

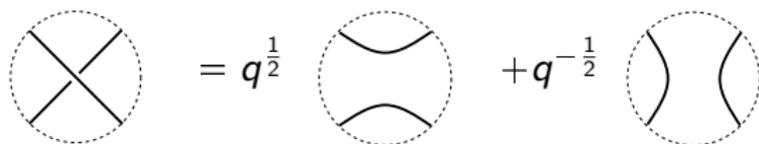
# Skein modules of mapping tori of the 2-torus

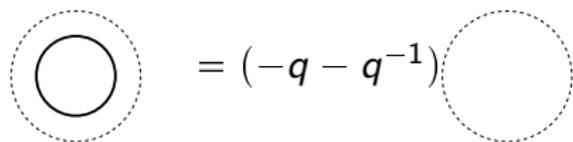
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Intelligence of Low Dimensional Topology  
26th May 2025

## Skein modules

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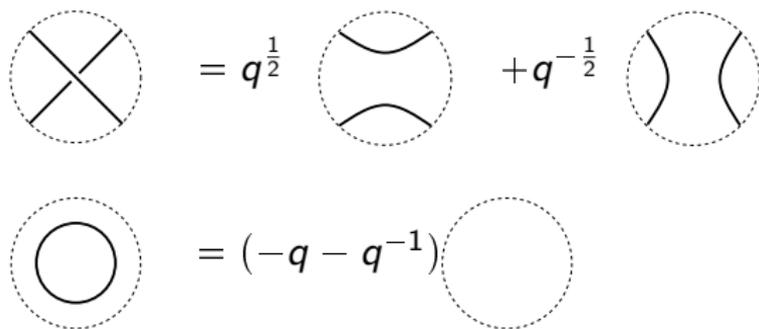
$$\text{Crossing} = q^{\frac{1}{2}} \text{Positive Crossing} + q^{-\frac{1}{2}} \text{Negative Crossing}$$


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We denote it  $\text{Sk}(M)$ .

**Note:** We can interpret framed links locally as morphisms between tensor powers of the defining representation in  $\text{Rep } U_q \mathfrak{sl}_2$ .

Considering a different Lie group  $G$  gives a more general notion of skein module  $\text{Sk}_G(M)$ .

★ = we know how to generalize to  $G = \text{GL}_N, \text{SL}_N$ .

## Skein module dimensions

Let  $M$  be compact, oriented, and without boundary.

It was shown in [Gunningham-Jordan-Safronov 23] that

$$\dim \text{Sk}(M) < \infty$$

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Some computations:

$$M = L(p, q) \quad \dim \text{Sk}(M) = \lfloor \frac{p}{2} \rfloor + 1 \quad [\text{Hoste-Przytycki 93}]$$

$$M = \mathbb{T}^3 \quad \dim \text{Sk}(M) \leq 9 \quad [\text{Carrega 17}]$$

$$\dim \text{Sk}(M) \geq 9 \quad [\text{Gilmer 18}]$$

$$M = \Sigma_g \times S^1 \quad \dim \text{Sk}(M) \geq 2^{2g+1} + 2g - 1 \quad [\text{Gilmer-Masbaum 19}]$$

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**This talk:** dimensions for mapping tori of  $\mathbb{T}^2$ , from [Kinnear 25].

## Mapping tori of $\mathbb{T}^2$

Let  $\gamma \in \text{Mod}(\mathbb{T}^2)$ . Consider

$$M_\gamma = (\mathbb{T}^2 \times [0, 1]) / ((a, 0) \sim (\gamma(a), 1)).$$

Note  $M_\gamma \cong M_\phi$  as oriented manifolds iff  $\gamma$  and  $\phi$  are conjugate ( $\sim$ ) in  $\text{Mod}(\mathbb{T}^2)$ . Recall

$$\text{Mod}(\mathbb{T}^2) \cong \text{SL}_2(\mathbb{Z}) = \langle S, T \mid S^4 = \text{Id}, (ST)^3 = S^2 \rangle.$$

