvearrowright \mathbb{H}^3$, $\quad z \mapsto u_1^2 z$ on $\mathbb{C} = \partial\mathbb{H}^3$.

$\rho(g_{23})$ acts as a stretch and rotation along the thick vertical line.

Action of g_{12}

ρ' : geometric representation of π_1 with diagonal $\rho'(g_{12})$.



$p_1 = 1$, $p_4 = -u_1$, $p_2 =$ square of the eigenvalue of h_2,

$\rho'(g_{12}) = \begin{pmatrix} u_2 & 0 \\ 0 & u_2^{-1} \end{pmatrix} \curvearrowright \mathbb{H}^3, \quad z \mapsto u_2^2 z$ on $\mathbb{C} = \partial\mathbb{H}^3$.

$\rho'(g_{12})$ acts as a stretch and rotation along the orange vertical line.

Volume Conjecture

- K : knot or link in S^3,
- $V_{N-1}(K)$: colored Jones inv. for $N-1$ parallel at $q = e^{\pi i/N}$,
- $\mathrm{Vol}(K)$: hyperbolic volume of $S^3 \setminus K$,
- $\mathrm{CS}(K)$: Chern-Simons inv. of $S^3 \setminus K$.

Conjecture. The following holds.

$$2\pi \lim_{N\to\infty} \frac{\log\left(V_{N-1}(K)\right)}{N} = \mathrm{Vol}(K) + i\,\mathrm{CS}(K).$$

Proof for knots with a few crossings: (established by T. Ohtsuki)

1. Replace quantum factorials by quantum dilogarithm functions.
2. Convert sum to integral by using Poisson sum formula.
3. Apply the saddle point method (have to check the **condition**).

T. Ohtsuki, *On the asymptotic expansion of the Kashaev invariant of the* 5_2 *knot,* Quantum Topol. **7** (2016), no. 4, 669–735.
and two more papers for prime knots up to seven crossings.

ADO (Akutsu-Deguchi-Ohtsuki) invariant

The ADO invariant is an nvariant for oriented trivalent knotted graph. In the rest, we use the ADO invariant since

$$\boxed{V_{N-1}(K) = \mathrm{ADO}_N(K^{\frac{N-1}{2}}).}$$

Notations:

$$N \in \mathbb{N}, \qquad q^a = \exp\frac{\pi i a}{N} \quad (a \in \mathbb{C}),$$

$$\{a\} = q^a - q^{-a}, \qquad \{n\}! = \{n\}\{n-1\}\dots\{1\},$$

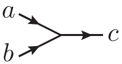
$$\{a,k\} = \{a\}\{a-1\}\dots\{a-k+1\},$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\{a, a-b\}}{\{a-b\}!} \quad (a - b \in \{0, 1, \dots, N-1\}),$$

$$t_a = a(a+1-N) = (a - \tfrac{N-1}{2})^2 - \tfrac{(N-1)^2}{4}.$$

Color of edges: $(\mathbb{C} \setminus (\mathbb{Z}/2)) \cup N(\mathbb{Z}/2)$.

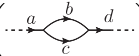
Admissibility at vertex: $a + b - c = 0, 1, 2, \cdots, N-1$.

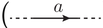


Y. Akutsu, T. Deguchi, T. Ohtsuki, *Invariant of colored links,* Journal of Knot Theory and its Ramifications **1** (1992), 161–184.

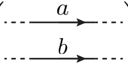
F. Costantino, J. M. *On* $\mathrm{SL}(2,\mathbb{C})$ *quantum 6j-symbol and its relation to the hyperbolic volume,* Quantum Topology **4** (2013), 303–351.

