# The orbit classification of $\mathbb{Z}^m$ by the $m$-braid group

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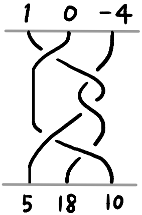
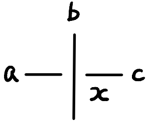


# Fox $\mathbb{Z}$-colorings for knots and tangles

$D$ : a knot diagram

- A *$\mathbb{Z}$-coloring* of $D$ is a map $\mathcal{C} : \{\text{arcs of } D\} \to \mathbb{Z}$ satisfying $a + c = 2b$ at each crossing $x$ of $D$.
- $\mathcal{C}$ is *trivial* $\overset{\text{def}}{\Longleftrightarrow}$ $\mathcal{C}$ is constant.



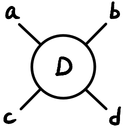

**Folklore fact**

Any knot diagram cannot have a nontrivial $\mathbb{Z}$-coloring.

$T$ : a rational tangle

$D$ : a diagram of $T$, $\mathcal{C}$ : a nontrivial $\mathbb{Z}$-coloring of $D$

- $f(T) = f(D, \mathcal{C}) := \dfrac{b-a}{b-d} \in \mathbb{Q} \cup \{\infty\}$ : the *fraction* of $T$




**Theorem (Conway '70, Kauffman-Lambropoulou '04)**

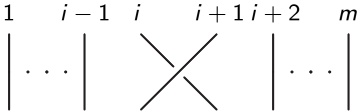
$T, T'$ : rational tangles

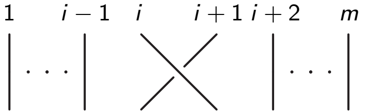
$T$ and $T'$ are ambient isotopic $\Longleftrightarrow$ $f(T) = f(T') \in \mathbb{Q} \cup \{\infty\}$

$\mathbb{Z}$-colorings are powerful tools for rational tangles!

# An action of the $m$-braid group on $\mathbb{Z}^m$

$\mathcal{B}_m \, (m \geq 2)$ : the $m$-braid group with the standard generators $\sigma_1, \ldots, \sigma_{m-1}$

$\sigma_i =$ 


$\sigma_i^{-1} =$ 


A right action of $\mathcal{B}_m$ on $\mathbb{Z}^m$ is defined, for $v = (a_1, \ldots, a_m) \in \mathbb{Z}^m$, by

- $v \cdot \sigma_i = (a_1, \ldots, a_{i-1}, a_{i+1}, 2a_{i+1} - a_i, a_{i+2}, \ldots, a_m)$
- $v \cdot \sigma_i^{-1} = (a_1, \ldots, a_{i-1}, 2a_i - a_{i+1}, a_i, a_{i+2}, \ldots, a_m)$

This action corresponds to $\mathbb{Z}$-colorings for $m$-braids.

$v, w \in \mathbb{Z}^m$ are *equivalent* (written $v \sim w$) $\overset{\text{def}}{\iff}$ $\exists \beta \in \mathcal{B}_m$ s.t. $v \cdot \beta = w$.

**Problem**

- Find a necessary and sufficient condition for $v, w \in \mathbb{Z}^m$ to be $v \sim w$.
- Find the orbit decomposition of $\mathbb{Z}^m$ under the action of $\mathcal{B}_m$.

# Main result

**Theorem**

For $v, w \in \mathbb{Z}^m$ $(m \geq 2)$, the following are equivalent:

(i) $v \sim w$.

(ii) $\Delta(v) = \Delta(w) \in \mathbb{Z}$, $d(v) = d(w) \in \mathbb{Z}_{\geq 0}$, $M(v)_{2d(v)} = M(w)_{2d(w)}$.

The proof is divided into three cases: $m \geq 3$ odd, $m \geq 4$ even and $m = 2$.

