# Non-trivial cycles of the spaces of long embeddings detected by 2-loop graphs

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# Keyword 1: the space of long embbedings

- $\mathcal{K}_{n,j} := \mathsf{Emb}(\mathbb{R}^j, \mathbb{R}^n)$ : the *space of long embeddings* $\mathbb{R}^j \to \mathbb{R}^n$
- $\overline{\mathcal{K}}_{n,j} := \overline{\mathsf{Emb}}(\mathbb{R}^j, \mathbb{R}^n) := \mathsf{hofib}_\iota(\mathsf{Emb}(\mathbb{R}^j, \mathbb{R}^n) \to \mathsf{Imm}(\mathbb{R}^j, \mathbb{R}^n))$

## Problem

- *Compute* $\pi_*(\mathcal{K}_{n,j})$.
- *Compare* $n - j = 2$ *and* $n - j \geqslant 3$.

# Keyword 1: the space of long embbedings

- $\mathcal{K}_{n,j} := \mathsf{Emb}(\mathbb{R}^j, \mathbb{R}^n)$
- In 1966, Haefliger computed

$$\pi_0 \mathcal{K}_{n,j} \quad (2n - 3j - 3 = 0,\ n - j \geqslant 3).$$

- The result depends only on *parities* of $n$ and $j$.
- In 2004, Budney, by using Goodwillie's result, further showed

$$\pi_{2n-3j-3} \mathcal{K}_{n,j} \quad (n - j \geqslant 3,\ j \neq 1)$$

depends only on parities of $n$ and $j$. (*bi-periodicity*)

# Keyword 1: the space of long embeddings

$\overline{\mathcal{K}}_{n,j} := \overline{\mathsf{Emb}}(\mathbb{R}^j, \mathbb{R}^n)$: the space of long embeddings mod immersions.

**Thm. (Fresse-Turchin-Willwacher 2017)**

*For $n - j \geqslant 3$, $j \geqslant 1$, $\pi_* \overline{\mathcal{K}}_{n,j} \otimes \mathbb{Q}$ depends only on the parities of $n, j$ up to degree shifts.*

**Proof.**

By a homotopy theoretical approach (called *Goodwillie–Weiss embedding calculus*). □

[i]

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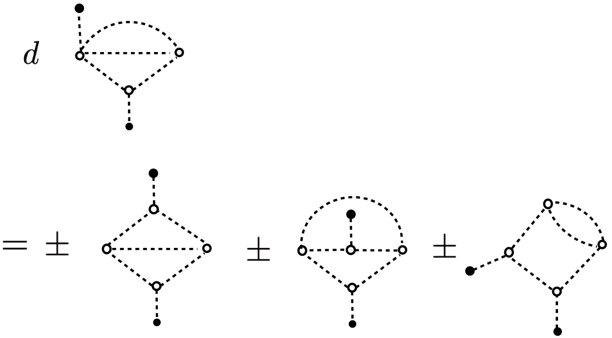
[i] Embedding calculus gives $\overline{\mathcal{K}}_{n,j} \to T_k \overline{\mathcal{K}}_{n,j}$, which is, if $n - j \geqslant 3$, higher and higher connected when $k$ increases.

# Keyword 2: the hairy graph complex

- $HGC_{n,j}$ : the *hairy graph complex* (defined later)
- $HGC_{n,j}$ depends on the parities of $n$ and $j$ only.
- $\mathcal{B}_{n,j} := H_{\text{"top"}}(HGC_{n,j})$
- "top" $=$ any white vertex has exactly three edges.



# Keyword 2: the hairy graph complex

- $HGC_{n,j}$ : the *hairy graph complex* (defined later)
- $\mathcal{B}_{n,j} := H_{\text{“top”}}(HGC_{n,j})$

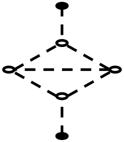
**Thm. (Fresse–Turchin–Willwacher 2017)**

*For* $n - j \geqslant 3$, $j \geqslant 1$, *there is an isom*

$$\pi_* \overline{\mathcal{K}}_{n,j} \otimes \mathbb{Q} \cong H_*(HGC_{n,j})$$

[ii]

**Ex.** ($n, j$: odd, $n - j \geqslant 3$)

*Since*  $\neq 0 \in \mathcal{B}_{n,j}$, $\pi_{3(n-j-2)+(j-1)}(\overline{\mathcal{K}}_{n,j}) \otimes \mathbb{Q} \neq 0$.

[ii] All path-components of $\overline{\mathcal{K}}_{n,j}$ $(n - j \geqslant 3)$ are homotopy equivalent.

# Keyword 2: the hairy graph complex

- Dually, there exists a zigzag of quasi-isoms (in $\mathbf{CDGA}_{\mathbb{Q}}$)

$$\bigwedge \widetilde{HGC}_{n,j} \overset{\simeq}{\leftarrow} \ldots \overset{\simeq}{\rightarrow} A^*_{PL}(\overline{\mathcal{K}}_{n,j})$$

- In other words, $\bigwedge \widetilde{HGC}_{n,j}$ is a *rational model* of $\overline{\mathcal{K}}_{n,j}$ $(n - j \geqslant 3)$.

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## Problem

- *Give a geometric meaning to* $I : H^*(\widetilde{HGC}_{n,j}) \to H^*(\overline{\mathcal{K}}_{n,j}, \mathbb{Q})$.
- *Does* $H^*(\widetilde{HGC}_{n,j})$ *"survive" when* $n - j = 2$ *?*

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## Rem. (Vassiliev–Kontsevich–Bar-Natan)

*When* $(n, j) = (3, 1)$*, the cohomology* $H^{top}(\widetilde{HGC}_{n,j})$ *"survives" as the space of Vassiliev invariants.*

# Keyword 3: configuration space integrals

- Formally, *configuration space integrals* (defined later) give a map

$$I : \bigwedge \widetilde{GC}_{n,j} \longrightarrow A^*_{dR}(\overline{\mathcal{K}}_{n,j}) \quad (n - j \geqslant 2,\ j \geqslant 2)^{\text{iii}}$$

from another graph complex $\widetilde{GC}_{n,j}$. [iv]

[iii] The case $j = 1$ is developed by Bott, Taubes, Kohno, Cattaneo, Cotta-Ramusino, Longoni, Sakai, and some others, though little is known when $* \neq$ top.