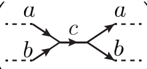
Properties of the ADO invariant

ADO_N (

) $= \delta_{ad} \begin{bmatrix} 2a+N \\ 2a+1 \end{bmatrix}$ ADO_N (

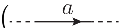
) ,

ADO_N (

) $= \sum_{a+b-c=0,1,\cdots,N-1} \begin{bmatrix} 2c+N \\ 2c+1 \end{bmatrix}^{-1}$ ADO_N (

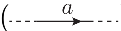
) ,

ADO_N (

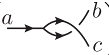
) $= q^{2t_a}$ ADO_N (

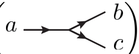
) ,

ADO_N (

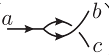
) $= q^{-2t_a}$ ADO_N (

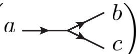
) ,

ADO_N (

) $= q^{t_c-t_a-t_b}$ ADO_N (

) ,

ADO_N (

) $= q^{-(t_c-t_a-t_b)}$ ADO_N (

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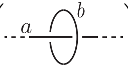
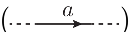
ADO_N (

) $= i^{N-1} \{2a+N, N-1\} \, q^{(2a+1-N)(2b+1-N)}$ ADO_N (

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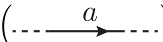
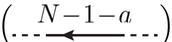
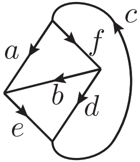
ADO_N (

) $=$ ADO_N (

) (dual representation).

Quantum $6j$ symbol for ADO invariant

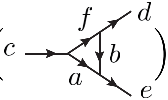
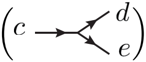


Let $A_{xyz} = x + y + z, \quad B_{xyz} = x + y - z.$

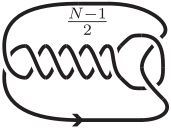
The $6j$ symbol which is the ADO invariant of the above tetrahedral graph is the following.

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q = (-1)^{N-1} \frac{\{B_{dec}\}!\{B_{abe}\}!}{\{B_{bdf}\}!\{B_{afc}\}!} \begin{bmatrix} 2e \\ A_{abe}+1-N \end{bmatrix} \begin{bmatrix} 2e \\ B_{ced} \end{bmatrix}^{-1} \times$$

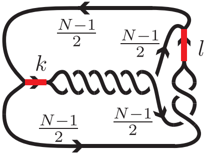
$$\sum_{s=\max(0,-B_{bdf}+B_{dec})}^{\min(B_{dec},B_{afc})} \begin{bmatrix} A_{acf}+1-N \\ 2c+s+1-N \end{bmatrix} \begin{bmatrix} B_{acf}+s \\ B_{acf} \end{bmatrix} \begin{bmatrix} B_{bfd}+B_{dec}-s \\ B_{bfd} \end{bmatrix} \begin{bmatrix} B_{cde}+s \\ B_{dfb} \end{bmatrix}.$$

- $\mathrm{ADO}_N\Bigg($  $\Bigg) = \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q \mathrm{ADO}_N\Bigg($  $\Bigg)$

ADO invariant of double twist knots



$$= \sum_{k,l=0}^{N-1} \begin{bmatrix} 2k+N \\ 2k+1 \end{bmatrix}^{-1} \begin{bmatrix} 2l+N \\ 2l+1 \end{bmatrix}^{-1}$$



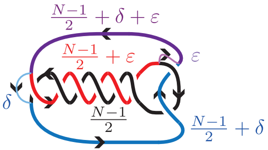
$$\mathrm{ADO}_N(K_{6,2}^{\frac{N-1}{2}}) =$$

$$q^{-(N-1)^2} N^2 \sum_{k,l=0}^{N-1} \frac{\{2k+1\}\{2l+1\} q^{6(k-\frac{N-1}{2})^2 - 2(l-\frac{N-1}{2})^2}}{\{2l+N, N\}\{2k+N, N\}} \begin{Bmatrix} \frac{N-1}{2} & \frac{N-1}{2} & l \\ \frac{N-1}{2} & \frac{N-1}{2} & k \end{Bmatrix}_q .$$

Caution! This is $0/0$. **How to fix it?** Ans. Use *L'Hopital*

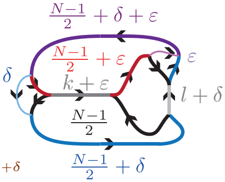
1. **For the colored Jones invariant:**
   Perturb the parameter q.
   The differential with respect to q is complicated.
2. **For the ADO invariant:**
   Perturb the colors $\frac{N-1}{2}$, k, l as follows.