**Remark**

- $v, w \in \mathbb{Z}^2$
  $v \sim w \iff \Delta(v) = \Delta(w),\ M(v)_{2d(v)} = M(w)_{2d(w)}$
  ∵ $d(v) = |\Delta(v)|$
- $v, w \in \mathbb{Z}^3$
  $v \sim w \iff \Delta(v) = \Delta(w),\ d(v) = d(w)$
  ∵ $M(v)_{2d(v)} = \{\Delta(v), \Delta(v) + d(v), \Delta(v) + d(v)\}_{2d(v)}$

# The orbit decomposition of $\mathbb{Z}^m$ under the action of $\mathcal{B}_m$

**Theorem**

(i) A complete system of representatives of $\mathbb{Z}^{2k-1}/\sim$ $(k \geq 2)$ is given by

$$\mathcal{C}_{2k-1} = \{(x, \ldots, x) \mid x \in \mathbb{Z}\} \sqcup \{(\underbrace{x, \ldots, x}_{2p-1}, \underbrace{y, \ldots, y}_{2k-2p}) \mid x < y,\ 1 \leq p \leq k-1\}.$$

(ii) A complete system of representatives of $\mathbb{Z}^{2k}/\sim$ $(k \geq 2)$ is given by

$$\begin{aligned}\mathcal{C}_{2k} = &\{(x, \ldots, x) \mid x \in \mathbb{Z}\} \\ &\sqcup \{(x, (1-\lambda)x + \lambda y, \underbrace{y, \ldots, y}_{2k-2}) \mid 0 \leq x < \tfrac{1}{2}y,\ \lambda : \text{odd}\} \\ &\sqcup \{(\underbrace{x, \ldots, x}_{p}, (1-\lambda)x + \lambda y, \underbrace{y, \ldots, y}_{2k-p-1}) \\ &\qquad \mid 0 \leq x < \tfrac{1}{2}y,\ 1 \leq p \leq 2k-2,\ \lambda : \text{even}\}.\end{aligned}$$

# Three kinds of invariants (1/2): Two integers $\Delta(v)$, $d(v)$

$v = (a_1, \dots, a_m) \in \mathbb{Z}^m$

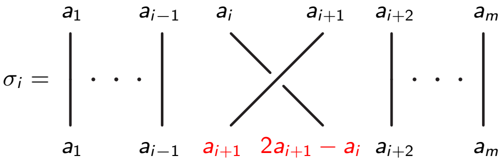
- $\Delta(v) := \sum_{i=1}^{m} (-1)^{i-1} a_i = a_1 - a_2 + \cdots + (-1)^{m-1} a_m$
- $d(v) := \gcd\{a_2 - a_1, a_3 - a_1, \dots, a_m - a_1\} \geq 0 \rightsquigarrow a_i \equiv a_j \pmod{d(v)}$

**Lemma**

For $v = (a_1, \dots, a_m) \in \mathbb{Z}^m$ and $\beta \in \mathcal{B}_m$, we set $v \cdot \beta = (b_1, \dots, b_m)$.
$\pi_\beta \in \mathfrak{S}_m$ : the permutation on $\{1, \dots, m\}$ associated with $\beta$

Then $b_{\pi_\beta(i)} \equiv a_i \pmod{2d(v)}$ for any $i = 1, \dots, m$.

∵ It suffices to consider the case $\beta = \sigma_i$ $(i = 1, \dots, m)$.




Then $b_i = a_{i+1}$ and $b_{i+1} = 2a_{i+1} - a_i = a_i + 2(a_{i+1} - a_i) \equiv a_i \pmod{2d(v)}$. □

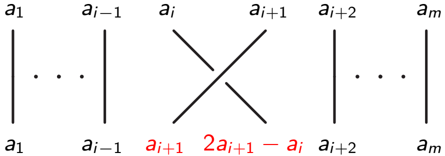
# Three kinds of invariants (2/2): A multi-set $M(v)_{2d(v)}$

$v = (a_1, \ldots, a_m) \in \mathbb{Z}^m$, $n \neq 1$ : a nonnegative integer
$M(v)_n := \{[a_1]_n, \ldots, [a_m]_n\}$ as a multi-set, where $[a_i]_n$ denotes the congruence class of $a_i$ modulo $n$.

**Lemma**

$v \sim w \implies \Delta(v) = \Delta(w),\ d(v) = d(w),\ M(v)_{2d(v)} = M(w)_{2d(w)}$

$\because$ For $v = (a_1, \ldots, a_m)$, it suffices to consider the case $w = v \cdot \sigma_i$.