[iv] Define $A^*_{dR}(X)$ by $\mathbf{Sset}(Sing^\infty_*(X), A^*_{dR}(\Delta))$.

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from another graph complex $\widetilde{GC}_{n,j}$. [iv]

**Difficulty (Refer also to Leturcq's problem session in ILDT 2021)**

(1) *Deal with obstructions for $I$ to be a cochain map.*

(2) *Give dual cycles of $\overline{\mathcal{K}}_{n,j}$.*

(3) *Compute $H^*(\widetilde{GC}_{n,j})$.*

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[iv] Define $A^*_{dR}(X)$ by $\mathbf{Sset}(Sing^{\infty}_*(X), A^*_{dR}(\Delta))$.

# Keyword 3: configuration space integrals

- Configuration space integrals give

$$I : \bigwedge \widetilde{GC}_{n,j} \longrightarrow A^*_{dR}(\overline{\mathcal{K}}_{n,j}) \quad (n-j \geqslant 2,\ j \geqslant 2)$$

- We have a decomposition

$$\widetilde{GC}^*_{n,j} = \bigoplus_{g \geqslant 0} \widetilde{GC}^*_{n,j}(g),$$

where $g$ is the first Betti number of graphs.

- $g = 0, 1$ and $* =$ top: [Bott, Cattaneo–Rossi, Sakai, Watanabe]
- $g = 2$ and $* =$ top: Today !

# Our ultimate goal

## Problem

*Give an* *explicit* *zigzag of quasi-isoms between* $\bigwedge HGC$ *and* $A^*_{dR}(\overline{\mathcal{K}}_{n,j})$. [a]

[a] For simplicity, we write $HGC$ for $\widetilde{HGC} \otimes \mathbb{R}$.

- HGC has too few graphs to be a source of morphisms.

# Our ultimate goal

## Problem

*Give an explicit zigzag of quasi-isoms between $\bigwedge HGC$ and $A^*_{dR}(\overline{\mathcal{K}}_{n,j})$.* [a]

[a] For simplicity, we write $HGC$ for $\widetilde{HGC} \otimes \mathbb{R}$.

- HGC has too few graphs to be a source of morphisms.

## Strategy (Y.)

- *For $n - j \geqslant 2$, $j \geqslant 2$, give a new graph complex $GC$ and a zigzag*

$$HGC \underset{p}{\overset{\simeq}{\longleftarrow}} GC \underset{I}{\longrightarrow} A^*_{dR}(\overline{\mathcal{K}}_{n,j}).$$

  *of cochain maps.*
- *Show $I^* \circ (p^*)^{-1}$ is injective.*

- The map $I$ will be given by configuration space integrals.

# Main Result

- $\overline{\mathcal{K}}_{n,j} := \overline{\mathsf{Emb}}(\mathbb{R}^j, \mathbb{R}^n)$.
- $g$: the first Betti number of graphs

**Thm. (Y.)**

*Assume $n-j$ is even, $n-j \geqslant 2$ and $j \geqslant 2$. Then there exists a graph complex $DGC$ and a zigzag*

$$HGC \underset{p}{\leftarrow} DGC \underset{I}{\rightarrow} A^*_{dR}(\overline{\mathcal{K}}_{n,j})$$

*of cochain maps such that*

(1) $p^* : H^{top}(DGC) \to H^{top}(HGC)$ *is surjective.*

(2) *If $H \in H^{top}(DGC(g=2))$ and $I^*(H) = 0$, then $p^*(H) = 0$.*

# Main Result

- We have a decomposition

$$\widetilde{HGC}^*_{n,j} = \bigoplus_{g\geqslant 0, k\geqslant 1} \widetilde{HGC}^*_{n,j}(k,g),$$

where $k$ is the *order* of graphs defined by |edges| − |white vertices|.

**Cor. (Y.)**

*If $n-j$ is even, $n-j \geqslant 2$ and $j \geqslant 2$,*

$$\mathit{dim}\, H_{k(n-j-2)+(j-1)}(\overline{\mathcal{K}}_{n,j}, \mathbb{Q}) \geqslant \mathit{dim}\, \mathcal{B}_{n,j}(k, g=2).$$

**Cor. (Y.)**

*If $j \geqslant 2$, $\pi_{(j-1)}(\overline{\mathcal{K}}_{j+2,j})_u \otimes \mathbb{Q}$ is infinite dim. ($u$: unknot component)*

# Background: Is our result new ?

**Rem.**

$\pi_{j-1}(\overline{\mathcal{K}}_{j+2,j})_u \otimes \mathbb{Q}$: *infinite dim* $\Rightarrow \pi_{j-1}(\mathcal{K}_{j+2,j})_u \otimes \mathbb{Q}$: *infinite dim*

[v]

**Thm. (Hatcher '83)**

$\pi_*(\mathcal{K}_{3,1})_u$ *is trivial.*

**Thm. (Budney-Gabai '19, Watanabe '20)**

*For* $j \geqslant 2$, $\pi_{j-1}(\mathcal{K}_{j+2,j})_u \otimes \mathbb{Q}$ *is infinite dim.*

[vi]

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[v] (RHS) $\Rightarrow \pi_{j-1}(\mathrm{Emb}(S^j, S^{j+2})_u) \otimes \mathbb{Q}$: infinite dim

[vi] They developed

- Embedding calculus for $\mathrm{Emb}_\partial(D^1, D^{j+1} \times S^1)$, and
- Kontsevich characteristic classes of $\mathrm{BDiff}_\partial(D^{j+1} \times S^1)$ respectively.

# Graph complexes and graph homologies

- We introduce two graph complexes $PGC$ and $HGC$.
- There is a projection $p : PGC \to HGC$.

