Pure cactus groups and configuration spaces of points on the circle

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Based on joint works with

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As an analogue of the braid group, the following cactus group J_n was introduced (motivated by the study of quantum groups).

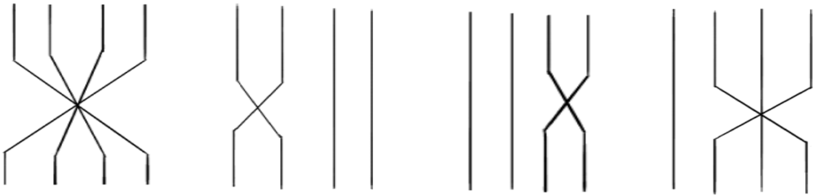
Definition [Henriques-Kamnitzer,2006]

For $n \in \mathbb{Z}_{\geq 2}$, the cactus group J_n of degree n is defined by the following presentation.
Generators: $s_{p,q}$ with $1 \leq p < q \leq n$
Relations:

- $s_{p,q}^2 = e$
- $s_{p,q}s_{m,r} = s_{m,r}s_{p,q}$ $([p,q] \cap [m,r] = \varnothing)$
- $s_{p,q}s_{m,r} = s_{p+q-r,p+q-m}s_{p,q}$ $([m,r] \subset [p,q])$

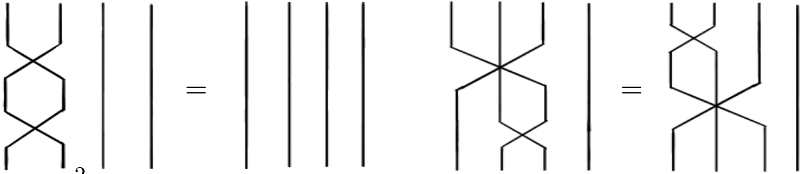
Here, $[p,q]$ denotes the set $\{p, p+1, \ldots, q-1, q\}$ of integers for positive integers p, q with $p < q$.

Diagrams of J_4



s_{14} $\quad$ s_{12} $\quad$ s_{34} $\quad$ s_{24}

$\pi(s_{14}) = (14)(23)$ $\quad$ $\pi(s_{12}) = (12)$ $\quad$ $\pi(s_{34}) = (34)$ $\quad$ $\pi(s_{24}) = (24)$



$s_{12}^2 = e$ $\qquad$ $s_{13}s_{23} = s_{12}s_{13}$

We can consider the natural surjection π from J_n to the symmetric group of degree n.

Definition

The pure cactus group PJ_n of degree n is defined by the kernel of the natural surjection π.

Known facts

- $PJ_4 \cong \pi_1(\overline{M_{0,5}}(\mathbb{R}))$ [Henriques-Kamnitzer]
- $\overline{M_{0,5}}(\mathbb{R}) \cong \overset{5}{\#}\mathbb{RP}^2$ [c.f. Etingof-Henriques-Kamnitzer-Rains]
- $PJ_4 \cong < \alpha, \beta, \gamma, \delta, \epsilon \mid \alpha\gamma\epsilon\beta\epsilon\alpha^{-1}\delta^{-1}\beta\gamma\delta^{-1} >$ [Bellingeri-Chemin-Lebed, 2022] (by Reidemeister-Schreier method.)

We obtain the following (not mentioning the moduli spaces) by using geometric group theory and hyperbolic geometry.

Theorem 1.[Hama-I., arXiv.2504.11852]

PJ_4 admits the following presentation.

$$\left\langle g_1, \cdots, g_{10} \;\middle|\; \begin{array}{l} g_1 g_{10}^{-1} g_2^{-1},\ g_9 g_5^{-1} g_4,\ g_5 g_1 g_6^{-1}, \\ g_8 g_{10} g_7^{-1},\ g_8 g_3^{-1} g_4,\ g_2 g_9 g_7^{-1} g_6 g_3^{-1} \end{array} \right\rangle$$

This presentation is transformed to the next one.

$$\langle g_2, g_4, g_8, g_9, g_{10} \mid g_2 g_9 g_{10}^{-1} g_8^{-1} g_4 g_9 g_2 g_{10} g_8^{-1} g_4^{-1} \rangle$$