$\sigma_i =$ 


- $\Delta(v) - \Delta(w) = (-1)^{i-1}(a_i - a_{i+1}) + (-1)^i(a_{i+1} - (2a_{i+1} - a_i)) = 0$
- $(2a_{i+1} - a_i) - a_1 = 2(a_{i+1} - a_1) - (a_i - a_1) \equiv 0 \pmod{d(v)}$
  $\rightsquigarrow d(v) \mid d(w) \quad \therefore d(v) \leq d(w)$
  Since $w \sim v$ holds, we have $d(w) \leq d(v)$, and hence $d(v) = d(w)$.
- $M(v)_{2d(v)} = M(w)_{2d(w)}$ follows from the previous lemma (p. 6). □

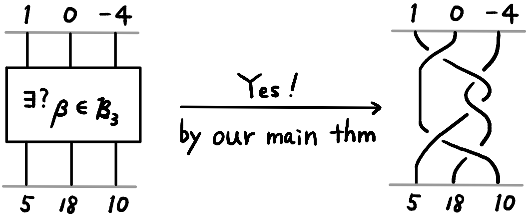
# Example

- $v \in \mathbb{Z}^m$ is *trivial* $\overset{\text{def}}{\iff}$ $v = (a, \ldots, a)$ for some $a \in \mathbb{Z}$ $\iff$ $d(v) = 0$
  For a trivial element $v \in \mathbb{Z}^m$, its orbit $\{v \cdot \beta \mid \beta \in \mathcal{B}_m\} = \{v\}$.
- $v = (-6, 0, 12, 9, 3) \not\sim w = (0, 3, 3, 3, 3) \in \mathbb{Z}^5$

$$\Delta(v) = \Delta(w) = 0, \quad d(v) = d(w) = 3$$
$$M(v)_6 = \{0, 0, 0, 3, 3\}_6 \neq \{0, 3, 3, 3, 3\}_6 = M(w)_6$$

  We remark that $v = (-6, 0, 12, 9, 3) \sim (0, 0, 0, 3, 3)$.
- $v = (1, 0, -4) \sim w = (5, 18, 10) \in \mathbb{Z}^3$

  $\Delta(v) = \Delta(w) = -3,\ d(v) = d(w) = 1,\ M(v)_2 = M(w)_2 = \{0, 0, 1\}_2$



# Normal form (1/2): The case of $m = 2k-1 \geq 3$ odd

$v = (a_1, \ldots, a_{2k-1}) \in \mathbb{Z}^{2k-1}$

$$\Delta(v) = \sum_{i=1}^{2k-1} (-1)^{i-1} a_i \equiv \sum_{i=1}^{2k-1} (-1)^{i-1} a_1 = a_1 \equiv \cdots \equiv a_{2k-1} \pmod{d(v)}$$

- $a_i \equiv \Delta(v)$ or $\Delta(v) + d(v) \pmod{2d(v)}$
- $\#\{a_i \mid a_i \equiv \Delta(v) \pmod{2d(v)}\}$ is odd.

## Proposition

Assume that an element $v \in \mathbb{Z}^{2k-1}$ has

$$M(v)_{2d} = \{\underbrace{\Delta(v), \ldots, \Delta(v)}_{2p-1}, \underbrace{\Delta(v)+d(v), \ldots, \Delta(v)+d(v)}_{2k-2p}\}_{2d(v)}$$

for some $1 \leq p \leq k$. Then

$$v \sim (\underbrace{\Delta(v), \ldots, \Delta(v)}_{2p-1}, \underbrace{\Delta(v)+d(v), \ldots, \Delta(v)+d(v)}_{2k-2p}).$$

# Normal form (2/2): The case of $m = 2k \geq 4$ even

**Proposition**

Assume that a nontrivial element $v \in \mathbb{Z}^{2k}$ ($\iff d(v) \neq 0$) has

$$M(v)_{2d} = \{\underbrace{r, \dots, r}_{p}, \underbrace{r + d(v), \dots, r + d(v)}_{2k-p}\}_{2d(v)}$$

for some $0 \leq r < d(v)$ and $1 \leq p \leq 2k - 1$.

(i) $p \geq 2 \Rightarrow \exists \lambda$ : an even integer (which is determined by $\Delta(v)$) s.t.

$$v \sim (\underbrace{r, \dots, r}_{p-1}, r + \lambda d(v), \underbrace{r + d(v), \dots, r + d(v)}_{2k-p}).$$

(ii) $2k - p \geq 2 \Rightarrow \exists \lambda$ : an odd integer (which is determined by $\Delta(v)$) s.t.

$$v \sim (\underbrace{r, \dots, r}_{p}, r + \lambda d(v), \underbrace{r + d(v), \dots, r + d(v)}_{2k-p-1}).$$

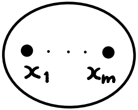
# $(m, m)$-tangles with or without loops (1/3)