- Several parts of $H^*(HGC)$ is already computed.
  (For $g = 2$, refer to [Conant–Costello–Turchin–Weed '14])
- PGC has more graphs and hence is suited as a source of morphisms.

# The plain graph complex $PGC$

- $PGC$ is a cochain complex generated by connected *plain graphs*.
- *Plain graphs* have two types of vertices and two types of edges.
  - White vertices have at least three dashed edges and no solid edges.
  - Black vertices have an arbitrary number of solid and dashed edges.
  - Each component has at least one black vertex.
  - No loop-edge is allowed.

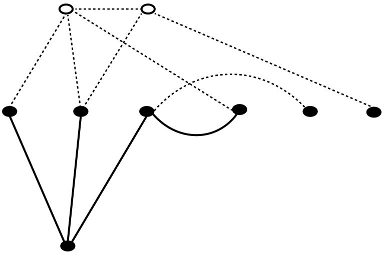


Figure: Example of a plain graph

- $deg(\Gamma)$ by $(n-1)|E_{\cdots\cdots}| + (j-1)|E_{\_\_}| - n|V_\circ| - j|V_\bullet|$.

# The plain graph complex PGC

- $deg(\Gamma) = (n-1)|E_{\cdots\cdots}| + (j-1)|E_{\_\_}| - n|V_\circ| - j|V_\bullet|$.

A plain graph is *admissible* if it satisfies both of the following.

(I) Every black vertex without dashed edges must have at least three solid edges. [vii]

(II) The restriction to solid edges consists of disjoint broken lines.

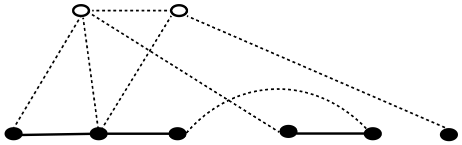


Figure: Example of an admissible plain graph

[vii](I) + (II) ⇒Every black vertex has at least one dashed edge and at most two solid edges.

# The plain graph complex PGC

- A label of a plain graph gives an orientation of a graph.
- The orientation depends only on parities of $n$ and $j$.

**Def.**

As a vector space,

$$PGC_{n,j} = \frac{\mathbb{Q}\{\text{Labeled, admissible plain graphs}\}}{\text{Ori relations}}.$$

**Def.**

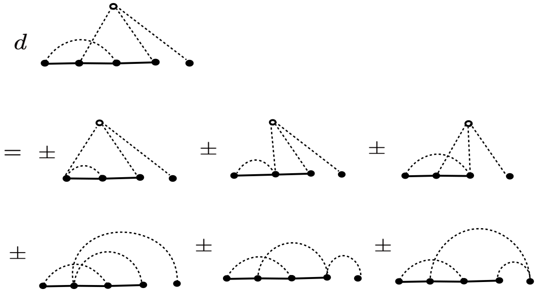
The differential $d_{PGC}$ of $PGC$ is defined by

$$d_{PGC}(\Gamma) = \sum_{\substack{e \in E(\Gamma) \\ e \neq \bullet\!-\!-\!-\!\bullet}} \pm \Gamma / e.$$

# The plain graph complex PGC

- $d_{PGC}(\Gamma) = \sum_{\substack{e \in E(\Gamma) \\ e \neq \bullet - - - \bullet}} \pm \Gamma / e$
- $(PGC, d_{PGC})$ is a cochain complex.

## Example




- The signs arise when labels of vertices and edges are permuted and when $d$ "jumps" vertices.

# The hairy graph complex HGC

- $HGC$ is a cochain complex generated by *hairy graphs*.
- Hairy graphs are admissible plain graphs that satisfy the following.
  - No solid edge exists.
  - Each black vertex has exactly one dashed edge.
- A segment ●- - - is called a *hair*.

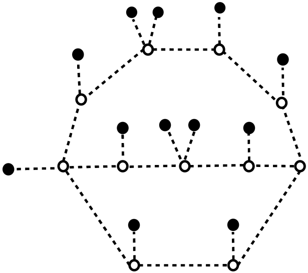


Figure: Example of a hairy graph

# The hairy graph complex HGC

**Def.**

As a vector space,

$$HGC_{n,j} = \frac{\mathbb{Q}\{\text{Labeled hairy graphs}\}}{\text{Ori relations}}.$$

**Def.**

The differential $d_{HGC}$ of $HGC$ is defined by

$$d_{HGC} = \sum_{\substack{e \in E(\Gamma) \\ e = \circ - - - \circ}} \pm \Gamma / e.$$

# Relasionship between $PGC$ and $HGC$

## Thm. (Y.)

*The projection $PGC^{top} \to HGC^{top}$ induces an epimorphism between the top cohomologies.*

## Proof.

(Sketch)

- Dually, we show the map: $\chi_* : H_{top}(HGC) \to H_{top}(PGC)$ induced by the inclusion is injective.
- In fact, we can construct a left inverse

$$\sigma_* : H_{top}(PGC) \to H_{top}(HGC), \quad \sigma_* \chi_* = id$$

  by induction on the number of black vertices.
- This construction of $\sigma_*$ is motivated by Bar-Natan's construction of $\chi^{-1} : \mathcal{A}(S^1) \to \mathcal{B}$.

□

# Cycles: ribbon presentations

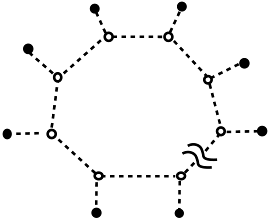
(i) 1-loop ($g = 1$) [Sakai, Watanabe '12]

- What: $k(n-j-2)$-cycle

$$c(W_k) : (S^{n-j-2})^k \to \mathcal{K}_{n,j}$$

- How: Perturb the *wheel-like ribbon presentation*

- The cycle $c(W_k)$ is detected by the graph $W_k$ with $k$ hairs:



# Cycles: ribbon presentations

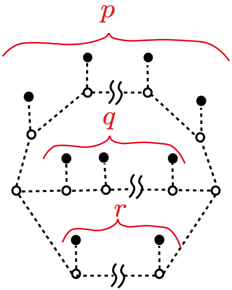
(ii) 2-loop ($g = 2$) [Y.]