Corollary.

$$PJ_4 \cong < \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \mid \alpha_1^2 \alpha_2^2 \alpha_3^2 \alpha_4^2 \alpha_5^2 > .$$

Poincaré's polygon theorem [c.f. Maskit, 1971]

Let D be a polygon with side-identifications on $\mathbb{H}^2$ satisfying angle conditions. Let G be the group generated by the side-identifications. Then D is a fundamental polygon for the action $G \curvearrowright \mathbb{H}^2$, and the cycle relations form the complete set of relations for G.

Sketch of proof

- Consider an action $PJ_4 \curvearrowright \mathcal{C} \cong \mathbb{H}^2$.
- Find a Dirichlet polygon D with respect to the action.
- Applying Poincaré's polygon theorem.

Definition. [Genevois 2022]

$$J_4' := \left\langle \begin{array}{l} s_{12}, s_{23}, s_{34}, \\ s_{13}, s_{24} \end{array} \middle| \begin{array}{l} s_{ij}^2 = e,\ s_{13}s_{12} = s_{23}s_{13}, \\ s_{24}s_{23} = s_{34}s_{24},\ s_{12}s_{34} = s_{34}s_{12} \end{array} \right\rangle$$

Let $\mathcal{C}$ denote the Cayley complex of J_4'.

Fact. [Genevois, 2022]

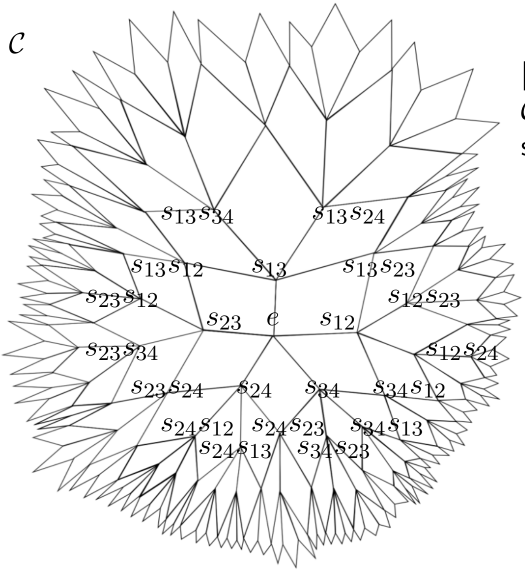
The map Γ induced from the following Γ_0 implies an action of PJ_4 on $\mathcal{C}$. (Note that $\mathcal{C}^{(0)} = J_4'$.)

$$\Gamma_0 : PJ_4 \times \mathcal{C}^{(0)} \longrightarrow \mathcal{C}^{(0)}$$

$$(g, h) \longmapsto \begin{cases} gh & gh \in J_4' \\ ghs_{14} & gh \notin J_4', \end{cases}$$

Moreover, Γ acts on $\mathcal{C}$ freely and cocompactly.