L'Hopital's rule



$$=$$

$$\sum_{k,l=0}^{N-1} \begin{bmatrix}2k+2\varepsilon+N\\2k+2\varepsilon+1\end{bmatrix}^{-1}\begin{bmatrix}2l+2\delta+N\\2l+2\delta+1\end{bmatrix}^{-1} q^{6t_{k+\varepsilon}-6t_{\frac{N-1}{2}+\varepsilon}-2t_{l+\delta}+2t_{\frac{N-1}{2}+\delta}}$$



$$= N^2 \sum_{k,l=0}^{N-1} \frac{\{2k+2\varepsilon+1\}\{2l+2\delta+1\}q^{6t_{k+\varepsilon}-6t_{\frac{N-1}{2}+\varepsilon}-2t_{l+\delta}+2t_{\frac{N-1}{2}+\delta}}}{\{2k+2\varepsilon+N,N\}\{2l+2\delta+N,N\}} \times$$

$$\begin{Bmatrix}\frac{N-1}{2}+\delta & \frac{N-1}{2} & l+\delta\\ \frac{N-1}{2}+\varepsilon & \frac{N-1}{2}+\varepsilon+\delta & k+\varepsilon\end{Bmatrix}_q .$$

Use l'Hopital's rule to get the limit $\lim_{\varepsilon,\delta\to 0}$ of the above.

Key Lemma. $\frac{\partial^2}{\partial\varepsilon\partial\delta}$ of tne numerator of the above is equal to

$$N^2\sum_{k,l=0}^{N-1}\frac{\partial^2}{\partial\alpha\partial\beta}\{2\alpha+1\}\{2\beta+1\}q^{6t_\alpha-2t_\beta}\begin{Bmatrix}\frac{N-1}{2} & \frac{N-1}{2} & \beta\\ \frac{N-1}{2} & \frac{N-1}{2} & \alpha\end{Bmatrix}_q\Bigg|_{\alpha=k,\beta=l} .$$

- $\frac{\partial^2}{\partial\varepsilon\partial\delta}$ of the denominator is $8(-1)^{k+l+N}\pi^2$.

Integral by part

- **Continuation:** There is an analytic functions $\Phi(x,y)$ such that

$$e^{\frac{N}{2\pi i}\Phi_N(2\pi\frac{2k+1}{2N},2\pi\frac{2l+1}{2N})} = \begin{Bmatrix} \frac{N-1}{2} & \frac{N-1}{2} & l \\ \frac{N-1}{2} & \frac{N-1}{2} & k \end{Bmatrix}_q, \quad \Phi_N(x,y) \underset{N\to\infty}{\longrightarrow} \Phi(x,y).$$

- **Poisson sum formula:** $\sum_{k\in\mathbb{Z}} f(k) = \sum_{m\in\mathbb{Z}} \widehat{f}(m), \quad \widehat{f}(x) = \int_{\mathbb{R}} e^{-2\pi ixy} f(y)\,dy.$

- **Integral by part:** $\Psi_{\varepsilon_1,\varepsilon_2}(2\pi\frac{2\alpha+1}{2N}, 2\pi\frac{2\beta+1}{N}\beta) =$
$e^{\varepsilon_1\frac{2\alpha+1}{2N}\pi i+\varepsilon_2\frac{2\beta+1}{2N}\pi i+\frac{N}{2\pi i}\left(6(\frac{2\alpha+1-N}{2N}\pi i)^2-2(\frac{2\beta+1-N}{2N}\pi i)^2+\Phi_N(2\pi\frac{2\alpha+1}{2N},2\pi\frac{2\beta+1}{2N})\right)}$,