- What: $(k(n-j-2) + (j-1))$-cycle

$$d(\Theta(p,q,r)) : (S^{n-j-2})^k \times S^{j-1} \to \mathcal{K}_{n,j}$$

$(p, r \geqslant 1,\ q \geqslant 0,\ p+q+r+1 = k)$

- How: Perturb a *ribbon presentation with one node*
- The cycle $d(\Theta(p,q,r))$ is detected by the graph $\Theta(p,q,r)$:



# The process for giving a cycle from $\Theta(p, q, r)$

Recall $p, r \geqslant 1$, $q \geqslant 0$, $p + q + r + 1 = k$.

(1) Diagram $D(\Theta(p, q, r))$
$\downarrow$
(2) Ribbon presentation $P(\Theta(p, q, r))$
$\downarrow$
(3) $S^{j-1} \times (S^{n-j-2})^{\times k}$ cycle of submanifolds ($\approx \mathbb{R}^j$) in $\mathbb{R}^n$
$\downarrow$
(4) Desired cycle $d(\Theta(p, q, r)) : S^{j-1} \times (S^{n-j-2})^{\times k} \to \overline{\mathcal{K}}_{n,j}$

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(3) $S^{j-1} \times (S^{n-j-2})^{\times k}$ cycle of submanifolds ($\approx \mathbb{R}^j$) in $\mathbb{R}^n$
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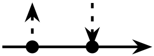
# The diagram associated with $\Theta(p, q, r)$

A diagram $D(\Theta(p, q, r))$ is obtained from $\Theta(p, q, r)$ as follows.

- Orient three edges of $\Theta$.
- Replace each hair with the oriented line with two open chords



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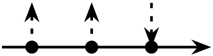
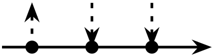
- Exceptionally replace the leftmost (resp. rightmost) hair of the upper (resp. lower) edge with



(resp. ).

- Connect ends of chords as expected.

# The diagram associated with $\Theta(p, q, r)$

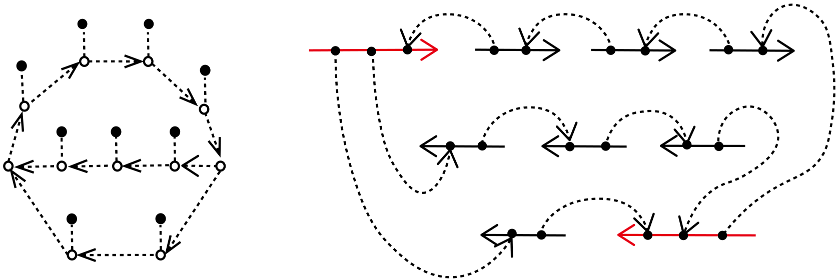


Figure: Graph $\Theta(4, 3, 2)$ and Diagram $D(\Theta(4, 3, 2)))$

# The process for giving a cycle from $\Theta(p, q, r)$

Recall $p, r \geqslant 1$, $q \geqslant 0$, $p + q + r + 1 = k$.

(1) Diagram $D(\Theta(p, q, r))$
$\downarrow$
(2) Ribbon presentation $P(\Theta(p, q, r))$
$\downarrow$
(3) $S^{j-1} \times (S^{n-j-2})^{\times k}$ cycle of submanifolds ($\approx \mathbb{R}^j$) in $\mathbb{R}^n$
$\downarrow$
(4) Desired cycle $d(\Theta(p, q, r)) : S^{j-1} \times (S^{n-j-2})^{\times k} \to \overline{\mathcal{K}}_{n,j}$

# Ribbon presentations

**Def. (Habiro-Kanenobu-Shima ('99) ('01))**

A *ribbon presentation* $P = \mathcal{D} \cup \mathcal{B}$ is an oriented immersed $2$-disk in $\mathbb{R}^3$ s.t.

- $\mathcal{D} = (D_0, *) \cup D_1 \cdots \cup D_l$: disks ($D_i \approx D^2$).
- $\mathcal{B} = B_1 \cup \cdots \cup B_l$: bands ($B_i \approx I \times I$).
- Each band connects two disks.
- Each band can intersect with the interiors of disks except for $D_0$.

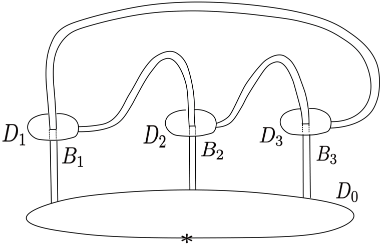



Figure: Example of a ribbon presentation

# The long embedding associated with a ribbon presentation

- $V_P := \mathcal{B} \times [-1/4, 1/4]^{j-1} \bigcup \mathcal{D} \times [-1/2, 1/2]^{j-1}$ (thickening)
- $\psi(P) := \partial V_P \# \iota(\mathbb{R}^j) \subset \mathbb{R}^n$ is a long embedding with $k$ *crossings*.
- A crossing is a link of of $\hat{D}_i \approx S^j \backslash pt$ and $\hat{B}_i \approx I \times S^{j-1}$.

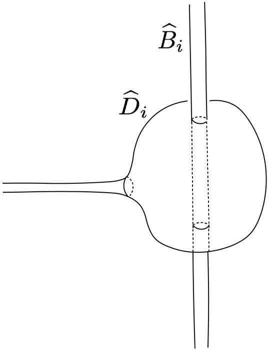



Figure: The $i$-th crossing

# Moves of ribbon presentations

- Habiro, Kanenobu, Shima introduced the following moves.
- These moves do not change isotopy classes of corresponding embeddings.