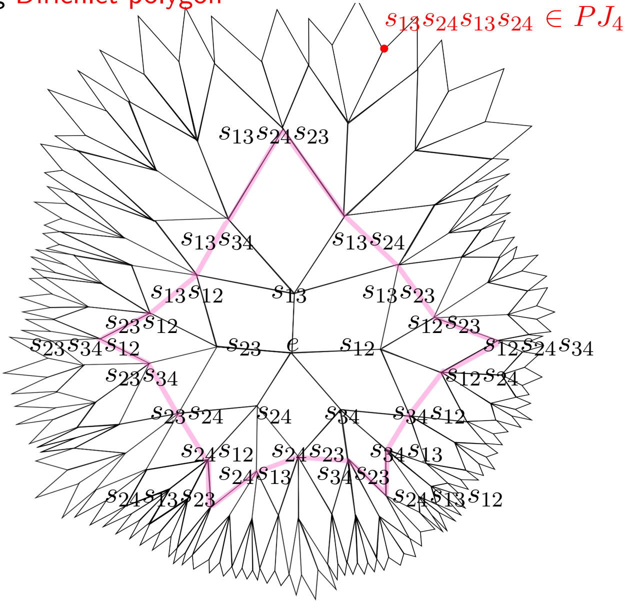
$\mathcal{C}$

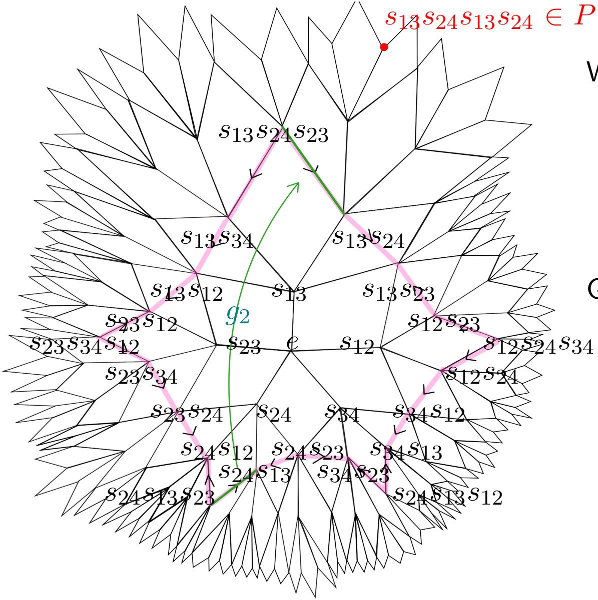


[Fact]
$\mathcal{C}$ is isometric to $\mathbb{H}^2$ up to scaling.

$$s_{ij}^2 = e$$
$$s_{13}s_{12} = s_{23}s_{13}$$
$$s_{24}s_{23} = s_{34}s_{24}$$
$$s_{12}s_{34} = s_{34}s_{12}$$

Finding Dirichlet polygon





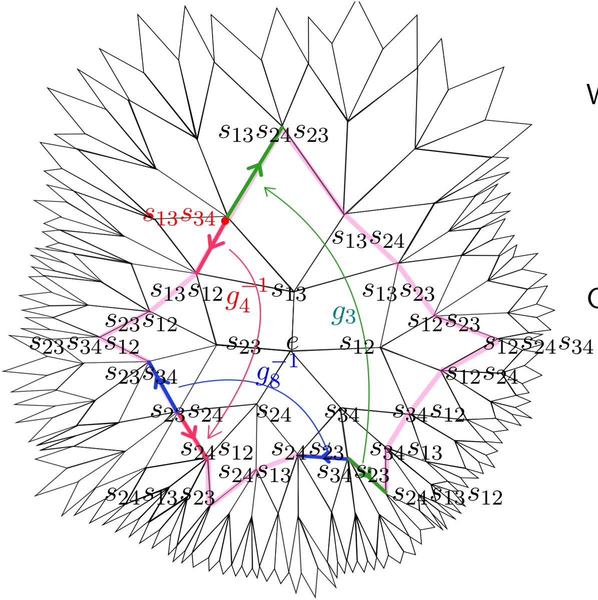
We obtain a generator:

g_2

$\downarrow$

Generators:

$g_1, g_2, g_3, g_4, g_5,$
$g_6, g_7, g_8, g_9, g_{10}$



We obtain a cycle relation:

$$g_3 g_8^{-1} g_4^{-1}$$

$$\downarrow$$

Complete set of relations:

$$g_3 g_8^{-1} g_4^{-1},$$
$$g_5 g_9^{-1} g_4^{-1},$$
$$g_5 g_1 g_6^{-1},$$
$$g_8 g_{10} g_7^{-1},$$
$$g_{10} g_1^{-1} g_2,$$
$$g_3 g_6^{-1} g_7 g_9^{-1} g_2^{-1} \qquad \square$$