$$\sum_{\varepsilon_1,\varepsilon_2=\pm1}\sum_{k,l=0}^{N-1}\varepsilon_1\varepsilon_2\frac{\partial^2}{\partial\alpha\partial\beta}\ \Psi_{\varepsilon_1,\varepsilon_2}(2\pi\frac{2\alpha+1}{2N}, 2\pi\frac{2\beta+1}{N})\Big|_{\substack{\alpha\,=\,k\\ \beta\,=\,l}}$$

$$\sim \sum_{\varepsilon_1,\varepsilon_2=\pm1}\varepsilon_1\varepsilon_2\iint_{[0,2\pi]^2} e^{-Ni\left(m_1(x-\frac{\pi}{N})+m_2(y-\frac{\pi}{N})\right)}\frac{\partial^2}{\partial x\partial y}\Psi_{\varepsilon_1,\varepsilon_2}(x,y)\,dxdy$$

$$\underset{\text{integral by part}}{\sim} m_1 m_2 \iint_{[0,2\pi]^2} e^{\frac{N}{2\pi i}\left(2\pi(m_1x+m_2y)-3(x-\pi)^2+(y-\pi)^2+\Phi(x,y)\right)}\,dxdy$$

- **Big cancellation:** Terms with $m_1 = 0$ or $m_2 = 0$ vanish.

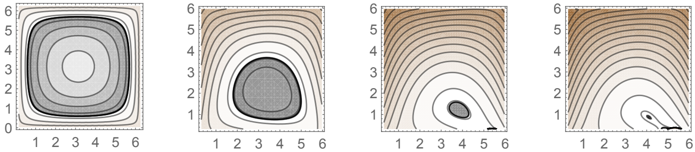
Saddle point method

Values at the saddle points:

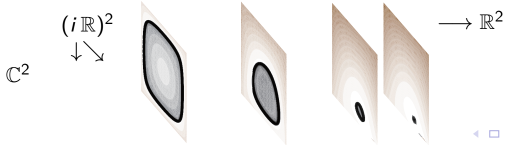
$m_1 = 0,\ m_2 = 0 :\ 7.3277\ldots\ (\mathrm{Vol}(S^3 \setminus B)),$
$m_1 = 1,\ m_2 = 0 :\ 6.7847\ldots\ (\mathrm{Vol}(S^3 \setminus W_6)),$ vanish by big cancellation
$m_1 = 0,\ m_2 = 1 :\ 3.6638\ldots\ (\mathrm{Vol}(S^3 \setminus W_2)),$

$m_1 = 1,\ m_2 = 1 :\ 3.4272\ldots\ (\mathrm{Vol}(S^3 \setminus K_{6,2})?).$ Largest surviving value.

Check the condition for the saddle point method.
Push the integral region along real axes toward the saddle point.



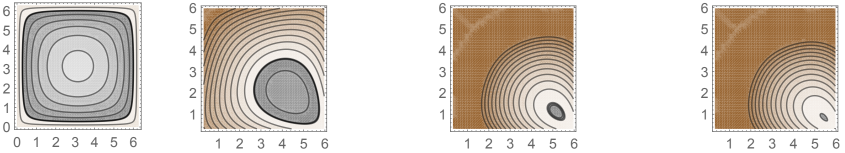
$x = 0 + i\alpha$, $y = 0 + i\beta$ | $x = 0.04 + i\alpha$, $y = 0.5 + i\beta$ | $x = 0.07 + i\alpha$, $y = 0.9 + i\beta$ | $x = 0.076 + i\alpha$, $y = 0.988 + i\beta$



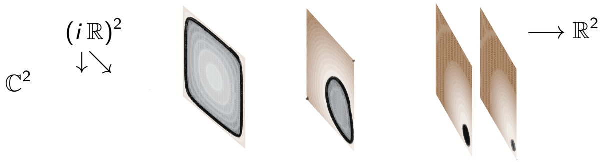
Most extremal case $K_{2,2}$ $(= 4_1)$

$m_1 = 1,\ m_2 = 1\ :\ 2.0298...$ $(\mathrm{Vol}(S^3 \setminus K_{2,2})?)$. Largest surviving value.

Check the condition by pushing the integral region.