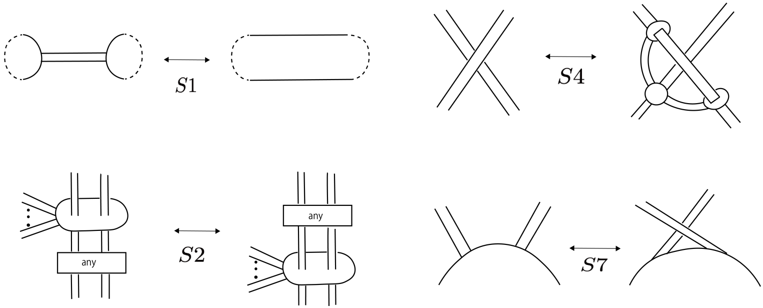
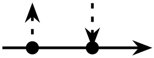
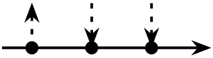
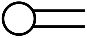
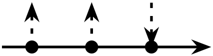
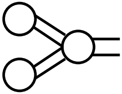
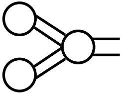



Figure: Example of moves of ribbon presentations

# The ribbon presentation associated with $D(\Theta(p, q, r))$

A ribbon presentation $P(\Theta(p, q, r))$ is obtained from $D(\Theta(p, q, r))$ .

- Replace  and  with 
- Exceptionally, replace  with 
- Intersect a disk with a band if they are connected by chords. Assign the label $\star$ to this crossing.
- Connect the end of each band to the based disk.
- Two disks of  must intersect with bands in opposite orientation.

# The diagram associated with $\Theta(p, q, r)$

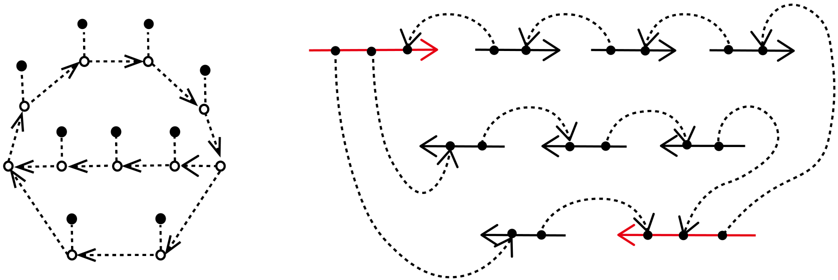


Figure: Diagram $D(\Theta(4, 3, 2)))$

# The ribbon presentation associated with $D(\Theta(p,q,r))$

- node = a disk which does not intersect with bands
- $P(\Theta(p,q,r))$ has $k = p+q+r+1$ crossings and one node.

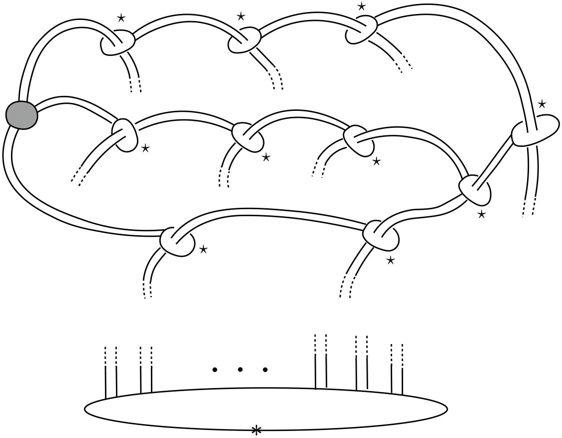
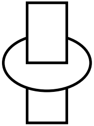
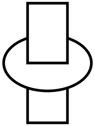
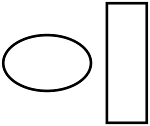


Figure: Ribbon presentation $P(\Theta(4,3,2))$. The node is drawn in gray.

# Properties of $P(\Theta(p,q,r))$

## Notation

*Let $\varepsilon_i = \pm 1$. Write $P(\Theta(p,q,r))(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ for the ribbon presentation obtained by changing the $j$th crossing*  $\star$ *to*  *when $\varepsilon_j = 1$, and to*  *when $\varepsilon_j = -1$.*

## Prop. (Y.)

*After several cross-change moves are performed to $P(\Theta(p,q,r))$, $P(\Theta(p,q,r))(1,1,\ldots,1)$ is equivalent to the trivial presentation.*

# Cross-change moves

- We perform the following cross-change move to $P(\Theta(p, q, r))$.

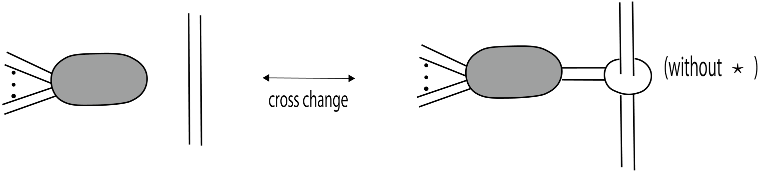



Figure: Cross-change move

- The move might change the cycles we later define.
- However, the move does not affect the pairing argument we later discuss.
- cf. Vassiliev invariants of order $\leqslant k$ vanish for singular knots with $(k+1)$ singular points.

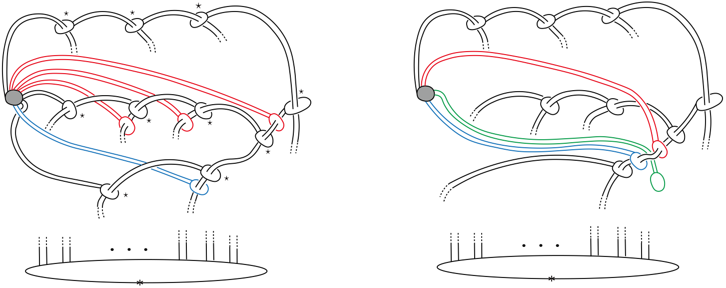
# Properties of $P(\Theta(p, q, r))$

## Prop. (Y.)

*After several cross-change moves are performed to* $P(\Theta(p, q, r))$*,* $P(\Theta(p, q, r))(1, 1, \ldots, 1)$ *is equivalent to the trivial presentation.*

## Proof.

We perform cross-changes as in the figure. Then the resulting presentation becomes trivial, after moves including $S4$. □



# The process for giving a cycle from $\Theta(p,q,r)$

Recall $p,r \geqslant 1$, $q \geqslant 0$, $p+q+r+1=k$.