Explicit isomorphism

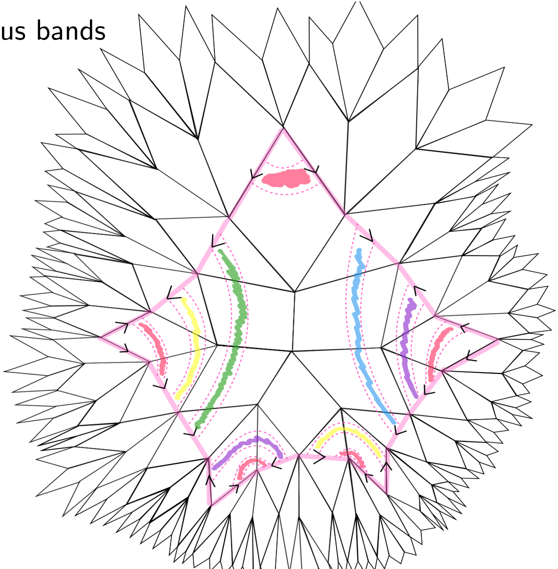
The following map is shown to be an isomorphism.

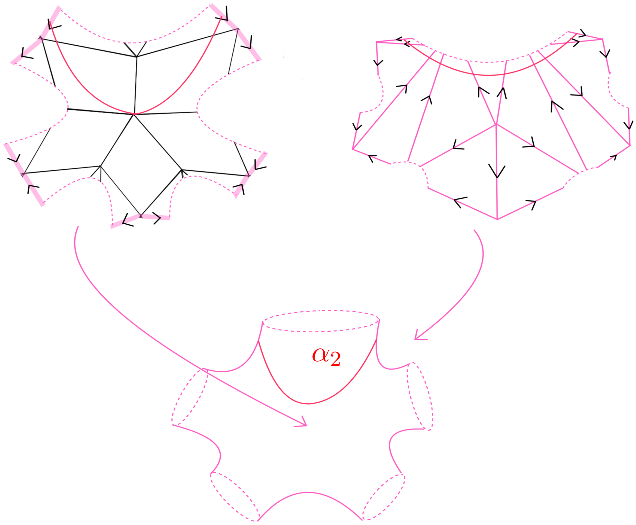
$$f : \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 | \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5^2 \rangle$$
$$\longrightarrow \left\langle g_1, \cdots, g_{10} \;\middle|\; \begin{array}{l} g_1 g_{10}^{-1} g_2^{-1}, g_9 g_5^{-1} g_4, g_5 g_1 g_6^{-1}, \\ g_8 g_{10} g_7^{-1}, g_8 g_3^{-1} g_4, g_2 g_9 g_7^{-1} g_6 g_3^{-1} \end{array} \right\rangle$$

$$\begin{aligned}
\alpha_1 &\longmapsto g_1^{-1} = g_{10}^{-1} g_2^{-1} \\
\alpha_2 &\longmapsto g_2 g_{10} g_5^{-1} g_8 g_3^{-1} = g_2 g_{10} g_9^{-1} g_4^{-2} \\
\alpha_3 &\longmapsto g_4 \\
\alpha_4 &\longmapsto g_9 g_{10}^{-1} \\
\alpha_5 &\longmapsto g_8^{-1} g_6 = g_8^{-1} g_4 g_9 g_2 g_{10}
\end{aligned}$$

How to find the map?

Five Möbius bands




α_2

Introduction

Presentation of PJ_4

Proof 1

PJ_n and Configuration space of points on S^1

Proof 2

Fact. [Henriques-Kamnitzer, 2006]

- $PJ_n \cong \pi_1(\overline{M_{0,n+1}}(\mathbb{R}))$

$M_{0,n+1}(\mathbb{R})$: the real points of the moduli space of genus 0 curves with $n+1$ marked points.

$\overline{M_{0,n+1}}(\mathbb{R})$: Deligne-Mumford compactification of $M_{0,n+1}(\mathbb{R})$.

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It is known that $M_{0,n+1}(\mathbb{R}) \simeq X(n+1)$.

$X(n+1)$: the configuration space of $n+1$ points on the circle.

$\overline{X(n+1)}$: the "natural" compactification introduced by [Yoshida, 1996].

But we do **NOT** know $\overline{M_{0,n+1}}(\mathbb{R})$ and $\overline{X(n+1)}$ are homeomorphic or not $(n \geq 6)$.