$m \geq 2,\ m' \geq 0$

$\mathbb{D}^2$ : the unit 2-disk with $m$ points $x_1, \ldots, x_m$ in $\mathrm{Int}\,\mathbb{D}^2$

$I_1, \ldots, I_m$ : $m$ copies of $[0, 1]$

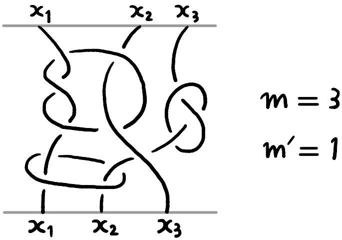
$S_1, \ldots, S_{m'}$ : $m'$ copies of $\mathbb{S}^1$




- An *$(m, m)$-tangle* is the image of a proper embedding

$$f : (I_1 \cup \cdots \cup I_m) \cup (S_1 \cup \cdots \cup S_{m'}) \to \mathbb{D}^2 \times [0, 1]$$

satisfying each string $f(I_i)$ $(i = 1, \ldots, m)$ connects a top point $x_{i'} \in \mathbb{D}^2 \times \{0\}$ and a bottom one $x_{i''} \in \mathbb{D}^2 \times \{1\}$.



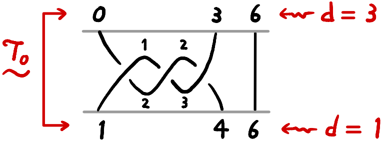
# $(m, m)$-tangles with or without loops (2/3)

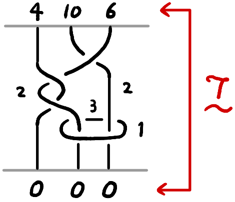
$$\mathcal{B}_m = \{m\text{-braids}\} \subset \boxed{\{(m, m)\text{-tangles with or without loops}\}}$$

For $v, w \in \mathbb{Z}^m$, we write

- $v \overset{\mathcal{T}_0}{\sim} w$ if $\exists D$ : a diagram of an $(m, m)$-tangle without loops admitting a $\mathbb{Z}$-coloring s.t. the top and bottom points of $D$ receive $v$ and $w$, respectively
- $v \overset{\mathcal{T}}{\sim} w$ if $\exists D$ : a diagram of an $(m, m)$-tangle possibly with some loops admitting a $\mathbb{Z}$-coloring s.t. the top and bottom points of $D$ receive $v$ and $w$, respectively

$$v \sim w \implies v \overset{\mathcal{T}_0}{\sim} w \implies v \overset{\mathcal{T}}{\sim} w$$



# $(m, m)$-tangles with or without loops (3/3)

$v \in \mathbb{Z}^m$

- If $v$ is nontrivial with $d(v) = 2^s t \geq 1$ for $s \geq 0$ and $t$ odd, then we set $d_2(v) := 2^s \geq 1$.
- If $v$ is trivial, then we set $d_2(v) := 0$.

**Theorem**

$v, w \in \mathbb{Z}^m$ $(m \geq 2)$

- $v \overset{\mathcal{T}_0}{\sim} w \Leftrightarrow \Delta(v) = \Delta(w), d_2(v) = d_2(w), M(v)_{2d_2(v)} = M(w)_{2d_2(w)}$
- $v \overset{\mathcal{T}}{\sim} w \Leftrightarrow \Delta(v) = \Delta(w), M(v)_2 = M(w)_2$

**Theorem (Recall)**

$v, w \in \mathbb{Z}^m$ $(m \geq 2)$

$v \sim w \iff \Delta(v) = \Delta(w), d(v) = d(w), M(v)_{2d(v)} = M(w)_{2d(w)}$

# Pure braids and String links with or without loops (1/2)

$$\begin{array}{ccc} \mathcal{B}_m = \{m\text{-braids}\} & \subset & \{(m,m)\text{-tangles with or without loops}\} \\ \cup & & \cup \\ \boxed{\mathcal{PB}_m = \{\text{pure } m\text{-braids}\}} & \subset & \boxed{\{m\text{-string links with or without loops}\}} \end{array}$$

- A *pure m-braid* is an $m$-braid $\beta \in \mathcal{B}_m$ s.t. $\pi_\beta = e \in \mathfrak{S}_m$.
- An *m-string link* is an $(m,m)$-tangle $T$ s.t. $\pi_T = e \in \mathfrak{S}_m$.