(1) Diagram $D(\Theta(p,q,r))$
$\downarrow$
(2) Ribbon presentation $P(\Theta(p,q,r))$
$\downarrow$
(3) $S^{j-1} \times (S^{n-j-2})^{\times k}$ cycle of submanifolds ($\approx \mathbb{R}^j$) in $\mathbb{R}^n$
$\downarrow$
(4) Desired cycle $d(\Theta(p,q,r)) : S^{j-1} \times (S^{n-j-2})^{\times k} \to \overline{\mathcal{K}}_{n,j}$

# The cycle of submanifolds associated with $P(\Theta(p, q, r))$

- $\mathbb{R}^n = \mathbb{R}^3 \times \mathbb{R}^{n-j-2} \times \mathbb{R}^{j-1}$
- See the parameter space $S^{n-j-2}$ as $\{(x_3, \ldots, x_{n-j+1}) \in \mathbb{R}^{n-j-1} \mid (x_3 - 1)^2 + x_4^2 + \cdots + x_{n-j+1}^2 = 1\}$.[viii]
- In particular, $S^0 = \{x_3 = 0, x_3 = +2\}$.

**Def. (Watanabe '06)**

The *perturbation of a crossing* (with $\star$) is the operation to replace the band $B$ with the perturbed band $B(v)$ ($v \in S^{n-j-2}$).

[viii] Assume the $x_3$ coordinate is perpendicular to bands, near crossings.

# Perturbation of a crossing

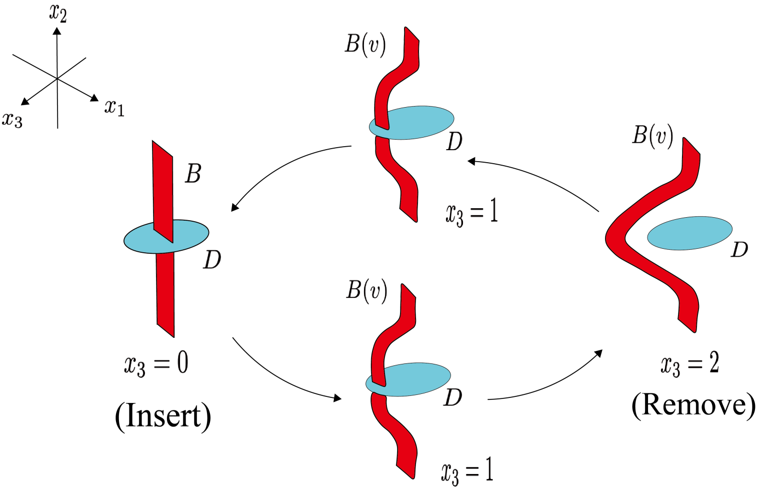



Figure: Perturbation of a crossing $(n - j = 3)$

## The cycle of submanifolds associated with $P(\Theta(p,q,r))$

- $P = P(\Theta(p,q,r))$ has $k$ crossings.
- Each band $B_j$ has one or two crossings.
- For each $\mathbf{v} = (v_1, \dots, v_k) \in (S^{n-j-2})^k$,

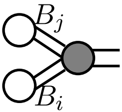
$$P_{\mathbf{v}} := \mathcal{D} \cup \mathcal{B}(\mathbf{v}) = \bigcup D_i \cup \bigcup B_j(v_1, v_2, \dots, v_k)$$

**Def.**

The cycle $c(\Theta(p,q,r)) : (S^{n-j-2})^k \to \mathcal{K}_{n,j}$ is defined by

$$\mathbf{v} \longmapsto \psi(P_{\mathbf{v}}) := \partial V_{P_{\mathbf{v}}} \# \iota(\mathbb{R}^j).$$

# The cycle of submanifolds associated with $P(\Theta(p, q, r))$

- Recall our ribbon presentation has a *node* 

.
- $S^{j-1}$ family : move one tube ($\hat{B}_i$) around the other tube ($\hat{B}_j$)
- Then we obtain a cycle $d(\Theta(p, q, r)) : (S^{n-j-2})^k \times S^{j-1} \to \mathcal{K}_{n,j}$

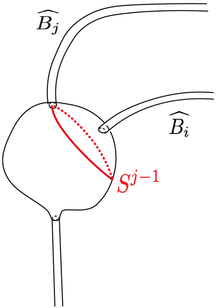



Figure: The additional $S^{j-1}$ family

# Cocycles: configuration space integrals

- We give a geometric correspondence

$$I : PGC \to A_{dR}(\overline{\mathcal{K}}_{n,j}).$$

- $I$ is given by configuration space integrals
(in the same way as [Bott, Cattaneo, Rossi, Sakai, Watanabe]).

# Cocycles: configuration space integrals

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- $I$ is given by configuration space integrals
(in the same way as [Bott, Cattaneo, Rossi, Sakai, Watanabe]).
- Notice: configuration space integrals may not give cochain maps.
($\exists$ obstruction called *hidden faces*)

**Thm. (Y. advised by Turchin)**
*$I$ is a cochain map "up to homotopy".*

# Cocycles: configuration space integrals

## Def. (*Configuration spaces*)

$$C_k(\mathbb{R}^n) \coloneqq (\mathbb{R}^n)^{\times k} \setminus \Delta, \quad (\Delta = \bigcup_{i \neq j} \{y_i = y_j\})^a$$

[a] Though $C_k(\mathbb{R}^n)$ is open, there is a canonical compactification of it.