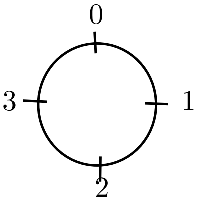
Definition.

The configuration space $X(n)$ of n points on S^1 is defined by

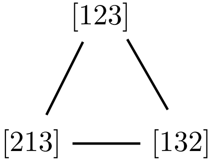
$$X(n) = PGL(2)\backslash\{(\mathbb{P}^1)^n - \Delta\}$$

where $\Delta = \{(x_1, \cdots, x_n) \mid x_i = x_j \text{ for some } i \neq j\ \}$ and the projective general linear group $PGL(2)$ acts diagonally and freely.

$X(4) \ni [123]$



$\overline{X(4)}$

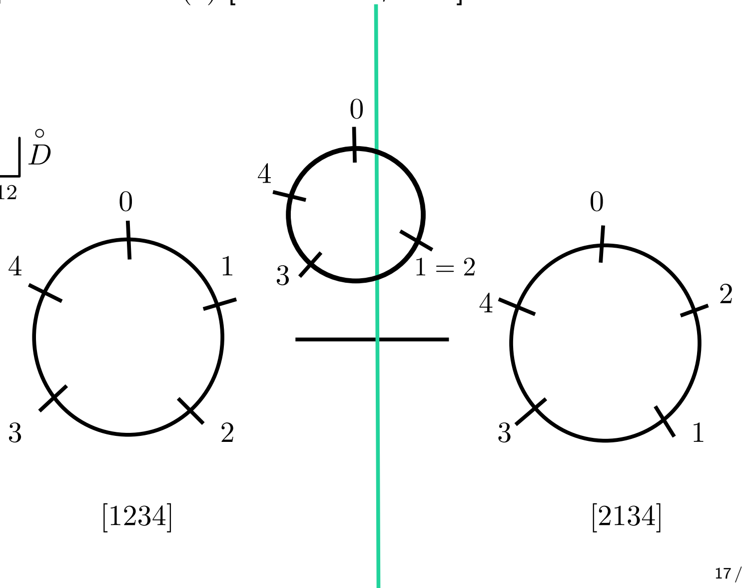


$$X(4) \simeq \bigsqcup_{3} (0,1)$$

$$\overline{X(4)} \simeq S^1$$

The compactification $\overline{X(5)}$ [c.f. Yoshida, 1996]

$$X(5) \simeq \bigsqcup_{12} \mathring{D}$$



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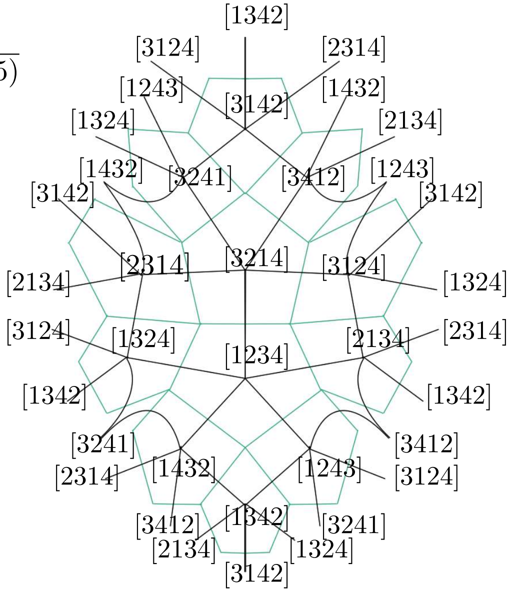

$$\overline{X(5)} \simeq \overset{5}{\#}\mathbb{RP}^2$$

Recall:

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- $X(n+1) \simeq M_{0,n+1}(\mathbb{R})$

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Question.

Is PJ_n isomorphic to $\pi_1(\overline{X(n+1)})$?

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Question.

Is PJ_n isomorphic to $\pi_1(\overline{X(n+1)})$?