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- ▶  $|\text{tr}(\gamma)| = 0$ : then  $\gamma \sim \pm S$  for  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .
- ▶  $|\text{tr}(\gamma)| = 1$ : then  $\gamma \sim \pm E^{\pm 1}$  for  $E = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ .
- ▶  $|\text{tr}(\gamma)| = 2$ : then  $\gamma \sim \pm T^n$  for  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ,  $n \in \mathbb{Z}$ .
- ▶  $|\text{tr}(\gamma)| > 2$ : classification by continued fractions.

# Main Theorem

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Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ . Then  $\dim \mathrm{Sk}(M_\gamma)$  is as follows:

- ▶  $|\mathrm{tr}(\gamma)| = 0$ : then  $\dim \mathrm{Sk}(M_\gamma) = 6$ .
- ▶  $|\mathrm{tr}(\gamma)| = 1$ : then  $\dim \mathrm{Sk}(M_\gamma) = 4$ .
- ▶  $|\mathrm{tr}(\gamma)| = 2$ : then  $\gamma \sim \pm T^n$ , and

$$\dim \mathrm{Sk}(M_\gamma) = \begin{cases} 9 + k & n = 2k \\ 6 + k & n = 2k + 1 \end{cases}.$$

- ▶  $|\mathrm{tr}(\gamma)| > 2$ : then

$$\dim \mathrm{Sk}(M_\gamma) = |\mathrm{tr}(\gamma)| + 2^{c(\gamma)+1}$$

where

$$c(\gamma) = \#\{m \in \{\mathrm{gcd}(a-1, b, c, d-1), \mathrm{tr}(\gamma)\} : m \text{ even}\}.$$

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$$\dim \mathrm{Sk}(M_\gamma) = 2^{c(\gamma)} + \frac{\prod_{i=1}^{r_+} a_i^+ + 2^{p_+}}{2} + \frac{\prod_{i=1}^{r_-} a_i^- + 2^{p_-}}{2}$$

where  $r_\pm$  is the rank of

$$\mathrm{Id} \mp \gamma = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

and

$$a_1^\pm = \gcd(w, x, y, z), \quad a_2^\pm = \frac{|\mathrm{tr}(\mathrm{Id} \mp \gamma)|}{a_1^\pm}$$

are the invariant factors of  $\mathrm{Id} \mp \gamma$ , and

$$p_\pm = \#\{a_i^\pm \text{ even} : 1 \leq i \leq r_\pm\}.$$

## Elements of proof

Lemma (★ [Carrega 17])

*For  $M$  a 3-manifold,  $\text{Sk}(M)$  is graded by  $H_1(M; \mathbb{Z}/2\mathbb{Z})$ .*

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So

$$\text{Sk}(M_\gamma) = S_0 \oplus S_1.$$

## Twisted Hochschild homology

Recall  $\text{Sk}(\mathbb{T}^2 \times [0, 1])$  is an algebra. Denote this  $\text{SkAlg}(\mathbb{T}^2)$ .

We see that

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Can give an explicit presentation of  $E$  as a 4-dimensional algebra, and calculate

$$\dim S_1 = \dim \text{HH}_0^\gamma(E) = 2^{c(\gamma)}.$$

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All together gives:

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- ▶ But  $\dim \text{Sk}(M_{\gamma})$  is not bounded above.

## Further questions

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What about  $\text{HH}_\bullet^\gamma$ ? (Bai, Kinnear)

- ▶ We considered  $\text{coker}(\text{Id} \mp \gamma)$ . In the paper [Chun-Gukov-Park-Sopenko 20] this is identified with the (almost) abelian flat  $SL_2(\mathbb{C})$ -connections on  $M_\gamma$ , and is used to define the so-called  $\hat{Z}$ -invariant. Is there a connection with skein theory?

## References I

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