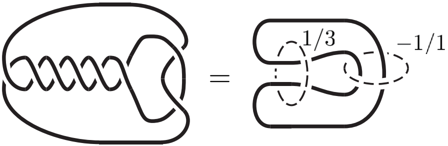
$x = 0 + i\alpha$, $y = 0 + i\beta$ | $x = 0.2 + 0.1(\alpha - 0.861) + i\alpha$, $y = 0.2 - 0.1(\beta - 5.421) + i\beta$ | $x = 0.4 + 0.3(\alpha - 0.861) + i\alpha$, $y = 0.4 - 0.3(\beta - 5.421) + i\beta$ | $x = 0.5435 + 0.3(\alpha - 0.861) + i\alpha$, $y = 0.5435 - 0.3(\beta - 5.421) + i\beta$



For general $K_{s,t}$: Deform s and t continuously, then we can deform the potential function and the integral region continuously.

Neumann-Zagier function

$K_{6,2}$ is obtained from the borromean rings with $1/3$, $-1/1$ surgeries.



$\Phi(x, y)$: potential function of the borromean rings

$x,\ y \leftrightarrow$ log of e.v. $-\lambda_1,\ -\lambda_2$ of longitudes of the surgery comp.

$\frac{\partial}{\partial x}\Phi(x, y)$, $\frac{\partial}{\partial y}\Phi(x, y)$ $\leftrightarrow 2 \times$ log of e.v. $\mu_1,\ \mu_2$ of meridians of the surgery comp.

Then we have
$$\frac{\partial}{\partial \mu_i}\big(\lambda_1\mu_1 + \lambda_2\mu_2 + \Phi(\lambda_1, \lambda_2)\big) = \lambda_i.$$

So $\lambda_1\mu_1+\lambda_2\mu_2+\Phi(\lambda_1, \lambda_2)$ coincides with the **Neumann-Zagier-Yoshita** function up to a constant. For $\lambda_1 = \lambda_2 = -1$, the value is equal to the volume of the borromean rings and so the constant is equal to 0. By the surgery, we have $6\lambda_1 - 2\mu_1 = 2\pi$ and $2\lambda_1 + 2\mu_2 = 2\pi$, which correspond to the saddle point equation. Therefore, the value at the saddle point coincieds with the complex volume.

Volume Conjecture for double twist knots (summary)

- **Representation**

$$\begin{array}{cccccc} & B & \longrightarrow & \text{ideal regular octahedron} & & \\ \text{surgery} & \downarrow & & \downarrow & & \text{deformation} \\ & K_{s,t} & \longrightarrow & \textbf{complexified tetrahedron} & & \end{array}$$

The meridians and longitudes for the surgery comes from SL_2 repr.

- **Invariant**

$$\begin{array}{ccc} V_N(K_{s,t}) & \longrightarrow & \mathrm{ADO}_N(K_{s,t}) \\ & & \downarrow \text{ deform} \\ N \to \infty \downarrow & & \mathrm{ADO}_N(K_{s,t})^{\varepsilon,\delta} \\ & & \downarrow \text{ l'Hopital} \\ \text{value at the saddle point} & \longleftarrow & \textbf{big cancellation} \quad \text{Poisson sum} \end{array}$$

The parameters of the potential function correspond to the longitudes of the surgery components. Differentials by these parameters correspond to the meridians of the surgery components.

- **Neumann-Zagier-Yoshida** function

Value at the saddle point of our potential function matches the complex volume since our function coincides with the Nouman-Zagier-Yoshida function.

Problems

1. Realize the faces of the complexified tetrahedron.
   For example, the minimal surface spanning the edges of the boundary.

2. Generalize the proof to two bridge knots and links.

3. Generalize the proof to the surgeries of fully augmented links.
   Hopf links are connected summed at some edges of a planar graph.

4. Generalize the proof to all knots and links.
   Use the face model for the colored Jones and for the ADO invariant.
   The parameter at the face may be the eigenvalue of the matrix of the element of π_1 assigned to the face.