## Def. (*Configuration space bundles*)

$E_{s,t}$ is the bundle over $\overline{\mathcal{K}}_{n,j}$ defined by the pullback

$$\begin{array}{ccc} E_{s,t} & \longrightarrow & C_{s+t}(\mathbb{R}^n) \\ \downarrow & & \downarrow \text{restriction} \\ \overline{\mathcal{K}}_{n,j} \times C_s(\mathbb{R}^j) & \xrightarrow{\text{evaluation at } u=1} & C_s(\mathbb{R}^n) \end{array}$$

(Recall $\overline{\mathcal{K}}_{n,j}$ consists of $\{\overline{\psi}\}_{u \in [0,1]}$ s.t. $\overline{\psi}_u \in \mathrm{Imm}(\mathbb{R}^j, \mathbb{R}^n)$, $\overline{\psi}_1 \in \mathcal{K}_{n,j}$. )

# Cocycles: configuration space integrals

- If $\Gamma$ has $s$ black vertices $\overset{(-j)}{\bullet}$ and $t$ white vertices $\overset{(-n)}{\circ}$, use $E_{s,t}$.
- Each oriented dashed (resp. solid) edge $e$ gives a direction map

$$P_e : E_{s,t} \to S^{n-1} \quad \text{(resp. } P_e : E_{s,t} \to S^{j-1}\text{)}.$$

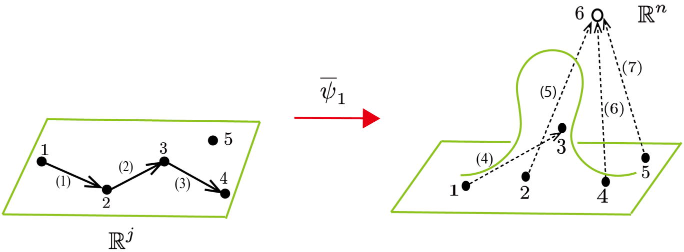


Figure: Example of the direction map

# Cocycles: configuration space integrals

$\Gamma$: a (labeled) plain graph

**Def.**

Define a form $I(\Gamma) \in A_{dR}(\overline{\mathcal{K}}_{n,j})$ as follows. For a simplex $f : \Delta_m \to \overline{\mathcal{K}}_{n,j}$,

$$I(\Gamma)(f) := \pi_* \Omega_f(\Gamma) = \int_{\overline{C}_{s,t}} \Omega_f(\Gamma) \in A_{dR}(\Delta^m),$$

where $\Omega_f(\Gamma) := (P(\Gamma) \circ f)^* (\bigwedge \omega_{S^{j-1}} \wedge \bigwedge \omega_{S^{n-1}})$.

$$\begin{array}{ccccc}
f^* E_{s,t} & \xrightarrow{\quad f \quad} & E_{s,t} & \xrightarrow{P(\Gamma) = \prod_e P_e} & \prod S^{j-1} \times \prod S^{n-1} \\
\downarrow \pi & & \downarrow \pi & & \\
\Delta^m & \xrightarrow{f\text{: a ``smooth'' simplex}} & \overline{\mathcal{K}}_{n,j} & &
\end{array}$$

# Cocycles: configuration space integrals

- The correspondence

$$I : PGC \to A_{dR}(\overline{\mathcal{K}}_{n,j})$$

is not necessarily a cochain map.

- In fact,

$$(-1)^{|\Gamma|+1} dI(\Gamma) = \int_{\partial \overline{C}_{s,t}} \Omega(\Gamma) = \sum_{\substack{S \subseteq V(\Gamma) \cup \infty \\ |S| \geqslant 2}} \int_{\widetilde{C}_S} \Omega(\Gamma),$$

where $\widetilde{C}_S$ is the configuration s.t. the vertices of $S$ are infinitely close.

- The obstructions

$$dI(\Gamma) - I(d\Gamma) = \Big( \sum_{\substack{S \subseteq V(\Gamma) \\ |S| \geqslant 3}} + \sum_{S = V(\bullet - - - \bullet)} \Big) \int_{\widetilde{C}_S} \Omega(\Gamma)$$

are called *hidden face contributions*.

# How to cancel hidden faces ?

- Some hidden faces vanish by symmetries and rescaling of the faces.
- Other faces are canceled by introducing correction terms.
- We interpret adding correction terms as replacing graph complexes.

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- Some hidden faces vanish by symmetries and rescaling of the faces.
- Other faces are canceled by introducing correction terms.
- We interpret adding correction terms as replacing graph complexes.

**Thm. (Y. advised by Turchin)**

*When $n - j \geqslant 2$, there exists a graph complex $DGC$ and a zigzag*

$$PGC \underset{p}{\overset{\simeq}{\leftarrow}} DGC \underset{I}{\rightarrow} A^*_{dR}(\overline{\mathcal{K}}_{n,j}),$$

*of cohain maps.*

# Cocycle-cycle pairngs

- Recall we have a zigzag

$$HGC \underset{p}{\leftarrow} PGC \overset{\simeq}{\leftarrow} DGC \underset{I}{\rightarrow} A^*_{dR}(\overline{\mathcal{K}}_{n,j}).$$

- We showed $p^* : H^{top}(PGC) \to H^{top}(HGC)$ is surjective.
- We know $H^{top}(HGC(g=2))$ is infinite-dim [Conant–Costello–Turchin–Weed '14].

**Thm. (Y. )**

*If* $H \in H^{top}(DGC(g=2))$ *and* $I^*(H) = 0$*, we have* $p^*(H) = 0$

# Cocycle-cycle pairngs

**Example (simplest odd case)**

Suppose $(n, j) = (\text{odd}, \text{odd})$, $n - j \geqslant 2$ and $j \geqslant 3$. There exists a non-trivial graph cocycle in $HGC^{top}(k = 3, g = 2)$ that includes $\Theta(1, 0, 1)$. Hence we have

$$H^{3(n-j-2)+(j-1)}(\overline{\mathcal{K}}_{n,j}, \mathbb{Q}) \neq 0.$$

**Example (simplest even case)**

Suppose $(n, j) = (\text{even}, \text{even})$, $n - j \geqslant 2$ and $j \geqslant 2$. There exists a non-trivial graph cocycle in $HGC^{top}(k = 7, g = 2)$ that includes $\Theta(3, 2, 1)$. Hence we have

$$H^{7(n-j-2)+(j-1)}(\overline{\mathcal{K}}_{n,j}, \mathbb{Q}) \neq 0.$$

# Cocycle-cycle pairings

- Suppose $p + q + r + 1 = k$, $p, r \geqslant 1$, $q \geqslant 0$.
- $H$: a 2-loop, top graph cocycle of order $\leqslant k$,

$$H = \sum_i \frac{w(\Gamma_i)}{|\mathrm{Aut}(\Gamma_i)|}\Gamma_i.$$ [ix]

- Let $p_k^* : DGC^{top}(g = 2) \to HGC^{top}(g = 2, k)$ be the projection.