For $v, w \in \mathbb{Z}^m$, we write

- $v \overset{\mathcal{P}}{\sim} w$ if $\exists \beta \in \mathcal{PB}_m$ s.t. $v \cdot \beta = w$
- $v \overset{\mathcal{L}_0}{\sim} w$ if $\exists D$ : a diagram of an $m$-string link without loops admitting a $\mathbb{Z}$-coloring s.t. the top and bottom points of $D$ receive $v$ and $w$, respectively
- $v \overset{\mathcal{L}}{\sim} w$ if $\exists D$ : a diagram of an $m$-string link possibly with some loops admitting a $\mathbb{Z}$-coloring s.t. the top and bottom points of $D$ receive $v$ and $w$, respectively

$$v \overset{\mathcal{P}}{\sim} w \implies v \overset{\mathcal{L}_0}{\sim} w \implies v \overset{\mathcal{L}}{\sim} w$$

# Pure braids and String links with or without loops (2/2)

$v = (a_1, \ldots, a_m), w = (b_1, \ldots, b_m) \in \mathbb{Z}^m$
$n \neq 1$ : a nonnegative integer
We write $v \equiv w \pmod{n}$ if $a_i \equiv b_i \pmod{n}$ for any $i = 1, \ldots, m$.

## Theorem

$v, w \in \mathbb{Z}^m$ $(m \geq 2)$

- $v \overset{\mathcal{P}}{\sim} w \iff \Delta(v) = \Delta(w), d(v) = d(w), v \equiv w \pmod{2d(v)}$
- $v \overset{\mathcal{L}_0}{\sim} w \iff \Delta(v) = \Delta(w), d_2(v) = d_2(w), v \equiv w \pmod{2d_2(v)}$
- $v \overset{\mathcal{L}}{\sim} w \iff \Delta(v) = \Delta(w), v \equiv w \pmod{2}$

## Theorem (Recall)

- $v \sim w \Leftrightarrow \Delta(v) = \Delta(w), d(v) = d(w), M(v)_{2d(v)} = M(w)_{2d(w)}$
- $v \overset{\mathcal{T}_0}{\sim} w \Leftrightarrow \Delta(v) = \Delta(w), d_2(v) = d_2(w), M(v)_{2d_2(v)} = M(w)_{2d_2(w)}$
- $v \overset{\mathcal{T}}{\sim} w \Leftrightarrow \Delta(v) = \Delta(w), M(v)_2 = M(w)_2$

# Summary

$$
\begin{array}{ccccc}
v \sim w & \Longrightarrow & v \overset{\mathcal{T}_0}{\sim} w & \Longrightarrow & v \overset{\mathcal{T}}{\sim} w \\
\Uparrow & & \Uparrow & & \Uparrow \\
v \overset{\mathcal{P}}{\sim} w & \Longrightarrow & v \overset{\mathcal{L}_0}{\sim} w & \Longrightarrow & v \overset{\mathcal{L}}{\sim} w
\end{array}
$$

| | classical case | | | | | | virtual case | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\sim$ | $\overset{\mathcal{T}_0}{\sim}$ | $\overset{\mathcal{T}}{\sim}$ | $\overset{\mathcal{P}}{\sim}$ | $\overset{\mathcal{L}_0}{\sim}$ | $\overset{\mathcal{L}}{\sim}$ | $\overset{v}{\sim}$ | $\overset{v\mathcal{T}_0}{\sim}$ | $\overset{v\mathcal{T}}{\sim}$ | $\overset{v\mathcal{P}}{\sim}$ | $\overset{v\mathcal{L}_0}{\sim}$ | $\overset{v\mathcal{L}}{\sim}$ |
| $\Delta(v) = \Delta(w)$ | ○ | ○ | ○ | ○ | ○ | ○ | | | | | | |
| $d(v) = d(w)$ | ○ | | | ○ | | | ○ | | | ○ | | |
| $d_2(v) = d_2(w)$ | | ○ | | | ○ | | | ○ | | | ○ | |
| $M(v)_{2d} = M(w)_{2d}$ | ○ | | | | | | ○ | | | | | |
| $M(v)_{2d_2} = M(w)_{2d_2}$ | | ○ | | | | | | ○ | | | | |
| $M(v)_2 = M(w)_2$ | | | ○ | | | | | | ○ | | | |
| $v \equiv w \pmod{2d}$ | | | | ○ | | | | | | ○ | | |
| $v \equiv w \pmod{2d_2}$ | | | | | ○ | | | | | | ○ | |
| $v \equiv w \pmod{2}$ | | | | | | ○ | | | | | | ○ |