YES. for $n = 4, 5$

- $\overline{M_{0,4}}(\mathbb{R}) \simeq S^1$ and $\overline{X(4)} \simeq S^1$
- $\overline{M_{0,5}}(\mathbb{R}) \simeq \overset{5}{\#}\mathbb{RP}^2$ and $\overline{X(5)} \simeq \overset{5}{\#}\mathbb{RP}^2$

Recall:

- $PJ_n \cong \pi_1(\overline{M_{0,n+1}}(\mathbb{R}))$
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Question.

Can we show $PJ_n \simeq \pi_1(\overline{X(n+1)})$ directly? $(n = 3, 4)$

PJ_3

Proposition. [Genevois, 2022]

PJ_3 acts on $C_3^{\{2\}}(\simeq \mathbb{R})$.

Theorem 2. [Hama-I., P.I.N.S., Nihon Univ.]

$\exists \varphi : \widetilde{X(4)} \to C_3^{\{2\}}$ so that $\forall g \in PJ_3$, $\exists \tilde{g} \in \pi_1(\overline{X(4)})$ s.t.

$$\begin{array}{ccc} \widetilde{X(4)} & \xrightarrow{\widetilde{\Gamma}_{\tilde{g}}} & \widetilde{X(4)} \\ \varphi \downarrow & & \downarrow \varphi \\ {C_3}^{\{2\}} & \xrightarrow{\Gamma_g} & {C_3}^{\{2\}} \end{array}$$

Corollary.

PJ_3 is isomorphic $\pi_1(\overline{X(4)})$.

PJ_4

Proposition. [Genevois, 2022]

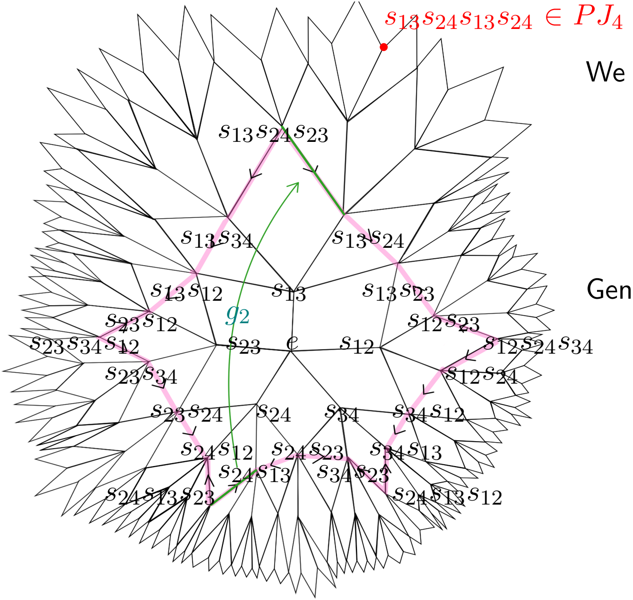
PJ_4 acts on $\mathcal{C}(\simeq \mathbb{H}^2)$.

Theorem 3. [Hama-I.]

$\overline{X(5)}$ is homeomorphic to $\mathcal{C}/PJ_4$.

Corollary.

$\pi_1\left(\overline{X(5)}\right)$ is isomorphic to PJ_4.



We obtain a generator:

$$g_2 = s_{13}s_{24}s_{13}s_{24}$$

$$\downarrow$$

Generators:

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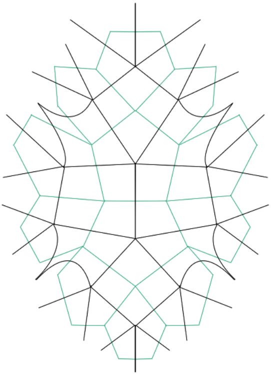

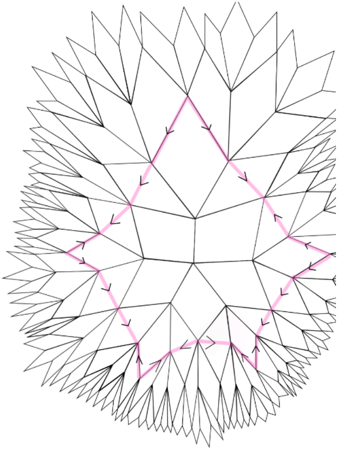

Thank you
for your attention.