## Key prop. (Counting formula)

$$< I(H), d(\Theta(p, q, r)) >= \pm w(\Theta(p, q, r)).$$

*where $\pm$ depends only on the oriented graph $\Theta(p, q, r)$.*

## Proof of Thm.

Assume Key prop. Then if $I(H)$ is exact, $p_k^*(H) = 0$. □

[ix] Assume $\Gamma_i \not\cong \Gamma_j$ if $i \neq j$, and assume $\Gamma_i$ has no ori. reversing auto.

# Cocycle-cycle pairings

- Consider the four graphs $D_1$, $D_2$, $D_3$, $D_4$ without white vertices obtained by performing $STU$ relations to $\Theta(p,q,r)$ as follows.
- Take orientations of the graphs so that

$$w(\Theta(p,q,r)) = w(D_1) + w(D_2) + w(D_3) + w(D_4).$$

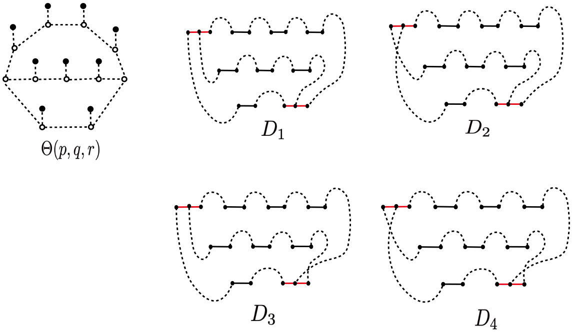



Figure: Graph $D_1$, $D_2$, $D_3$,$D_4$

# Cocycle-cycle pairings

- Recall $H = \sum_i \frac{w(\Gamma_i)}{|\mathrm{Aut}(\Gamma_i)|}\Gamma_i$.
- $w(\Theta(p,q,r)) = w(D_1) + w(D_2) + w(D_3) + w(D_4)$.

## Notation

$$< \Gamma_i, D_j > := \begin{cases} 0 & (\Gamma_i \textit{ is not isom to } D_j) \\ \pm 1 & (\Gamma_i \textit{ is isom to } D_j) \end{cases}$$

## Lemma

*If a graph $\Gamma_i$ has order $\leqslant k$,*

$$< I(\Gamma_i), d(\Theta(p,q,r)) > = \pm \sum_{j=1,2,3,4} |Aut(D_j)| < \Gamma_i, D_j >,$$

# Cocycle-cycle pairings

### Proof of Key prop.

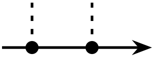
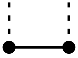
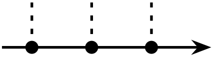
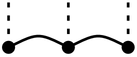
Assuming lemma, we have

$$
\begin{aligned}
< I(H), d(\Theta(p,q,r)) > &= \pm \sum_i \frac{w(\Gamma_i)}{|\mathrm{Aut}(\Gamma_i)|} \sum_{j=1,2,3,4} |\mathrm{Aut}(D_j)| < \Gamma_i, D_j > \\
&= \pm(w(D_1) + w(D_2) + w(D_3) + w(D_4)) \\
&= \pm w(\Theta(p,q,r)).
\end{aligned}
$$

□

# Cocycle-cycle pairings

## Proof of Lemma.

- The pairing $< I(\Gamma_i), d(\Theta(p,q,r)) >$ is equal to counting graphs on the diagram $D = D(\Theta(p,q,r))$.
- On the segment , only  is allowed.
- On , only  or  is allowed.
- We can show decorated graphs are not counted.
- There are four plain graphs counted, which are $D_1, \dots, D_4$.

□

# Cocycle-cycle pairings

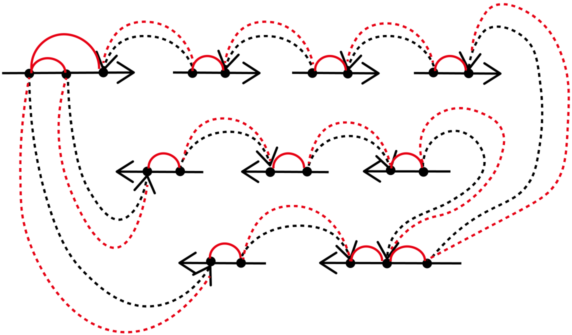


Figure: Graph $D_2$ is counted on the diagram $D(\Theta(p, q, r))$

# Some questions

(Q1) Detect torsions of $\pi_* \mathcal{K}_{n,j}$ by the geometric approach.

**Thm. (Haefliger, Budney)**

*Let $n - j \geqslant 3$ and $2n - 3j - 3 \geqslant 0$. Then*

$$\pi_{2n-3j-3}\,\mathcal{K}_{n,j} \simeq \begin{cases} \mathbb{Z} & (j = 1 \text{ or } n - j \text{ odd}) \\ \mathbb{Z}_2 & (j > 1 \text{ and } n - j \text{ even}) \end{cases}$$

(Q2) Establish configuration space integrals for $\mathrm{Emb}(M, \mathbb{R}^n)$.

**Thm. (Fresse-Turchin-Willwacher '20)**

*$M$: a complement of a compact mfd in $\mathbb{R}^j$, $M \neq \mathbb{R}^j$ if $n - j = 2$. Then $\pi_* \left(\overline{Emb}(M, \mathbb{R}^n), \mathbb{Q}\right)$ is controlled by $R$-decorated hairy graph complex, where $R \simeq A_{PL}(M \cup \infty)$.*